

Quantum fields and Noether charges for κ -spacetime symmetries

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- **Motivation**
- **Intro to κ -Poincaré**
- **Translation symmetries and Noether charges (classical fields)**
- **Quantum fields: path integral (and canonical) quantization**
- **Conclusions and outlook**

Motivation

Historical

- **Early 90's (Lukierski et al):** use “quantum groups” (non-co-commutative Hopf algebra) to describe “quantization” of standard relativistic symmetries (analogous to Moyal quantization of Poisson manifolds: **CM**→**QM**)
- **1994 (Majid and Ruegg):** κ -Poincaré and its relation to κ -Minkowski **NCST**
- **2000 (Amelino-Camelia):** κ -Poincaré as a way to introduce $L_p \sim 1/\kappa$ as an **observer-independent** scale in SR (DSR)
- **2004 (Amelino, Smolin, Starodubstev):** κ -Poincaré from **low-energy limit** of 2+1 QG (and speculations for 3+1)
- **2006 (Freidel and Livine):** κ -Poincaré-type non-commutative **effective field theory** emerging from 3d Ponzano-Regge+massive particles

Consider κ -Poincaré and κ -Minkowski as a “window” on QG and try to understand how the usual physics in flat space looks like in this new scenario

κ -Poincare' in a nutshell I: Hopf algebras

Hopf algebras generalize algebras

algebra (*unital, associative*) (A, m, η) :

$$m : A \otimes A \rightarrow A; \quad \eta : \mathbb{C} \rightarrow A$$

add

$$\Delta : A \rightarrow A \otimes A; \quad \varepsilon : A \rightarrow \mathbb{C}$$

and (together with appropriate properties) you get

bialgebra $(A, m, \eta, \Delta, \varepsilon)$

add a map (antipode)

$$S : A \rightarrow A$$

and you have a **Hopf algebra** $(A, m, \eta, \Delta, \varepsilon, S)$

the additional structure introduced is motivated by the definition of *tensor product representations of algebras*...

κ -Poincaré' in a nutshell I: Hopf algebras (appendix)

Algebra axioms:

$$m(m \otimes id) = m(id \otimes m) \quad \text{associativity}$$

$$m(id \otimes \eta) = m(\eta \otimes id) = id \quad \text{unit}$$

Co-algebra axioms

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \quad \text{co-associativity}$$

$$(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id \quad \text{co-unit}$$

Antipode

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta \circ \varepsilon$$

(id identity map on A)

κ -Poincaré' in a nutshell II: the bi-crossproduct basis

(Majid-Ruegg 1994) "Quantum" deformation of Poincaré algebra (def. parameter $1/\kappa$)
Coproducts for P_0, P_i translations, M_i rotations and N_i boosts

$$\begin{aligned}\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 & \Delta(P_j) &= P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j \\ \Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j \\ \Delta(N_j) &= N_j \otimes 1 + e^{-P_0/\kappa} \otimes N_j + \frac{\epsilon_{jkl}}{\kappa} P_k \otimes N_l.\end{aligned}$$

antipodes

$$\begin{aligned}S(M_l) &= -M_l \\ S(P_0) &= -P_0 \\ S(P_i) &= -e^{\frac{P_0}{\kappa}} P_i \\ S(N_l) &= -e^{\frac{P_0}{\kappa}} N_l + \frac{1}{\kappa} \epsilon_{ljk} e^{\frac{P_0}{\kappa}} P_j M_k,\end{aligned}$$

co-units

$$\epsilon(P_\mu) = \epsilon(M_j) = \epsilon(N_k) = 0.$$

κ -Poincaré' in a nutshell III: the bi-crossproduct basis

The **Hopf algebra multiplication** is implicitly defined through the **commutators** (standard "commutator bracket" construction from an associative algebra)

$$\begin{aligned} [P_0, P_j] &= 0 & [M_j, M_k] &= i\epsilon_{jkl}M_l & [M_j, N_k] &= i\epsilon_{jkl}N_l & [N_j, N_k] &= i\epsilon_{jkl}M_l \\ [P_0, N_l] &= -iP_l & [P_l, N_j] &= -i\delta_{lj}\left(\frac{\kappa}{2}\left(1 - e^{-\frac{2P_0}{\kappa}}\right) + \frac{1}{2\kappa}\vec{P}^2\right) + \frac{i}{\kappa}P_lP_j \\ [P_0, M_k] &= 0 & [P_j, M_k] &= i\epsilon_{jkl}P_l \end{aligned}$$

