Some Surprising Consequences of Background Independence in Canonical Quantum Gravity

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Motivation: Some perplexing aspects of Diff invariance

• At the foundation of LQG kinematics lie uniqueness theorems: “There is a unique diff invariant state on the holonomy-flux quantum algebra.” (Lewandowski, Okolow, Sahlmann, Thiemann (LOST); Fleishhack). But this seems very strange to experts in string theory and quantum geometrodynamics (QGD) (e.g., the recent KITP workshop).

• Reaction: “ How can this be? Surely quantum gravity admits infinitely many diff invariance states! Take, e.g., $\Psi(q_{ab}) = \int f(R_{ab}R^{ab})\, dv_q$ in QGD. Is this uniqueness perhaps a peculiarity of the connection framework of LQG?”

Related questions:

• Is there then a unique gauge invariant state on the kinematic algebra of gauge theories? If not, why not? Is the uniqueness tied to the non-Abelian character of Diff? Is there a difference between Abelian and non-Abelian gauge theories?

• Is Diff Invariance tied to the non-existence of the connection operator? Non-separability of the kinematical Hilbert space? Would these features persist in QGD?

Goal:

To clarify these issues. Should help in discussions with people outside LQG. Also a few new mathematical problems for the experts.

Presentation will be pedagogical and will skip some technicalities. Primary focus: Conceptual issues. For concreteness, will focus on the LOST framework rather than Fleishhack's.

Conclusion: Diff invariance is an *extremely* strong requirement on the kinematical algebra also in QGD. Very different from Gauge invariance. Infinitely many Diff invariant states do exist but on a *different algebra*.

Organization:

1. General Framework
2. Quantum Geometrodynamics
3. Loop Quantum Gravity
4. Quantum Geometrodynamics Revisited
5. Gauge versus Diff invariance
6. Discussion

1. General Framework

Canonical approach a la Dirac (or BRST).
Kinematical framework: Home for quantum constraint operators (or BRST charges) .

• Start with a $\star$-algebra $\mathfrak{a}$ generated by 'canonically conjugate operators'. Represent $\mathfrak{a}$ by operators on a Hilbert space $\mathcal{H}$.
• Represent constraints by self-adjoint operators so the gauge transformations they generate are unitary operators on $\mathcal{H}$
• Pass to $\mathcal{H}_{\rm phys}$ by, e.g., group averaging.

• A 'royal road to obtain representations of $\mathfrak{a}$:
The Gel'fand-Naimark-Segal (GNS) construction

The GNS construction

• A state F on $\mathfrak{a}$ is a positive linear functional ('expectation-value' of the operators in $\mathfrak{a}$): For any $a \in \mathfrak{a}$, $F(a)$ is a complex number such that:
$F(a + \lambda b) = F(a) + \lambda F(b) \ \forall \lambda \in \mathbb{C}; \quad F(I) = 1; \quad F(a^\star a) \geq 0.$

• Given any F, the GNS construction provides a Hilbert space $\mathcal{H}$ and a representation of $\mathfrak{a}$ by operators on $\mathcal{H}$ such that:
i) the Rep is cyclic; i.e. there exists a vector Ψ_F in $\mathcal{H}$ s.t. $\{\mathfrak{a} \cdot \Psi_F\}$ is dense in $\mathcal{H}$; and
ii) $F(a) = (\Psi_F, \, a\Psi_F) \ \forall a \in \mathfrak{a}$

• Very general procedure. e.g., Every IRR of $\mathfrak{a}$ is cyclic.
If θ is an automorphism on $\mathfrak{a}$ (i.e. a structure preserving map from $\mathfrak{a}$ to itself), and if $F[\theta(a)] = F[a]$ then θ is unitarily implemented on $\mathcal{H}$: There exists an unitary operator U_θ on $\mathcal{H}$ such that $(\theta(a))\Psi = (U_\theta^{-1} \, a \, U_\theta)\Psi, \ \forall \Psi \in \mathcal{H}$ and $U_\theta \, \Psi_F = \Psi_F$

• A powerful and economic way to ensure that (gauge-)symmetries are unitarily implemented. In Minkowskian field theories, $F(a) = \langle 0|a|0\rangle$ is Poincaré invariant.

