Discretisation and Diffeomorphisms

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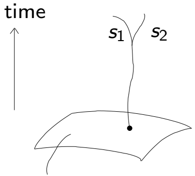
Outline:

- Introduction
  - Gauge symmetries: Reparametrisation invariance in GR
  - Discretisations and (gauge) symmetries
- Reparametrisation-invariant systems in $1D$
  - Breaking of rep-inv
  - Improved and perfect actions
- Gauge symmetries in GR discretised a la Regge
  - Diffeos vs. Vertex displacement symmetries
  - Improving the action: $3D$ with $\Lambda \neq 0$, $4D$
- Discretised canonical formulation
  - Matches (breaking of) symmetries in discrete covariant setting
  - Constraints $\Leftrightarrow$ 'pseudo-constraints'
- Triangulations and triangulation-independence
- Summary

Diff-invariance in GR as gauge symmetry

- '12: realisation of Einstein that initial conditions do not determine solution to his field equations uniquely (hole argument).
- Dirac: Different solutions with same initial data should be *physically* the same.
- Solutions unique up to space-time diffeomorphisms!



Phase space (space of initial data)

- Principle of background-independence: No metric singled out ($Diff(\mathcal{M})$ as gauge group).
- Canonical formulation: Constraint algebra ("hypersurface-deformation algebra")

$$
\begin{aligned}
\{D(X), D(Y)\} &= D([X, Y]), \qquad \{D(X), H(N)\} = H(\mathcal{L}_X N), \\
\{H(N), H(M)\} &= D(q^{-1}(NdM - MdN))
\end{aligned}
$$

$\Rightarrow$ Diffeo-symmetry intimately related to dynamics of GR!

Discretisation

Before quantization, one usually discretises systems (easier to quantise, deal with product of "fields at a point")

Unfortunately, upon discretisation, Poisson algebra relations are usually changed. Note: not always bad! For example the E-fields in LQG:

- Continuum: $\{E_I(x), E_J(y)\} = 0$
- Integrated over face f: $\{E_I(f), E_J(f)\} = \epsilon_{IJK} E_K(f)$

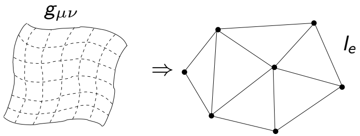
But when the fields are associated to (gauge) symmetries, change of Poisson structure usually viewed as *anomaly*

- In QFT: Lattice discretisation changes algebra of the P_μ $\Leftrightarrow$ breaks Poincaré-invariance
- In $3+1$ GR: No anomaly-free version of discretised Hypersurface-deformation algebra known (e.g. Loll gr-qc/9708025, Thiemann gr-qc/9705017, hep-th/0005232, Dittrich 0810.3594 [gr-qc])
- In $2+1$ LQG with $\Lambda \neq 0$: Discretisation along plaquettes produes "mild" anomaly (Perez, Pranzetti 1001.3292[gr-qc])

Discretising GR

For instance Regge gravity: GR on triangulation (starting point for Quantum Regge calculus, Spin foam quantization)

- Replace smooth metric $g_{\mu\nu}$ by piecewise linear-flat metric g_Δ (encoded in edge lengths l_e of triangulation)

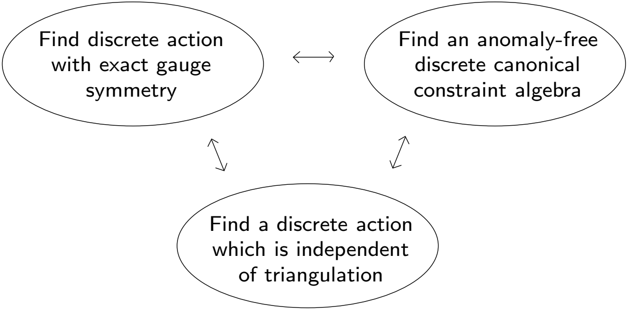


- Piecewise flat metrics approximate smooth ones, but have different orbit size under diffeos.

$\Rightarrow$ Fate of diffeomorphism-symmetry in the discrete theory?