The *mass Casimir invariant*

$$C_\kappa = \left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}} \quad (1)$$

In the limit $\kappa \rightarrow \infty$ recover the *trivial* Hopf algebra naturally associated to the Poincaré algebra, in particular in such limit one recovers **co-commutativity**

$$\sigma \circ \Delta = \Delta \circ id$$

($\sigma : A \otimes A \rightarrow A \otimes A$ "flip" map $\sigma(a \otimes b) = b \otimes a$)

Relation between κ -Minkowski NCST and κ -Poincaré:

$$[x_m, t] = \frac{i}{\kappa} x_m, \quad [x_m, x_l] = 0 \quad (2)$$

An “intuitive” argument:

- Consider plane waves e^{ipx} : $P_\mu \triangleright e^{ipx} = p_\mu e^{ipx}$
- Define a product $(*)$ for such functions, need to be compatible with co-product:

$$*(\Delta(P_\mu) \triangleright (e^{iq_1x} \otimes e^{iq_2x})) \equiv P_\mu \triangleright (e^{iq_1x} * e^{iq_2x}) \quad (3)$$

- Work out the LHS and obtain

$$P_\mu \triangleright (e^{iq_1x} * e^{iq_2x}) = (q_1 \dot{+} q_2) (e^{iq_1x} * e^{iq_2x}) \quad (4)$$

with $q_1 \dot{+} q_2 = (q_1^0 + q_2^0; \vec{q}_1 + e^{-q_1^0/\kappa} \vec{q}_2)$

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$$e^{iq_1x} * e^{iq_2x} \equiv e^{(q_1 \dot{+} q_2)x} \quad (5)$$

NOTE: $q_1 \dot{+} q_2 \neq q_2 \dot{+} q_1$ i.e. we have a **non-commutative algebra**

If we had started from NC plane-waves $e^{ip\hat{x}}$

- **Fix** a *normal ordering* for the non-commuting coordinates (e.g. all time coordinates to the right)

$$: e^{ip\hat{x}} : \equiv e^{ip_m \hat{x}_m} e^{-ip_0 \hat{x}_0}$$

- Wave exponentials combine in a non-trivial way: using the BCH formula

$$(: e^{iq_1 \hat{x}} :)(: e^{iq_2 \hat{x}} :) = : e^{i(q_1 + q_2) \hat{x}} :$$

same as above!

A choice of ordering is equivalent to a choice of **Weyl map** Ω

$$\begin{aligned}\Omega(e^{ipx}) &= e^{ip_m \hat{x}_m} e^{-ip_0 \hat{x}_0} \\ \Omega(e^{ipx}) \cdot \Omega(e^{ikx}) &= \Omega(e^{ipx} * e^{ikx})\end{aligned}$$

Such map is (obviously) **not unique**

$$\begin{aligned}\Omega_s(e^{ipx}) &= e^{-ip_0 \hat{x}_0/2} e^{ip_m \hat{x}_m} e^{-ip_0 \hat{x}_0/2} \\ \Omega_s(e^{ipx}) \cdot \Omega_s(e^{ikx}) &= \Omega_s(e^{ipx} *_s e^{ikx})\end{aligned}$$

these are **equivalent descriptions of the same field!**

The **action** of rotation and boost symmetries on such fields are well defined:

- **Rotations** are “classical”

$$M_j \triangleright \Omega(e^{ipx}) = \Omega(M_j^c \triangleright e^{ipx})$$
$$M_j \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(M_j^c \triangleright e^{ipx} * e^{ikx} + e^{ipx} * M_j^c \triangleright e^{ikx})$$

- **Boosts** are “deformed”

$$N_j \triangleright \Omega(e^{ipx}) = \Omega(N_j^\kappa \triangleright e^{ipx})$$
$$N_j \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(*(\Delta(N_j^\kappa) \triangleright (e^{ipx} \otimes e^{ikx})))$$

- **Translations** are **classical** on a single plane wave i.e.

$$P_\mu \triangleright \Omega(e^{ipx}) = \Omega(P_\mu^c \triangleright e^{ipx}) \quad (6)$$

but due to **non-trivial coproduct** action is **deformed** on products of waves e.g.:

$$P_i \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(P_i^c \triangleright e^{ipx} * e^{ikx} + e^{-P_0^c/\kappa} \triangleright e^{ipx} * P_i^c \triangleright e^{ikx}) \quad (7)$$

κ -Poincaré symmetries and choices of Weyl maps

Changing the choice of Weyl map **should not** change the way “symmetries” act on functions in κ -Minkowski...but

$$P_\mu \triangleright \Omega(e^{ipx} * e^{ikx})$$
$$P_\mu \triangleright \Omega_s(e^{ipx} *_s e^{ikx})$$

requires **different co-products for the P_μ s for each choice of the $*$ -product** i.e. different “bases” of the κ -Poincaré (Hopf) algebra (P_μ, M_i, N_i) and (P_μ^s, M_i^s, N_i^s) .