2. Quantum Geometrodynamics

• The $\star$-algebra $\mathfrak{a}$ generated by $\hat{q}(f) = \int \hat{q}_{ab}\tilde{f}^{ab}\mathrm{d}^3\mathrm{x}$ and $\hat{p}(g) = \int \hat{p}^{ab} g_{ab}\mathrm{d}^3\mathrm{x}$ subject to:

$$[\hat{q}(f),\, \hat{p}(g)] = -i\hbar \int \tilde{f}_{ab} g^{ab}\, \mathrm{d}^3\mathrm{x}; \quad (\hat{\mathrm{q}}(\mathrm{f}))^\star = \hat{\mathrm{q}}(\mathrm{f}),\ (\hat{\mathrm{p}}(\mathrm{g}))^\star = \hat{\mathrm{p}}(\mathrm{g})\,.$$

Each diffeo α naturally acts of (f, g) inducing an automorphism θ_α on $\mathfrak{a}$.
e.g. $\theta_\alpha(\hat{q}(f)) = \hat{q}(\alpha(f))$.

• Klauder's affine algebra. Viewpoint: Want the metric operator to be positive definite.
Change the algebra by replacing $\tilde{p}^{ab}$ with $\pi_a^b = p^{bc} q_{ac}$. CCRs replaced by Affine CRs:
$[\hat{q},\, \hat{q}] = 0$; $[\hat{\pi},\, \hat{q}] \sim \hat{q}$; $[\hat{\pi},\, \hat{\pi}] \sim \hat{\pi}$. Will comment on this at the end.

• To obtain a Diff covariant rep let us suppose $\mathfrak{a}$ admits a Diff invariant PLF F: $F(\theta_\alpha(a)) = F(a)$.
Then, the GNS construction would provide a desired rep of $\mathfrak{a}$ in which Diff acts unitarily.

QGD: Diff invariant states on $\mathfrak{a}$

- Now, $F[\hat{q}(f)] =: \int Q_{ab}\, \tilde{f}^{ab}(x)\mathrm{d}^3\mathrm{x}$ defines a distribution $Q_{ab}(x)$; $F[\hat{q}(f)\, \hat{q}(f')]$ defines a bi-distribution $Q_{aba'b'}(x, x')$; etc.

- All these distributions must be diff invariant.

(e.g.: $F[\hat{q}(\alpha \cdot f)] = F(\hat{q}(f)) \Rightarrow \int Q_{ab}(x)\ \alpha \cdot \tilde{f}_{ab} d^3x = \int Q_{ab}(x) \tilde{f}_{ab} d^3x\ \forall \tilde{f}_{ab}$,
i.e., $Q_{ab}(x)$ is a Diff invariant distribution.)
But there is no non-zero Diff invariant tensor distribution! So
$Q_{ab} = 0$, $Q_{aba'b'}(x, x') = 0$, etc

- Since $0 = F[\hat{q}(f)\, \hat{q}(f)] = (\Psi_F,\, \hat{q}(f)\hat{q}(f)\Psi_F) = (\hat{q}(f)\Psi_F,\, \hat{q}(f)\Psi_F)$,
we conclude: $\hat{q}(f)\Psi_F = 0,\ \forall f$.

Completely analogous reasoning gives $\hat{p}(g)\Psi_F = 0\ \forall g$. But this contradicts the CCR: $[\hat{q}(f),\, \hat{p}(g)]\Psi_F = -i\hbar\Psi_F$.

- Thus, the standard algebra $\mathfrak{a}$ of QGD does not admit a single Diff invariant state! Opposite of the naive expectation: Rather than infinitely many Diff invariant states there are none. Situation is the same with the more sophisticated affine algebra of Klauder's.

Loop Quantum Gravity

• Basic canonically conjugate pair $(A_a^i(x), \tilde{E}_i^a(x))$. (Let's drop hats on operators.) If we construct the algebra $\mathfrak{a}$ naively, i.e., with $A(f) = \int A_a^i \tilde{f}_i^a \mathrm{d}^3\mathrm{x}$ and $E(f) = \int \tilde{E}_i^a \mathrm{d}^3\mathrm{x}$ then again no diff invariant state. (Also issues of gauge covariance.)

• LQG algebra $\mathfrak{A}$: Generated by (gauge covariant) holonomies h_e and electric fluxes $E_{S,g} = \int_S \tilde{E}_i^a f_a^i(x)\mathrm{d}^2\mathrm{x}$. (Physically, directly useful only in the spatially compact case.) Now the situation is very different. The LOST theorem implies that there is a unique (SU(2)-gauge and) Diff invariant state.

