Quantum fields and Noether charges for $\kappa$-spacetime symmetries

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Outline

- Motivation
- Intro to $\kappa$-Poincaré
- Translation symmetries and Noether charges (classical fields)
- Quantum fields: path integral (and canonical) quantization
- Conclusions and outlook
Motivation

Historical

- **Early 90’s (Lukierski et al):** use “quantum groups” (non-co-commutative Hopf algebra) to describe “quantization” of standard relativistic symmetries (analogous to Moyal quantization of Poisson manifolds: $\text{CM} \rightarrow \text{QM}$)

- **1994 (Majid and Ruegg):** $\kappa$-Poincaré and its relation to $\kappa$-Minkowski NCST

- **2000 (Amelino-Camelia):** $\kappa$-Poincaré as a way to introduce $L_p \sim 1/\kappa$ as an observer-independent scale in SR (DSR)

- **2004 (Amelino, Smolin, Starodubstev):** $\kappa$-Poincaré from low-energy limit of 2+1 QG (and speculations for 3+1)

- **2006 (Freidel and Livine):** $\kappa$-Poincaré-type non-commutative effective field theory emerging from 3d Ponzano-Regge+massive particles

Consider $\kappa$-Poincaré and $\kappa$-Minkowski as a “window” on QG and try to understand how the usual physics in flat space looks like in this new scenario
Hopf algebras generalize algebras

\textbf{algebra} (\textit{unital, associative}) \((A, m, \eta)\):

\[ m : A \otimes A \to A; \quad \eta : \mathbb{C} \to A \]

add

\[ \Delta : A \to A \otimes A; \quad \varepsilon : A \to \mathbb{C} \]

and (together with appropriate properties) you get \textbf{bialgebra} \((A, m, \eta, \Delta, \varepsilon)\)

add a map (antipode)

\[ S : A \to A \]

and you have a \textbf{Hopf algebra} \((A, m, \eta, \Delta, \varepsilon, S)\)

the additional structure introduced is motivated by the definition of tensor product representations of algebras...
$\kappa$-Poincare’ in a nutshell I: Hopf algebras (appendix)

Algebra axioms:

\[
m(m \otimes id) = m(id \otimes m) \quad \text{associativity}
\]

\[
m(id \otimes \eta) = m(\eta \otimes id) = id \quad \text{unit}
\]

Co-algebra axioms

\[
(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \quad \text{co-associativity}
\]

\[
(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id \quad \text{co-unit}
\]

Antipode

\[
m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta \circ \varepsilon
\]

(id identity map on $A$)
κ-Poincare’ in a nutshell II: the bi-crossproduct basis

(Majid-Ruegg 1994) “Quantum” deformation of Poincaré algebra (def. parameter 1/κ)

Coproducts for $P_0$, $P_i$ translations, $M_i$ rotations and $N_i$ boosts

\[
\begin{align*}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 \\
\Delta(P_j) &= P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j \\
\Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j \\
\Delta(N_j) &= N_j \otimes 1 + e^{-P_0/\kappa} \otimes N_j + \frac{\epsilon_{jkl}}{\kappa} P_k \otimes N_l.
\end{align*}
\]

antipodes

\[
\begin{align*}
S(M_l) &= -M_l \\
S(P_0) &= -P_0 \\
S(P_l) &= -e^{P_0/\kappa} P_l \\
S(N_l) &= -e^{P_0/\kappa} N_l + \frac{1}{\kappa} \epsilon_{ljk} e^{P_0/\kappa} P_j M_k,
\end{align*}
\]

co-units

\[\epsilon(P_\mu) = \epsilon(M_j) = \epsilon(N_k) = 0.\]
κ-Poincare’ in a nutshell III: the bi-crossproduct basis

The **Hopf algebra multiplication** is implicitly defined through the **commutators** (standard "commutator bracket" construction from an associative algebra)

\[
[P_0, P_j] = 0 \quad [M_j, M_k] = i \epsilon_{jkl} M_l \quad [M_j, N_k] = i \epsilon_{jkl} N_l \quad [N_j, N_k] = i \epsilon_{jkl} M_l
\]

\[
[P_0, N_l] = -i P_l \quad [P_l, N_j] = -i \delta_{lj} \left( \frac{\kappa}{2} \left( 1 - e^{-\frac{2 P_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{P}^2 \right) + \frac{i}{\kappa} P_l P_j
\]

\[
[P_0, M_k] = 0 \quad [P_j, M_k] = i \epsilon_{jkl} P_l
\]

