Quantum fields and Noether charges for κ-spacetime symmetries

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March 13, 2007

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Outline

- **Motivation**
- **Intro to κ-Poincaré**
- **Translation symmetries and Noether charges (classical fields)**
- **Quantum fields: path integral (and canonical) quantization**
- **Conclusions and outlook**

Motivation

Historical

- **Early 90's (Lukierski et al):** use "quantum groups" (non-co-commutative Hopf algebra) to describe "quantization" of standard relativistic symmetries (analogous to Moyal quantization of Poisson manifolds: **CM→QM**)
- **1994 (Majid and Ruegg):** κ-Poincaré and its relation to κ-Minkowski NCST
- **2000 (Amelino-Camelia):** κ-Poincaré as a way to introduce $L_p \sim 1/\kappa$ as an observer-independent scale in SR (DSR)
- **2004 (Amelino, Smolin, Starodubstev):** κ-Poincaré from low-energy limit of 2+1 QG (and speculations for 3+1)
- **2006 (Freidel and Livine):** κ-Poincaré-type non-commutative effective field theory emerging from 3d Ponzano-Regge+massive particles

Consider κ-Poincaré and κ-Minkowski as a "window" on QG and try to understand how the usual physics in flat space looks like in this new scenario

κ-Poincare' in a nutshell I: Hopf algebras

Hopf algebras generalize algebras

algebra (*unital, associative*) (A, m, η):

$$m : A \otimes A \to A; \;\; \eta : \mathbb{C} \to A$$

add

$$\Delta : A \to A \otimes A; \;\; \varepsilon : A \to \mathbb{C}$$

and (together with appropriate properties) you get
bialgebra $(A, m, \eta, \Delta, \varepsilon)$

add a map (antipode)

$$S : A \to A$$

and you have a **Hopf algebra** $(A, m, \eta, \Delta, \varepsilon, S)$

the additional structure introduced is motivated by the definition of *tensor product representations* **of algebras...**

κ-Poincare' in a nutshell I: Hopf algebras (appendix)

Algebra axioms:

$$m(m \otimes id) = m(id \otimes m) \qquad \text{associativity}$$
$$m(id \otimes \eta) = m(\eta \otimes id) = id \qquad \text{unit}$$

Co-algebra axioms

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \qquad \mathrm{co-associativity}$$
$$(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id \qquad \mathrm{co-unit}$$

Antipode

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta \circ \varepsilon$$

(id identity map on A)

κ-Poincare' in a nutshell II: the bi-crossproduct basis

(Majid-Ruegg 1994) "Quantum" deformation of Poincaré algebra (def. parameter $1/\kappa$)
Coproducts for P_0, P_i translations, M_i rotations and N_i boosts

$$\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 \quad \Delta(P_j) = P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j \\
\Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j \\
\Delta(N_j) &= N_j \otimes 1 + e^{-P_0/\kappa} \otimes N_j + \frac{\epsilon_{jkl}}{\kappa} P_k \otimes N_l\,.
\end{aligned}$$

antipodes

$$\begin{aligned}
S(M_l) &= -M_l \\
S(P_0) &= -P_0 \\
S(P_l) &= -e^{\frac{P_0}{\kappa}} P_l \\
S(N_l) &= -e^{\frac{P_0}{\kappa}} N_l + \frac{1}{\kappa}\epsilon_{ljk} e^{\frac{P_0}{\kappa}} P_j M_k\,,
\end{aligned}$$

co-units

$$\epsilon(P_\mu) = \epsilon(M_j) = \epsilon(N_k) = 0.$$

κ-Poincare' in a nutshell III: the bi-crossproduct basis

The **Hopf algebra multiplication** is implicitly defined through the **commutators** (standard "commutator bracket" construction from an associative algebra)

$$[P_0, P_j] = 0 \qquad [M_j, M_k] = i\epsilon_{jkl}M_l \qquad [M_j, N_k] = i\epsilon_{jkl}N_l \qquad [N_j, N_k] = i\epsilon_{jkl}M_l$$

$$[P_0, N_l] = -iP_l \qquad [P_l, N_j] = -i\delta_{lj}\left(\frac{\kappa}{2}\left(1 - e^{-\frac{2P_0}{\kappa}}\right) + \frac{1}{2\kappa}\vec{P}^2\right) + \frac{i}{\kappa}P_l P_j$$

$$[P_0, M_k] = 0 \qquad [P_j, M_k] = i\epsilon_{jkl}P_l$$

The *mass* **Casimir invariant**

$$C_\kappa = \left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}} \tag{1}$$