• General Configuration operators: $C(A) = c(h_{e_1}(A), \cdots h_{e_n}(A))\ \ (\in \mathrm{Cyl})$.
Then, $F(C) = \int_{(\mathrm{SU(2)})^\mathrm{n}} c(g_1, \cdots g_n)\mathrm{d}\mu_\mathrm{H}$ and $F(a\, E_{S,f}) = 0$.
Defines F on $\mathfrak{A}$. This F is manifestly gauge and Diff invariant: Didn't use any background structures.

Elucidation: Properties of the rep

• Resulting Hilbert space: $\mathcal{H} = L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_{\mathrm{AL}})$ where $\bar{\mathcal{A}}$ is the space of generalized connections (quantum configuration space) and μ_{AL} a regular, Diff invariant, Borel measure thereon. $\mathcal{H}$ carries a natural unitary action of $\mathrm{SU}(2)_{\mathrm{loc}} \ltimes \mathrm{Diff}$. Used critically in solving the Gauss and Diff constraints by group averaging.

• Note: Only finite diffeos induce automorphisms on $\mathfrak{A}$ and these are unitarily implemented on $\mathcal{H}$. Infinitesimal Diffeos have no natural action on $\mathfrak{A}$ because $\mathfrak{A}$ is not equipped with the necessary topology. If we were to equip it and take limits, $\mathfrak{A}$ would be enlarged. Expectation: No Diff invariant state on the enlarged algebra.

• Use of holonomies —exponentiated connections— serves two purposes: i) Makes the algebra SU(2)-gauge covariant; and more importantly, ii) allows a non-trivial Diff invariant state.

Elucidation: Underlying Structure

• The algebra $\mathfrak{A}$ has an Abelian configuration part Cyl consisting of $C(A) = c(h_{e_1}(A), \cdots, h_{e_n}(A))$. Can be completed in sup norm to yield an Abelian $C^\star$ algebra $\overline{\mathrm{Cyl}}$.

Gel'fand theory then implies that in *any* cyclic representation of $\overline{\mathrm{Cyl}}$, $\mathcal{H} = L^2(\bar{\mathcal{A}}, \mathrm{d}\mu)$ for some measure μ and operators C act by multiplication.

Rep Diff invariant iff the measure μ on $\bar{\mathcal{A}}$.

• How many Diff invariant measures on $\bar{\mathcal{A}}$? Lots! The key achievement of LOST is to show that only one of them, μ_{AL} supports the representation of the momentum algebra. Technically subtle result.

• 'The semi-analytical category' plays a key role. Mathematically very natural and things fit together elegantly.

• Similar to the uniqueness of the Poincaré invariant vacuum in *free* Minkowskian QFTs. But here, no assumption on the details of dynamics!

Elucidation: Contrast with QGD of part 2.

• Why does $\mathfrak{A}$ of LQG admit a Diff invariant state while the algebra $\mathfrak{a}$ of QGD does not?

• The LQG algebra is generated by holonomies $h_e = \mathcal{P} \exp \int_e A$ rather than the smeared connection itself. $F(h_e) = 0$ if the edge e is non-trivial, and = 1 if e is a point edge. F is diff invariant but *discontinuous* in e. Hence an operator valued distribution $A(x)$ cannot be defined on $\mathcal{H}$ through weak limits of holonomy operators.

• In fact, $A(x)$ cannot be defined at all on $\mathcal{H}$ (assuming it commutes with all h_e)! Diff invariance of F and the fact that $F[a\, E_{(S,f)}] = 0$ implies that if it could be defined $A(x)$ would be the zero operator (valued distribution).

• The QGD algebra $\mathfrak{a}$ is analogous to the algebra generated directly by (A, E). This algebra does not admit a Diff invariant state either. Natural Question: Can we use an 'exponentiated algebra' in QGD?

4. QGD Revisited

• Following LQG, let us use exponentiated operators in QGD:
$W(f,g) := \exp i(q(f) + p(g))$ satisfying the Weyl commutation relations:

$$W(f_1, g_1)\, W(f_2, g_2) = \exp[(i/2)(f_2 g_1 - f_1 g_2)]\, W(f_1 + f_2, g_1 + g_2).$$

The general element of the Weyl algebra $\mathfrak{W}$ is $\sum_n a_n W(f_n, g_n)$.