The **mass Casimir invariant**

\[
C_\kappa = \left( 2\kappa \sinh \left( \frac{P_0}{2\kappa} \right) \right)^2 - \vec{P}^2 \epsilon \frac{P_0}{\kappa}
\] (1)

In the limit \( \kappa \to \infty \) recover the trivial Hopf algebra naturally associated to the Poincaré algebra, in particular in such limit one recovers **co-commutativity**

\[
\sigma \circ \Delta = \Delta \circ id
\]

\( (\sigma : A \otimes A \to A \otimes A "flip" \ map \sigma(a \otimes b) = b \otimes a) \)
Relation between $\kappa$-Minkowski NCST and $\kappa$-Poincaré:

\[
[x_m, t] = \frac{i}{\kappa} x_m, \quad [x_m, x_l] = 0
\] (2)

An “intuitive” argument:

- Consider plane waves $e^{ipx}$: $P_\mu \triangleright e^{ipx} = p_\mu e^{ipx}$
- Define a product ($\ast$) for such functions, need to be compatible with co-product:

\[
*(\Delta(P_\mu) \triangleright (e^{iq_1 x} \otimes e^{iq_2 x})) \equiv P_\mu \triangleright (e^{iq_1 x} \ast e^{iq_2 x})
\] (3)

- Work out the LHS and obtain

\[
P_\mu \triangleright (e^{iq_1 x} \ast e^{iq_2 x}) = (q_1^0 + q_2^0)(e^{iq_1 x} \ast e^{iq_2 x})
\] (4)

with $q_1^0 + q_2^0 = (q_1^0 + q_2^0; \vec{q}_1 + e^{-q_1^0/\kappa} \vec{q}_2)$

\[
e^{iq_1 x} \ast e^{iq_2 x} \equiv e^{(q_1^0 + q_2^0)x}
\] (5)

NOTE: $q_1^0 + q_2^0 \neq q_2^0 + q_1$ i.e. we have a non-commutative algebra
If we had started from NC plane-waves $e^{ip\hat{x}}$

- **Fix a normal ordering** for the non-commuting coordinates (e.g. all time coordinates to the right)

$$: e^{ip\hat{x}} : \equiv e^{ip_m\hat{x}_m} e^{-ip_0\hat{x}_0}$$

- Wave exponentials combine in a non-trivial way: using the BCH formula

$$(: e^{iq_1\hat{x}} :) (: e^{iq_2\hat{x}} :) = : e^{i(q_1 + q_2)\hat{x}} :$$

same as above!

A choice of ordering is equivalent to a choice of **Weyl map** $\Omega$

$$\Omega(e^{ipx}) = e^{ip_m\hat{x}_m} e^{-ip_0\hat{x}_0}$$

$$\Omega(e^{ipx}) \cdot \Omega(e^{ikx}) = \Omega(e^{ipx} \ast e^{ikx})$$

Such map is (obviously) **not unique**

$$\Omega_s(e^{ipx}) = e^{-ip_0\hat{x}_0/2} e^{ip_m\hat{x}_m} e^{-ip_0\hat{x}_0/2}$$

$$\Omega_s(e^{ipx}) \cdot \Omega_s(e^{ikx}) = \Omega_s(e^{ipx} \ast_s e^{ikx})$$

these are **equivalent descriptions** of the same field!
The **action** of rotation and boost symmetries on such fields are well defined:

- **Rotations** are **“classical”**
  \[
  M_j \triangleright \Omega(e^{ipx}) = \Omega(M_j^c \triangleright e^{ipx})
  \]
  \[
  M_j \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(M_j^c \triangleright e^{ipx} \ast e^{ikx} + e^{ipx} \ast M_j^c \triangleright e^{ikx})
  \]

- **Boosts** are **“deformed”**
  \[
  N_j \triangleright \Omega(e^{ipx}) = \Omega(N_j^\kappa \triangleright e^{ipx})
  \]
  \[
  N_j \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(\ast(\Delta(N_j^\kappa) \triangleright (e^{ipx} \otimes e^{ikx})))
  \]