In the limit $\kappa \to \infty$ recover the *trivial* Hopf algebra naturally associated to the Poincaré algebra, in particular in such limit one recovers **co-commutativity**

$$\sigma \circ \Delta = \Delta \circ id$$

($\sigma : A \otimes A \to A \otimes A$ "flip" map $\sigma(a \otimes b) = b \otimes a$)

Relation between κ-**Minkowski NCST and** κ-**Poincaré:**

$$[x_m, t] = \frac{i}{\kappa} x_m \;, \qquad [x_m, x_l] = 0 \tag{2}$$

An "intuitive" argument:

- Consider plane waves e^{ipx}: $P_\mu \triangleright e^{ipx} = p_\mu e^{ipx}$
- Define a product ($*$) for such functions, need to be compatible with co-product:

$$*(\Delta(P_\mu) \triangleright (e^{iq_1 x} \otimes e^{iq_2 x})) \equiv P_\mu \triangleright (e^{iq_1 x} * e^{iq_2 x}) \tag{3}$$

- Work out the LHS and obtain

$$P_\mu \triangleright (e^{iq_1 x} * e^{iq_2 x}) = (q_1 \dot{+} q_2)(e^{iq_1 x} * e^{iq_2 x}) \tag{4}$$

  with $q_1 \dot{+} q_2 = (q_1^0 + q_2^0; \vec{q}_1 + e^{-q_1^0/\kappa} \vec{q}_2)$

- $$e^{iq_1 x} * e^{iq_2 x} \equiv e^{(q_1 \dot{+} q_2)x} \tag{5}$$

  NOTE: $q_1 \dot{+} q_2 \neq q_2 \dot{+} q_1$ i.e. we have a **non-commutative algebra**

κ-Poincaré and κ-Minkowski NCST II

If we had started from NC plane-waves $e^{ip\hat{x}}$

- **Fix** a *normal ordering* for the non-commuting coordinates (e.g. all time coordinates to the right)
$$: e^{ip\hat{x}} :\equiv e^{ip_m \hat{x}_m} e^{-ip_0 \hat{x}_0}$$
- Wave exponentials combine in a non-trivial way: using the BCH formula
$$(: e^{iq_1\hat{x}} :)(: e^{iq_2\hat{x}} :) =: e^{i(q_1 \dot{+} q_2)\hat{x}} :$$

  same as above!

A choice of ordering is equivalent to a choice of **Weyl map** Ω

$$\begin{aligned} \Omega(e^{ipx}) &= e^{ip_m \hat{x}_m} e^{-ip_0 \hat{x}_0} \\ \Omega(e^{ipx}) \cdot \Omega(e^{ikx}) &= \Omega(e^{ipx} * e^{ikx}) \end{aligned}$$

Such map is (obviously) **not unique**

$$\begin{aligned} \Omega_s(e^{ipx}) &= e^{-ip_0 \hat{x}_0/2} e^{ip_m \hat{x}_m} e^{-ip_0 \hat{x}_0/2} \\ \Omega_s(e^{ipx}) \cdot \Omega_s(e^{ikx}) &= \Omega_s(e^{ipx} *_s e^{ikx}) \end{aligned}$$

these are **equivalent descriptions of the same field!**

κ-Poincaré symmetries

The **action** of rotation and boost symmetries on such fields are well defined:

- Rotations are **"classical"**

$$M_j \triangleright \Omega(e^{ipx}) = \Omega(M_j^c \triangleright e^{ipx})$$
$$M_j \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(M_j^c \triangleright e^{ipx} * e^{ikx} + e^{ipx} * M_j^c \triangleright e^{ikx})$$

- Boosts are **"deformed"**

$$N_j \triangleright \Omega(e^{ipx}) = \Omega(N_j^\kappa \triangleright e^{ipx})$$
$$N_j \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(*(\Delta(N_j^\kappa) \triangleright (e^{ipx} \otimes e^{ikx})))$$

- Translations are **classical** on a single plane wave i.e.

$$P_\mu \triangleright \Omega(e^{ipx}) = \Omega(P_\mu^c \triangleright e^{ipx}) \tag{6}$$

but due to non-trivial coproduct action is **deformed** on products of waves e.g.:

$$P_i \triangleright (\Omega(e^{ipx}) \cdot \Omega(e^{ikx})) = \Omega(P_i^c \triangleright e^{ipx} * e^{ikx} + e^{-P_0^c/\kappa} \triangleright e^{ipx} * P_i^c \triangleright e^{ikx}) \tag{7}$$

κ-Poincaré symmetries and choices of Weyl maps

Changing the choice of Weyl map **should not** change the way "symmetries" act on functions in κ-Minkowski...but

$$P_\mu \triangleright \Omega(e^{ipx} * e^{ikx})$$
$$P_\mu \triangleright \Omega_s(e^{ipx} *_s e^{ikx})$$

requires **different co-products for the P_μs for each choice of the $*$-product** i.e. different "bases" of the κ-Poincaré (Hopf) algebra (P_μ, M_i, N_i) and (P^s_μ, M^s_i, N^s_i).