• Diff invariant function along the lines of LQG:

$$F[W(f,g)] := \begin{cases} 1 & \text{if } g = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is continuous in f but not in g. Consequently, on the GNS $\mathcal{H}$, operators $q(f)$ are well-defined but $p(g)$ are not! Analogous to the fact that in LQG $E_{S,f}$ are well-defined but A are not. (There is an obvious dual representation.)

• In this QGD rep, $q(f)\Psi_F = 0$ (just as $E_{S,f}\Psi_F = 0$ in LQG.) Thus the cyclic state ("vacuum") in both representations corresponds to the 'zero metric'. Reminiscent of the 2+1 gravity (e.g. a la Witten).

This representation is unsuitable for Klauder's Affine Algebra because although q_{ab} does exist (as a distribution), it fails to be positive definite —the corner stone of that program.

Features of this QGD representation

• Gel'fand theory again applies. $\mathcal{H} = L^2(\mathcal{Q}, d\mu_o)$ where $\mathcal{Q}$ is a certain completion of the space of smooth metrics —the Gel'fand spectrum of the Abelian $C^\star$ algebra of configuration operators $\exp iq(f)$— and μ_o a regular Borel measure thereon. Structure of $\mathcal{Q}$ can again be explored using projective techniques. Not a standard space of distributions. Don't yet have explicit control on the integration theory.

• In this rep the Diff group has unitary action. But again infinitesimal diffeos not defined on $\mathfrak{W}$ nor on $\mathcal{H}$. The classical generator of infinitesimal diffeo along V^a is $\int (\mathcal{L}_V\, q_{ab})\, p^{ab}$ and operator p^{ab} does not exist. (Also the product has to be regularized!) So, same problems with the constraint algebra as in LQG.

• $\mathcal{H}$ is non-separable as in LQG. Again, seems a general feature of GNS reps arising from Diff invariant PLFs. A precise result along these lines?

• In the spatially compact case, does $\mathfrak{W}$ admit any diff invariant states other than F and its obvious 'dual'?

Major difficulty: Constraint Operators

- The scalar constraint reads:

$$S = p^{ab} p^{cd}\, G_{abcd}(q) + \sqrt{q} R$$

Significantly harder than in LQG because: (i) cannot treat curvature in terms of basic operators like holonomies; (ii) G_{abcd} would contain products of operator-valued distributions; (iii) P^{ab} themselves do not exist; and, most importantly, (iv) Discreteness which plays a key role in LQG a la Thiemann is not available because P^{ab} smeared with 3-d test fields not lower dimensional ones.

- No natural/obvious strategy to characterize solutions to the Diff constraint. Unlikely that states like $\int f(R_{ab}R^{ab})\, \mathrm{d}^3\mathrm{x}$ will result from group averaging. Indeed, the group averaged Hilbert space may be 'too small' to be useful. Is it?

- Quantum geometry also seems unmanageable. $\hat{q}(f)$ well-defined. But 3-d smearing implies: discreteness underlying 'polymer geometry' of LQG is lost. Quantization of geometric quantities (like areas and volumes) would be *much* more difficult. Is it even possible?

5. Gauge Invariance: Maxwell Theory

• Is Diff invariance very different from the more familiar gauge invariance? Are there gauge invariant states on the Maxwell kinematical algebra in Minkowski space-time?

• Decompose A, E into Longitudinal and Transverse parts. conjugate pairs: (A^T, E^T) and (A^T, E^T) generate the familiar $\star$ algebra $\mathfrak{a}$. Gauge transformations: $A^L \Rightarrow\!\rightarrow A^L + d\alpha$; (all other fields unchanged) generates automorphisms θ_α on $\mathfrak{a}$.

• $\theta_\alpha(\hat{A}^L) = \hat{A}^L + d\alpha\,\hat{I} \;\;\Rightarrow$ there is no PLF on $\mathfrak{a}$ which is invariant under these automorphisms. Thus, no gauge invariant state on the algebra $\mathfrak{a}$! Surprising because one would have naively expected infinitely many gauge invariant states.

• Problem associated with the longitudinal sector; the algebra generated just by A^T, E^T is gauge invariant $\Rightarrow$ it obviously admits infinitely many gauge invariant states.