- **Translations** are **classical** on a single plane wave i.e.
  \[
  P_\mu \triangleright \Omega(e^{ipx}) = \Omega(P_\mu^c \triangleright e^{ipx}) \tag{6}
  \]
  but due to non-trivial coproduct action is **deformed** on products of waves e.g.:
  \[
  P_i \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(P_i^c \triangleright e^{ipx} \ast e^{ikx} + e^{-P_0^c/\kappa} \triangleright e^{ipx} \ast P_i^c \triangleright e^{ikx}) \tag{7}
  \]
Changing the choice of Weyl map **should not** change the way “symmetries” act on functions in $\kappa$-Minkowski... but

$$P_{\mu} \triangleright \Omega(e^{ipx} \ast e^{ikx})$$

$$P_{\mu} \triangleright \Omega_s(e^{ipx} \ast_s e^{ikx})$$

requires **different co-products** for the $P_{\mu}$s for each choice of the $\ast$-product i.e. different ”bases” of the $\kappa$-Poincaré (Hopf) algebra $(P_{\mu}, M_i, N_i)$ and $(P_{\mu}^s, M_i^s, N_i^s)$.

**Rotations** and boosts are NOT affected by such degeneracy in the choice of Weyl map ¹ e.g.

$$M_i^s \triangleright \Omega(e^{ipx}) = M_i \triangleright \Omega(e^{ipx})$$

while instead for translations

$$P_{\mu}^s \triangleright \Omega(e^{ipx}) \neq P_{\mu} \triangleright \Omega(e^{ipx})$$

**What’s going on??** Ambiguity in defining the action of translation generators (??)...
Translations and Noether charges for classical fields

Take a more “pragmatical” approach: characterize the symmetries in terms of the infinitesimal variations of the fields:

classical case

\[ x_\mu \rightarrow x'_\mu = x_\mu + dx_\mu \]

\[ f(x) \rightarrow f'(x) = f(x) + iP_\mu f(x) dx_\mu = f(x) + df(x) \]

In the \( \kappa \)-deformed case we also need to specify \( dx_\mu s \) which must obey \(^2\)

\[ [x_j + dx_j, x_0 + dx_0] = i\kappa (x_j + dx_j), \quad [x_i + dx_i, x_j + dx_j] = 0 \]

A further ambiguity seem to emerge: \( df = iP_\mu f(x) dx_\mu \) or \( df = idx_\mu P_\mu f(x) \)?...Leibnitz \( d(fg) = (df)g + f(dg) \) restricts the choices to ONE, e.g. the unique choice for the “time-to-the-right” Weyl map is

\[ df = idx_\mu P_\mu f(x) \]

a different choice of the Weyl map affects the action of the \( P_\mu s \) but leads to the same \( df \)! (hep-th/0607221)

\(^2\) In dimension \( > 4 \) there exist other choices for the \( dx_\mu s \) see e.g. Freidel, Kowalski-Glikman and Nowak, hep-th/0612170
Translations and Noether charges for classical fields II

Translational symmetries at work: **free massless scalar field** Field on $\kappa$-Minkowski

\[
\Phi(\hat{x}) = \int d^4q \tilde{f}(q)\Omega(e^{iq\hat{x}}) = \int d^4q \tilde{\Phi}(q) e^{iq\hat{x}_i}e^{-iq_0\hat{x}_0}
\] (8)

Ordering prescription is **fixed**, drop $\sim$ from now on...

$\kappa$-Klein-Gordon e.o.m.

\[
C_\kappa(P_\mu)\Phi \equiv \left[ (2\kappa)^2 \sinh^2 \left( \frac{P_0}{2\kappa} \right) - e^{P_0/\kappa} \tilde{P}^2 \right] \Phi = 0
\] (9)

Derived from action

\[
S[\Phi] = \int d^4x \mathcal{L}[\Phi(x)] = \int d^4x \frac{1}{2} \tilde{P}_\mu \Phi \tilde{P}^\mu \Phi
\] (10)

with

\[
\tilde{P}_0 = (2\kappa) \sinh(P_0/2\kappa) \quad \tilde{P}_j = e^{P_0/2\kappa} P_j
\] (11)

Noether analysis for translations is rather straightforward ⇒ **conserved charges**

\[
Q_\mu = \int d^4p \frac{e^{3p_0/\kappa}}{2} p_\mu \tilde{\Phi}(p_0, \vec{p})\tilde{\Phi}(-p_0, -e^{p_0/\kappa} \vec{p}) \frac{p_0}{|p_0|} \delta(C_\kappa(p_\mu))
\] (12)
Most interest in $\kappa$-Minkowski/Poincaré scenarios from the “phenomenological” side has been motivated by possible **deformations** of energy-momentum **dispersion relation** that one expects from the deformed Casimir

$$C_\kappa = \left( 2\kappa \sinh \left( \frac{P_0}{2\kappa} \right) \right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}}$$

which could lead to **testable predictions** (e.g. trans-GZK events, time-of-flight tests using GRB..)