Rotations and boosts are NOT affected by such degeneracy in the choice of Weyl map [1] e.g.

$$M^s_i \triangleright \Omega(e^{ipx}) = M_i \triangleright \Omega(e^{ipx})$$

while instead for translations

$$P^s_\mu \triangleright \Omega(e^{ipx}) \neq P_\mu \triangleright \Omega(e^{ipx})$$

What's going on?? Ambiguity in defining the action of translation generators (?)...

[1] hep-th/0306013

Translations and Noether charges for classical fields I

Take a more "pragmatical" approach:
characterize the symmetries in terms of the infinitesimal variations of the fields: classical case

$$\begin{aligned} x_\mu \to x'_\mu &= x_\mu + dx_\mu \\ f(x) \to f'(x) &= f(x) + iP_\mu f(x) dx_\mu = f(x) + df(x) \end{aligned}$$

In the κ-deformed case we also **need to specify** dx_μ**s** which must obey [2]

$$[x_j + dx_j, x_0 + dx_0] = \frac{i}{\kappa}(x_j + dx_j) \ , \quad [x_i + dx_i, x_j + dx_j] = 0$$

A *further ambiguity* seem to emerge: $df = iP_\mu f(x) dx_\mu$ or $df = i dx_\mu P_\mu f(x)$?...Leibnitz $d(fg) = (df)g + f(dg)$ restricts the choices to ONE, e.g. the **unique choice** for the "time-to-the-right" Weyl map is

$$df = i dx_\mu P_\mu f(x)$$

a different choice of the Weyl map affects the action of the P_μ**s but leads to the same** df**!** (hep-th/0607221)

[2] In dimension > 4 there exist other choices for the dx_μs see e.g. Freidel, Kowalski-Glikman and Nowak, hep-th/0612170

Translations and Noether charges for classical fields II

Translational symmetries at work: **free massless scalar field** Field on κ-Minkowski

$$\Phi(\hat{x}) = \int d^4q \tilde{f}(q)\Omega(e^{iqx}) = \int d^4q\, \tilde{\Phi}(q)\, e^{iq_i\hat{x}_i} e^{-iq_0\hat{x}_0} \tag{8}$$

Ordering prescription is **fixed**, drop ˆ from now on...

κ-Klein-Gordon e.o.m.

$$C_\kappa(P_\mu)\Phi \equiv \left[(2\kappa)^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - e^{P_0/\kappa}\vec{P}^2\right]\Phi = 0 \tag{9}$$

Derived from action

$$S[\Phi] = \int d^4x \mathcal{L}[\Phi(x)] = \int d^4x \frac{1}{2}\tilde{P}_\mu\Phi\tilde{P}^\mu\Phi \tag{10}$$

with

$$\tilde{P}_0 = (2\kappa)\sinh(P_0/2\kappa) \qquad \tilde{P}_j = e^{P_0/2\kappa}P_j \tag{11}$$

Noether analysis for translations is rather straightforward $\Rightarrow$ **conserved charges**

$$Q_\mu = \int d^4p\, \frac{e^{3p_0/\kappa}}{2} p_\mu \tilde{\Phi}(p_0,\vec{p})\tilde{\Phi}(-p_0,-e^{p_0/\kappa}\vec{p})\frac{p_0}{|p_0|}\delta(C_\kappa(p_\mu)) \tag{12}$$

Intermission

- Most interest in κ -Minkowski/Poincaré scenarios from the "phenomenological" side has been motivated by possible **deformations** of energy-momentum **dispersion relation** that one expects from the deformed Casimir

$$C_\kappa = \left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}}$$

  which could lead to testable predictions (e.g. trans-GZK events, time-of-flight tests using GRB..)

- HOWEVER ambiguity in defining the action for generators of translations has been cause for **concern** in the recent past...

- We defined translations **un-ambigously** and the Noether charges we found DO INDEED obey a deformed dispersion relation (see hep-th/0607221)

- All this was done with *classical fields*...need **quantum fields**...