Rotations and **boosts** are NOT affected by such degeneracy in the choice of Weyl map¹ e.g.

$$M_i^s \triangleright \Omega(e^{ipx}) = M_i \triangleright \Omega(e^{ipx})$$

while instead for **translations**

$$P_\mu^s \triangleright \Omega(e^{ipx}) \neq P_\mu \triangleright \Omega(e^{ipx})$$

What's going on?? Ambiguity in defining the action of translation generators (?)...

¹ hep-th/0306013

Translations and Noether charges for classical fields I

Take a more “pragmatical” approach:

characterize the symmetries in terms of the infinitesimal variations of the fields:

classical case

$$\begin{aligned}x_\mu \rightarrow x'_\mu &= x_\mu + dx_\mu \\ f(x) \rightarrow f'(x) &= f(x) + iP_\mu f(x) dx_\mu = f(x) + df(x)\end{aligned}$$

In the κ -deformed case we also **need to specify** dx_μ s which must obey ²

$$[x_j + dx_j, x_0 + dx_0] = \frac{i}{\kappa}(x_j + dx_j), \quad [x_i + dx_i, x_j + dx_j] = 0$$

A *further ambiguity* seem to emerge: $df = iP_\mu f(x) dx_\mu$ or $df = id x_\mu P_\mu f(x)$?...Leibnitz $d(fg) = (df)g + f(dg)$ restricts the choices to ONE, e.g. the **unique choice** for the “time-to-the-right” Weyl map is

$$df = id x_\mu P_\mu f(x)$$

a different choice of the Weyl map affects the action of the P_μ s but leads to the same df ! (hep-th/0607221)

²In dimension > 4 there exist other choices for the dx_μ s see e.g. Freidel, Kowalski-Glikman and Nowak, hep-th/0612170

Translations and Noether charges for classical fields II

Translational symmetries at work: **free massless scalar field** Field on κ -Minkowski

$$\Phi(\hat{x}) = \int d^4q \tilde{f}(q) \Omega(e^{iqx}) = \int d^4q \tilde{\Phi}(q) e^{iq_i \hat{x}_i} e^{-iq_0 \hat{x}_0} \quad (8)$$

Ordering prescription is **fixed**, drop $\hat{}$ from now on...

κ -Klein-Gordon e.o.m.

$$C_\kappa(P_\mu)\Phi \equiv \left[(2\kappa)^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - e^{P_0/\kappa} \vec{P}^2 \right] \Phi = 0 \quad (9)$$

Derived from action

$$S[\Phi] = \int d^4x \mathcal{L}[\Phi(x)] = \int d^4x \frac{1}{2} \tilde{P}_\mu \Phi \tilde{P}^\mu \Phi \quad (10)$$

with

$$\tilde{P}_0 = (2\kappa) \sinh(P_0/2\kappa) \quad \tilde{P}_j = e^{P_0/2\kappa} P_j \quad (11)$$

Noether analysis for translations is rather straightforward \Rightarrow **conserved charges**

$$Q_\mu = \int d^4p \frac{e^{3p_0/\kappa}}{2} p_\mu \tilde{\Phi}(p_0, \vec{p}) \tilde{\Phi}(-p_0, -e^{p_0/\kappa} \vec{p}) \frac{p_0}{|p_0|} \delta(C_\kappa(p_\mu)) \quad (12)$$

- Most interest in κ -Minkowski/Poincaré scenarios from the “phenomenological” side has been motivated by possible **deformations** of energy-momentum **dispersion relation** that one expects from the **deformed Casimir**

$$C_\kappa = \left(2\kappa \sinh \left(\frac{P_0}{2\kappa} \right) \right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}}$$

which could lead to **testable predictions** (e.g. trans-GZK events, time-of-flight tests using GRB..)

- HOWEVER ambiguity in defining the action for generators of translations has been cause for **concern** in the recent past...
- We defined translations **un-ambiguously** and the Noether charges we found DO INDEED obey a deformed dispersion relation (see hep-th/0607221)
- All this was done with *classical fields*...need **quantum fields**...