Maxwell Theory: Weyl algebra

• On the Weyl algebra $\mathfrak{W}$, situation is *very* different. Since longitudinal and transverse modes decouple, now $\mathfrak{W}$ is generated by products $W(f_L, g_L)\, W(f_T, g_T)$ of Weyl operators associated with the two sets of modes. ($W(f_L, g_L)$ commute with $W(f_T, g_T)$).

• $\mathfrak{W}$ admits infinitely many gauge invariant states, e.g.:

$$F[W(f_L, g_L)\, W(f_T, g_T)] = \begin{cases} \langle \Psi_{\text{Fock}} | W(f_T, g_T) | \Psi_{\text{Fock}} \rangle & \text{if } f_L = 0, \\ 0 & \text{otherwise} \end{cases}.$$

This state is tensor product of any Fock state Ψ_{Fock} on transverse modes with the polymer cyclic state for longitudinal modes. Uniqueness result on the polymer part.

• Since F is discontinuous in f_L, operators $A(f_L)$ do *not* exist in this representation. Only their exponentiate versions $W(f_L, 0)$ well defined. Operators $E(g^L)$ do exist. The kinematical Hilbert space $\mathcal{H}$ is non-separable. One can solve the Gauss constraint and the physical Hilbert space is separable.

• In the covariant approach, this procedure enables one to construct the Fock representation without having indefinite inner product (Thirring).

6. Discussion

• Diff invariance is an extremely powerful restriction on the kinematical algebra both for LQG and QGD.
Counter intuitive Result: In both cases there is no Diff invariant state on the 'naive' $\star$-algebra $\mathfrak{a}$.
There is precisely one Diff invariant state on the holonomy-flux algebra $\mathfrak{A}$ of LQG and at least two for the Weyl algebra $\mathfrak{W}$ of QGD. The uniqueness issue is open in QGD.

• The resulting GNS representation of $\mathfrak{A}$ yields polymer geometry in LQG because connections are smeared along 1-dimensional edges. In turn, this leads to well defined quantum geometry operators, exhibiting a natural, fundamental discreteness. Rich set of solutions to the Gauss and Diff constraints.

• In the resulting GNS representations in QGD either the metric operator is not defined *or* it fails to be positive definite even if one were to use the affine Weyl algebra a la Klauder. 3-d excitations of geometry $\Rightarrow$ quantization of geometric operators and constraints seems *much* more difficult. Diff constraints may not admit 'enough' solutions. (Open issue)

• The Maxwell Weyl algebra $\mathfrak{W}$ admits infinitely many gauge invariant states. Thus Diff invariance is a much stronger requirement than the Maxwell-gauge invariance. (In the non-Abelian theory, the Weyl algebra has to be suitably modified, e.g., to the holonomy-flux algebra. Again infinitely many gauge invariant states.)

• In the Maxwell case, infinitely many gauge invariant states because the algebra $\mathfrak{W}$ has an easily identifiable sub-algebra of gauge invariant ('transverse') observables. The LQG $\mathfrak{A}$ or QGD $\mathfrak{W}$ do not admit sub-algebras of diff invariant observables (in the spatially compact case)
Reason: Extreme non-locality of gauge invariant observables: Solutions to diff constraint are not states on the LQG $\mathfrak{A}$ (or QGD $\mathfrak{W}$).

• In all cases, the kinematical Hilbert spaces are non-separable. (But the gauge invariant subspaces in the Maxwell case and the (SU(2)-gauge &) Diff invariant Hilbert space in LQG can be separable.)
In all cases, no operator on $\mathcal{H}$ corresponding to one of the canonically conjugate variables (A_a A_a^i or $\tilde{p}^{ab}$).
Thus these two surprising features of LQG are in fact universal consequences of the invariance requirement.

Limitation

• We followed the 'royal road' by looking for a diff invariant cyclic state. Can we find other interesting diff covariant representations of $\mathfrak{A}$ (or of the QGD $\mathfrak{W}$) where is there no diff invariant cyclic state? Cyclic property not a physical restriction because every IRR is in particular cyclic. But the cyclic vector need not be diff invariant.

• By restricting the cyclic state to Cyl, one gets a cyclic rep of this Abelian algebra. This Hilbert space is necessarily of the form $\mathcal{H} = L^2(\bar{\mathcal{A}}, d\mu)$ for some Diff invariant measure μ. But the action of $E_{S,f}$ on the cyclic state could enlarge the rep, still providing an unitary implementation of Diff without having any Diff invariant cyclic state. Can it?