$\lambda\phi^4$ and κ-Poincare': a path integral approach I (setting)

An early proposal of a κ-QFT model (Amelino-Camelia and MA; Phys. Rev. D **65**,(2002)):

- Start with a partition function on κ-Minkowski

$$Z[J(x)] = \int \mathcal{D}[\Phi]\, e^{i\int d^4x\, [\mathcal{L}_0 - \frac{\lambda}{4!}\Phi^4(x) + \frac{1}{2}(J(x)\Phi(x)+\Phi(x)J(x))]}$$

with $\mathcal{L}_0 = \frac{1}{2}\left(\tilde{P}_\mu \Phi \tilde{P}^\mu \Phi - m^2\Phi^2\right)$

- go to momentum space, the **normalized free partition function**

$$\bar{Z}^0[J(k)] \equiv \frac{Z^0[J(k)]}{Z^0[0]} = exp\left(-\frac{i}{2}\int d^4k\, \frac{J(k)J(\dot{-}k)}{\mathcal{C}_\kappa(k)-m^2}\right) \tag{13}$$

with $\dot{-}k \equiv \left(-k_0, -e^{\frac{k_0}{\kappa}}\vec{k}\right)$ coming from the **antipode**

- using appropriate generalization of the functional derivative

$$\bar{Z}[J(k)] = \exp\left[i\frac{\lambda}{24}\int \delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right)\prod_{j=1}^{4}\frac{d^4k_j}{2\pi}\xi(k_{j,0})\frac{\delta}{\delta J(\dot{-}k_j)}\right]\bar{Z}^0[J(k)]$$

with

$$\xi(k_{j,0}) \equiv 2\left(1+e^{\frac{3k_{j,0}}{\kappa}}\right)^{-1} \qquad \dot{\sum_{k_1,k_2,k_3,k_4}} \equiv k_1\dot{+}k_2\dot{+}k_3\dot{+}k_4$$

$\lambda\phi^4$ and κ-Poincare': a path integral approach II (results)

Perturbative expansion in λ

- Role of non-trivial topology of Feynman graphs beyond tree-level

$$G_0^{(2)}(p\,,\dot{-}p') \sim \frac{\delta^{(4)}(p-p')}{\mathcal{C}_\kappa(p)-m^2} \tag{14}$$

$$G_\lambda^{(2)}(p\,,\dot{-}p')^{\text{connected}}_{\text{planar}} \sim \frac{\delta^{(4)}(p-p')}{(\mathcal{C}_\kappa(p)-m^2)(\mathcal{C}_\kappa(p')-m^2)}\int\frac{d^4q}{\mathcal{C}_\kappa(q)-m^2} \tag{15}$$

$$G_\lambda^{(2)}(p\,,\dot{-}p')^{\text{connected}}_{\text{non-planar}} \sim \int\frac{d^4q\;\delta(p_0-p_0')\;\delta^{(3)}(e^{-p_0/\kappa}\vec{q}-\vec{p}+\vec{q}+e^{-q_0/\kappa}\vec{p'})}{(\mathcal{C}_\kappa(q)-m^2)(\mathcal{C}_\kappa(p)-m^2)(\mathcal{C}_\kappa(p')-m^2)} \tag{16}$$

- non-trivial "scattering" kinematics for tree-level vertex

$$G_\lambda^{(4)}(p_1,p_2,\dot{-}p_3,\dot{-}p_4)^{\text{connected}} \sim \frac{\lambda}{4!}\sum_{\mathcal{P}(\dot{-}p_1,\dot{-}p_2,p_3,p_4)}\left[\delta^{(4)}(\dot{-}p_1\dot{+}\dot{-}p_2\dot{+}p_3\dot{+}p_4)\right]$$

very interesting but these are just basic results...need to roll up sleeves and find out more...

Symmetries and symplectic geometry of κ-fields I

Looking for **energy and momentum charges**...useful to resort to canonical quantization... standard "textbook" approach imposing ETCRs on κ-Minkowski is *problematic*...

An **alternative strategy** motivated by recent work...

hep-th/0701268 study the symmetries of classical κ-fields **borrowing the** *basic* **tools of "covariant phase space" formalism**:

- Start with a standard relativistic massless scalar field
- Key point: $\{\phi(x); \pi(x)\} \in \Gamma \longleftrightarrow \Phi \in \mathcal{S}$ **identify the phase space** Γ **with the space of solutions of the (Klein-Gordon) equation of motion** $\mathcal{S}$
- On $\mathcal{S}$ is defined[3] **a symplectic 2-form** ω (which on the standard phase space manifold Γ corresponds to the familiar $\omega = \frac{1}{2}\int_{\Sigma_t} \delta\Pi \wedge \delta\Phi$)