$\lambda\phi^4$ and κ -Poincare': a path integral approach I (setting)

An early proposal of a κ -QFT model (Amelino-Camelia and MA; Phys. Rev. D **65**,(2002)):

- Start with a partition function on κ -Minkowski

$$Z[J(x)] = \int \mathcal{D}[\Phi] e^{i \int d^4x [\mathcal{L}_0 - \frac{\lambda}{4!} \Phi^4(x) + \frac{1}{2} (J(x)\Phi(x) + \Phi(x)J(x))]}$$

with $\mathcal{L}_0 = \frac{1}{2} \left(\tilde{P}_\mu \Phi \tilde{P}^\mu \Phi - m^2 \Phi^2 \right)$

- go to momentum space, the **normalized free partition function**

$$\bar{Z}^0[J(k)] \equiv \frac{Z^0[J(k)]}{Z^0[0]} = \exp \left(-\frac{i}{2} \int d^4k \frac{J(k)J(\dot{-}k)}{\mathcal{C}_\kappa(k) - m^2} \right) \quad (13)$$

with $\dot{-}k \equiv \left(-k_0, -e^{\frac{k_0}{\kappa}} \vec{k} \right)$ coming from the **antipode**

- using appropriate **generalization of the functional derivative**

$$\bar{Z}[J(k)] = \exp \left[i \frac{\lambda}{24} \int \delta^{(4)} \left(\sum_{k_1, k_2, k_3, k_4} \dot{-} \right) \prod_{j=1}^4 \frac{d^4 k_j}{2\pi} \xi(k_{j,0}) \frac{\delta}{\delta J(\dot{-}k_j)} \right] \bar{Z}^0[J(k)]$$

with

$$\xi(k_{j,0}) \equiv 2 \left(1 + e^{\frac{3k_{j,0}}{\kappa}} \right)^{-1} \quad \sum_{k_1, k_2, k_3, k_4} \dot{-} \equiv k_1 + k_2 + k_3 + k_4$$

$\lambda\phi^4$ and κ -Poincare': a path integral approach II (results)

Perturbative expansion in λ

- Role of non-trivial topology of Feynman graphs beyond tree-level

$$G_0^{(2)}(p, \dot{-}p') \sim \frac{\delta^{(4)}(p - p')}{\mathcal{C}_\kappa(p) - m^2} \quad (14)$$

$$G_\lambda^{(2)}(p, \dot{-}p')_{\text{planar}}^{\text{connected}} \sim \frac{\delta^{(4)}(p - p')}{(\mathcal{C}_\kappa(p) - m^2)(\mathcal{C}_\kappa(p') - m^2)} \int \frac{d^4q}{\mathcal{C}_\kappa(q) - m^2} \quad (15)$$

$$G_\lambda^{(2)}(p, \dot{-}p')_{\text{non-planar}}^{\text{connected}} \sim \int \frac{d^4q \delta(p_0 - p'_0) \delta^{(3)}(e^{-p_0/\kappa} \vec{q} - \vec{p} + \vec{q} + e^{-q_0/\kappa} \vec{p}')}{(\mathcal{C}_\kappa(q) - m^2)(\mathcal{C}_\kappa(p) - m^2)(\mathcal{C}_\kappa(p') - m^2)} \quad (16)$$

- non-trivial “scattering” kinematics for tree-level vertex

$$G_\lambda^{(4)}(p_1, p_2, \dot{-}p_3, \dot{-}p_4)^{\text{connected}} \sim \frac{\lambda}{4!} \sum_{\mathcal{P}(\dot{-}p_1, \dot{-}p_2, p_3, p_4)} \left[\delta^{(4)}(\dot{-}p_1 + \dot{-}p_2 + p_3 + p_4) \right]$$

very interesting but these are just basic results...need to roll up sleeves and find out more...

Symmetries and symplectic geometry of κ -fields I

Looking for **energy and momentum charges**...useful to resort to **canonical quantization**... standard “textbook” approach imposing ETCRs on κ -Minkowski is *problematic*...