**HOWEVER** ambiguity in defining the action for generators of translations has been cause for **concern** in the recent past...

We defined translations **un-ambigously** and the Noether charges we found **DO INDEED** obey a deformed dispersion relation (see hep-th/0607221)

All this was done with **classical fields**...need **quantum fields**...
An early proposal of a $\kappa$-QFT model (Amelino-Camelia and MA; Phys. Rev. D 65,(2002)):

- Start with a partition function on $\kappa$-Minkowski

$$Z[J(x)] = \int \mathcal{D}[\Phi] e^{i \int d^4x \left[ \mathcal{L}_0 - \frac{\lambda}{4!} \phi^4(x) + \frac{1}{2} (J(x)\phi(x) + \phi(x)J(x)) \right]}$$

with $\mathcal{L}_0 = \frac{1}{2} \left( \tilde{P}_\mu \phi \tilde{P}^\mu \phi - m^2 \phi^2 \right)$

- Go to momentum space, the normalized free partition function

$$\bar{Z}^0[J(k)] \equiv \frac{Z^0[J(k)]}{Z^0[0]} = \exp \left( -\frac{i}{2} \int d^4k \frac{J(k)J(-k)}{C_\kappa(k) - m^2} \right)$$

with $\dot{k} \equiv \left( -k_0, -e^{\frac{k_0}{\kappa}} \vec{k} \right)$ coming from the antipode

- Using appropriate generalization of the functional derivative

$$\bar{Z}[J(k)] = \exp \left[ i \frac{\lambda}{24} \int \delta^{(4)} \left( \sum_{k_1,k_2,k_3,k_4} \prod_{j=1}^4 \frac{d^4k_j}{2\pi} \xi(k_j,0) \frac{\delta}{\delta J(-k_j)} \right) \bar{Z}^0[J(k)] \right]$$

with

$$\xi(k_j,0) \equiv 2 \left( 1 + e^{\frac{3k_j,0}{\kappa}} \right)^{-1}$$

$$\sum_{k_1,k_2,k_3,k_4} \equiv k_1 \dot{+} k_2 \dot{+} k_3 \dot{+} k_4$$
Perturbative expansion in $\lambda$

- Role of non-trivial topology of Feynman graphs beyond tree-level

$$G_0^{(2)}(p, p') \sim \frac{\delta^{(4)}(p - p')}{C_{\kappa}(p) - m^2}$$  \hspace{1cm} (14)

$$G^{(2)}_\lambda(p, p')_{\text{connected planar}} \sim \frac{\delta^{(4)}(p - p')}{(C_{\kappa}(p) - m^2)(C_{\kappa}(p') - m^2)} \int \frac{d^4q}{C_{\kappa}(q) - m^2}$$  \hspace{1cm} (15)

$$G^{(2)}_\lambda(p, p')_{\text{connected non-planar}} \sim \int \frac{d^4q \delta(p_0 - p'_0) \delta^{(3)}(e^{-p_0/\kappa} \vec{q} - \vec{p} + \vec{q} + e^{-q_0/\kappa} \vec{p}')}{(C_{\kappa}(q) - m^2)(C_{\kappa}(p) - m^2)(C_{\kappa}(p') - m^2)}$$  \hspace{1cm} (16)

- Non-trivial "scattering" kinematics for tree-level vertex

$$G^{(4)}_\lambda(p_1, p_2, p_3, p_4)_{\text{connected}} \sim \frac{\lambda}{4!} \sum_{\mathcal{P}(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4)} \left[ \delta^{(4)}(-\hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4) \right]$$

**Very interesting** but these are just basic results...need to roll up sleeves and find out more...
Looking for energy and momentum charges... useful to resort to canonical quantization... standard “textbook” approach imposing ETCRs on $\kappa$-Minkowski is problematic...