Our strategy:

(i) **use a map** $\mathfrak{m}$ **between** $\mathcal{S}$ **and** $\mathcal{S}_\kappa$ **to define a symplectic structure on** $\mathcal{S}_\kappa$

(ii) **express the conserved charges associated with** κ**-symmetries through the symplectic structure**

[3] see C. Crnkovic, Class. Quant. Grav. **5**, 1557 (1988)

Symmetries and symplectic geometry of κ-fields II

In the **standard framework** the symplectic structure of phase space defines an hermitian **inner product** on the space of complex solutions

$$(\Phi_1, \Phi_2) = -2i\,\omega(\Phi_1^*,\ \Phi_2) = \int \frac{d^4p}{(2\pi)^3}\delta(\mathcal{C}(p))\frac{p_0}{|p_0|}\tilde{\Phi^*}_1(-p)\tilde{\Phi}_2(p) \tag{17}$$

Noether charges associated with translations can be written as

$$Q_\mu = \frac{1}{2}(\Phi, P_\mu \triangleright \Phi) = \int \frac{d^4p}{2(2\pi)^3}\ \delta(\mathcal{C}(p))\ \frac{p_0}{|p_0|}\ p_\mu\ \tilde{\Phi^*}(-p)\tilde{\Phi}(p) \tag{18}$$

Using our construction *(with almost no sweat!)* **one gets**

$$Q_\mu^\kappa = \int \frac{d^4p}{2(2\pi)^3}\ \delta(\mathcal{C}_\kappa(p))\ \frac{p_0}{|p_0|}\ p_\mu\ e^{\frac{3p_0}{\kappa}}\ \tilde{\Phi^*}(\dot{-}p)\tilde{\Phi}(p) \tag{19}$$

our κ-Noether charges !
with a bonus... an **inner product**

$$(\Phi_1, \Phi_2)_\kappa = \int \frac{d^4p}{(2\pi)^3}\ \delta(\mathcal{C}_\kappa(p))\ \frac{p_0}{|p_0|}\ e^{\frac{3p_0}{\kappa}}\ \tilde{\Phi^*}_1(\dot{-}p)\tilde{\Phi}_2(p)\,, \tag{20}$$

Canonical quantization of free κ-fields

Caution!!: work in progress...

- Given an inner product standard construction of **one-particle** Hilbert space $\mathcal{H}_\kappa$ form $\mathcal{S}_\kappa$...

- in defining the **algebra of creation and annihilation operators** crucial role of **non-trivial coproduct** e.g.

$$\star(\Delta(P_\mu) \blacktriangleright (a^\dagger_{\vec{q}_1} \otimes a^\dagger_{\vec{q}_2})) \equiv P_\mu \blacktriangleright (a^\dagger_{\vec{q}_1} \star a^\dagger_{\vec{q}_2}) \tag{21}$$

  the standard composition of operators must be deformed to be compatible with the co-product!

$$P_\mu \triangleright (a^\dagger_{\vec{q}} | 0 >) = (P_\mu \blacktriangleright a^\dagger_{\vec{q}}) | 0 >= q_\mu (a^\dagger_{\vec{q}} | 0 >) \tag{22}$$

$$P_\mu \triangleright (a^\dagger_{\vec{q}_1} \star a^\dagger_{\vec{q}_2} | 0 >) = q_{1\mu} \dot{+} q_{2\mu} (a^\dagger_{\vec{q}_1} \star a^\dagger_{\vec{q}_2} | 0 >) \ldots \tag{23}$$

- To construct the **full Fock-space** of the theory from n-particle states need a symmetrization principle...in the standard case "bosonization" is put in by hand, in our case $a^\dagger_{\vec{q}_1} \star a^\dagger_{\vec{q}_2} | 0 > \neq a^\dagger_{\vec{q}_2} \star a^\dagger_{\vec{q}_1} | 0 >$ **symmetrization is subtler**...stay tuned!

Conclusions

- **Possible to give a physical characterization of κ-Poincaré translations in terms of Noether charges for classical fields** ("non-classical" Planck-scale symmetries were discussed only in terms of properties of the algebra of would-be-symmetry generators) need quantum fields...
- non-trivial task: **reviewed earlier attempt in constructing κ-QFT and showed some surprising results**
- **want to focus on the energy and momentum charges carried by quantum fields a canonical quantization approach is more useful**
- **recent work on geometrical interpretation of κ-Noether charges seems to lead to the right direction...**
- **and there we reached the border of the κ-Poincaré** *terra incognita*; **our exploration continues and soon we will be back (hopefully) with more news...**