An **alternative strategy** motivated by recent work...

hep-th/0701268 study the symmetries of classical κ -fields **borrowing the *basic tools* of “covariant phase space” formalism:**

- Start with a **standard relativistic massless scalar field**
- Key point: $\{\phi(x); \pi(x)\} \in \Gamma \longleftrightarrow \Phi \in \mathcal{S}$ **identify the phase space Γ with the space of solutions of the (Klein-Gordon) equation of motion \mathcal{S}**
- On \mathcal{S} is defined³ a **symplectic 2-form** ω (which on the standard phase space manifold Γ corresponds to the familiar $\omega = \frac{1}{2} \int_{\Sigma_t} \delta\Pi \wedge \delta\Phi$)

Our strategy:

- (i) **use a map m between \mathcal{S} and \mathcal{S}_κ to define a symplectic structure on \mathcal{S}_κ**
- (ii) **express the conserved charges associated with κ -symmetries through the symplectic structure**

³ see C. Crnkovic, *Class. Quant. Grav.* **5**, 1557 (1988)

Symmetries and symplectic geometry of κ -fields II

In the **standard framework** the symplectic structure of phase space defines an hermitian **inner product** on the space of complex solutions

$$(\Phi_1, \Phi_2) = -2i \omega(\Phi_1^*, \Phi_2) = \int \frac{d^4 p}{(2\pi)^3} \delta(\mathcal{C}(p)) \frac{p_0}{|p_0|} \tilde{\Phi}_1^*(-p) \tilde{\Phi}_2(p) \quad (17)$$

Noether charges associated with translations can be written as

$$Q_\mu = \frac{1}{2} (\Phi, P_\mu \triangleright \Phi) = \int \frac{d^4 p}{2(2\pi)^3} \delta(\mathcal{C}(p)) \frac{p_0}{|p_0|} p_\mu \tilde{\Phi}^*(-p) \tilde{\Phi}(p) \quad (18)$$

Using our construction (*with almost no sweat!*) **one gets**

$$Q_\mu^\kappa = \int \frac{d^4 p}{2(2\pi)^3} \delta(\mathcal{C}_\kappa(p)) \frac{p_0}{|p_0|} p_\mu e^{\frac{3p_0}{\kappa}} \tilde{\Phi}^*(-p) \tilde{\Phi}(p) \quad (19)$$

our κ -Noether charges !

with a bonus... an **inner product**

$$(\Phi_1, \Phi_2)_\kappa = \int \frac{d^4 p}{(2\pi)^3} \delta(\mathcal{C}_\kappa(p)) \frac{p_0}{|p_0|} e^{\frac{3p_0}{\kappa}} \tilde{\Phi}_1^*(-p) \tilde{\Phi}_2(p), \quad (20)$$

Canonical quantization of free κ -fields

Caution!!! work in progress...

- Given an inner product standard construction of **one-particle** Hilbert space \mathcal{H}_κ form $\mathcal{S}_\kappa \dots$
- in defining the **algebra of creation and annihilation operators** **crucial** role of **non-trivial coproduct** e.g.

$$\star(\Delta(P_\mu) \blacktriangleright (a_{\vec{q}_1}^\dagger \otimes a_{\vec{q}_2}^\dagger)) \equiv P_\mu \blacktriangleright (a_{\vec{q}_1}^\dagger \star a_{\vec{q}_2}^\dagger) \quad (21)$$

the standard composition of operators must be deformed to be compatible with the co-product!

$$P_\mu \triangleright (a_{\vec{q}}^\dagger |0\rangle) = (P_\mu \blacktriangleright a_{\vec{q}}^\dagger) |0\rangle = q_\mu (a_{\vec{q}}^\dagger |0\rangle) \quad (22)$$

$$P_\mu \triangleright (a_{\vec{q}_1}^\dagger \star a_{\vec{q}_2}^\dagger |0\rangle) = q_{1\mu} \dot{+} q_{2\mu} (a_{\vec{q}_1}^\dagger \star a_{\vec{q}_2}^\dagger |0\rangle) \dots \quad (23)$$

- To construct the **full Fock-space** of the theory from n -particle states need a **symmetrization principle**...in the standard case “bosonization” is put in by hand, in our case $a_{\vec{q}_1}^\dagger \star a_{\vec{q}_2}^\dagger |0\rangle \neq a_{\vec{q}_2}^\dagger \star a_{\vec{q}_1}^\dagger |0\rangle$ **symmetrization is subtler...stay tuned!**

Conclusions

- Possible to give a **physical** characterization of κ -Poincaré translations in terms of Noether charges for classical fields (“non-classical” Planck-scale symmetries were discussed only in terms of properties of the algebra of would-be-symmetry generators) **need quantum fields...**
- non-trivial task: reviewed earlier attempt in constructing κ -QFT and showed some surprising results
- want to focus on the energy and momentum charges carried by quantum fields a **canonical quantization** approach is more useful
- recent work on geometrical interpretation of κ -Noether charges seems to lead to the right direction...
- and there we reached the **border** of the κ -Poincaré *terra incognita*; our exploration continues and soon we will be back (hopefully) with more news...