An alternative strategy motivated by recent work...

hep-th/0701268 study the symmetries of classical $\kappa$-fields borrowing the basic tools of “covariant phase space” formalism:

- Start with a standard relativistic massless scalar field
- Key point: $\{\phi(x); \pi(x)\} \in \Gamma \longleftrightarrow \Phi \in S$ identify the phase space $\Gamma$ with the space of solutions of the (Klein-Gordon) equation of motion $S$
- On $S$ is defined a symplectic 2-form $\omega$ (which on the standard phase space manifold $\Gamma$ corresponds to the familiar $\omega = \frac{1}{2} \int_{\Sigma_t} \delta \Pi \wedge \delta \Phi$)

Our strategy:

(i) use a map $m$ between $S$ and $S_\kappa$ to define a symplectic structure on $S_\kappa$
(ii) express the conserved charges associated with $\kappa$-symmetries through the symplectic structure

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3 see C. Crnkovic, Class. Quant. Grav. 5, 1557 (1988)
Symmetries and symplectic geometry of $\kappa$-fields II

In the **standard framework** the symplectic structure of phase space defines an hermitian inner product on the space of complex solutions

$$\langle \Phi_1, \Phi_2 \rangle = -2i \omega (\Phi_1^*, \Phi_2) = \int \frac{d^4 p}{(2\pi)^3} \delta(C(p)) \frac{p_0}{|p_0|} \tilde{\Phi}_1^*(-p) \tilde{\Phi}_2(p)$$  \hspace{1cm} (17)

**Noether charges** associated with translations can be written as

$$Q_\mu = \frac{1}{2} (\Phi, P_\mu \triangleright \Phi) = \int \frac{d^4 p}{2(2\pi)^3} \delta(C(p)) \frac{p_0}{|p_0|} \tilde{\Phi}^* (-p) \tilde{\Phi}(p)$$  \hspace{1cm} (18)

Using our construction *(with almost no sweat!)* one gets

$$Q^\kappa_\mu = \int \frac{d^4 p}{2(2\pi)^3} \delta(C_\kappa(p)) \frac{p_0}{|p_0|} p_\mu \frac{e^{3p_0/\kappa}}{\kappa} \tilde{\Phi}^*( -p) \tilde{\Phi}(p)$$  \hspace{1cm} (19)

our $\kappa$-Noether charges !

with a bonus... an inner product

$$\langle \Phi_1, \Phi_2 \rangle_\kappa = \int \frac{d^4 p}{(2\pi)^3} \delta(C_\kappa(p)) \frac{p_0}{|p_0|} e^{3p_0/\kappa} \tilde{\Phi}_1^* (-p) \tilde{\Phi}_2(p),$$  \hspace{1cm} (20)
Given an inner product standard construction of one-particle Hilbert space $H_\kappa$ form $S_\kappa$...

in defining the algebra of creation and annihilation operators crucial role of non-trivial coproduct e.g.

$$
\star(\Delta(P_\mu) \triangleright (a_{q_1}^\dagger \otimes a_{q_2}^\dagger)) \equiv P_\mu \triangleright (a_{q_1}^\dagger \star a_{q_2}^\dagger)
$$

(21)

the standard composition of operators must be deformed to be compatible with the co-product!

$$
P_\mu \triangleright (a_{q_1}^\dagger |0>) = (P_\mu \triangleright a_{q_1}^\dagger)|0> = q_\mu (a_{q_1}^\dagger |0>)
$$

(22)

$$
P_\mu \triangleright (a_{q_1}^\dagger \star a_{q_2}^\dagger |0>) = q_1\mu + q_2\mu (a_{q_1}^\dagger \star a_{q_2}^\dagger |0>) \ldots
$$

(23)

To construct the full Fock-space of the theory from $n$-particle states need a symmetrization principle...in the standard case “bosonization” is put in by hand, in our case $a_{q_1}^\dagger \star a_{q_2}^\dagger |0> \neq a_{q_2}^\dagger \star a_{q_1}^\dagger |0>$ symmetrization is subtler...stay tuned!
Possible to give a **physical** characterization of $\kappa$-Poincaré translations in terms of Noether charges for classical fields (**“non-classical”** Planck-scale symmetries were discussed only in terms of properties of the algebra of would-be-symmetry generators) *need quantum fields*...

- non-trivial task: reviewed earlier attempt in constructing $\kappa$-QFT and showed some surprising results

- want to focus on the energy and momentum charges carried by quantum fields a **canonical quantization** approach is more useful

- recent work on geometrical interpretation of $\kappa$-Noether charges seems to lead to the right direction...

- and there we reached the **border** of the $\kappa$-Poincaré **terra incognita**; our exploration continues and soon we will be back (hopefully) with more news...