Aim of this talk

- Describe, in which way gauge symmetry is lost in discretised Gravity (Regge), and which problems this causes.
- Argue, that the following three problems are connected:



Parametrise $1D$ systems

Reparametrisation-invariant systems in $1D$: Close to situation in GR

- Take usual system in $1D$ without gauge symmetries: $L(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - V(q)$, $\det \frac{\partial^2 L}{\partial q_i \partial q_j} \neq 0$
- Parametrise: treat time t as variable and regard evolution w.r.t. auxilliary parameter s: $q(t) \to q(s), t(s)$. Dynamics is governed by action:

$$\tilde{L}(q,t,q',t') := \left(\frac{1}{2}\frac{(q')^2}{(t')^2} - V(q)\right) t'$$

- The new system now is reparametrisation-invariant: $q(s), t(s)$ solutions for certain boundary data, then also $q(\tilde{s}), t(\tilde{s})$ for any $\tilde{s} = \tilde{s}(s)$.

  $\Rightarrow$ Boundary data $q(s_i), t(s_i)$, $q(s_f), t(s_f)$ does *not* determine unique solution of equations of motion (only up to reparametrisation). System has gauge symmetry in that sense. $\Leftrightarrow$ Analogue of diff-symmetry in GR.

- Generically: Eom for $t(s)$ is satisfied automatically. $\Rightarrow$ Choose arbitrary $t(s)$, then $q(s)$ is uniquely determined.

Discretise parametrised $1D$ systems

- Discretise the variables: $q(s), t(s) \rightarrow t_n, q_n, n = 0, \ldots, N+1$. Discretise action:

$$S_{\rm discr} = \sum_{n=0}^{N} \left(\frac{1}{2} \frac{(q_{n+1} - q_n)^2}{(t_{n+1} - t_n)^2} - V\left(\frac{q_n + q_{n+1}}{2}\right)\right)(t_{n+1} - t_n)$$

(Equivalent to replace smooth solutions by piecewise linear ones).

- Equations of motion:

$$\frac{\partial S_{\rm discr}}{\partial t_n} = \frac{\partial S_{\rm discr}}{\partial q_n} \stackrel{!}{=} 0 \qquad \text{unique solution!}$$

$\Rightarrow$ gauge-invariance broken! Unlike continuum $t(s)$ (which is arbitrary), t_n are fixed!

- Therefore the Hessian $H = \frac{\partial^2 S_{\rm discr}}{\partial x_n \partial x_m}$ has *no* zero Eigenvalue (where $x = \{t, q\}$). However: In limit $N \rightarrow \infty$, N out of the $2N$ eigenvalues of H approach zero. In continuum limit, gauge symmetry is restored in that sense.

Exception:

- There is an exception, for when also the discretised action S_{discr} exhibits gauge symmetry: Whenever the piecewise linear dynamics coincides with the continuum dynamics, i.e. for $V = 0$ ($\Leftrightarrow$ linear dynamics $\dot{q} = const$):

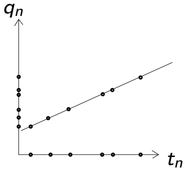


Figure: $V = 0$: t_n arbitrary, q_n uniquely fixed by boundary data and t_n

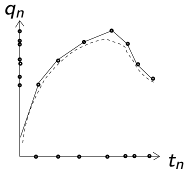


Figure: $V \neq 0$: t_n and q_n uniquely fixed by boundary data.

- If $V = 0$, then the discrete dynamics coincides with the continuum one. In this case the eom for t_n is automatically satisfied. $\Rightarrow$ S_{discr} has exact gauge symmetries from the continuum (t_n arbitrary, q_n fixed by boundary data *and* t_n).

The perfect action

- The breaking of gauge symmetry in discrete theory can be traced back to the choice of the discrete action S_{discr}, which effectively replaces smooth dynamics by piecewise linear ones. In fact, one can write down a discrete action which exhibits exact gauge symmetry (perfect action): (see e.g. Marsden, West Act.Num.10 (2001))

$$S_{\mathrm{perf}} = \sum_{n=0}^{N} \int_{s_n}^{s_{n+1}} ds\, \tilde{L}\big(q(s), t(s), q'(s), t'(s)\big)$$

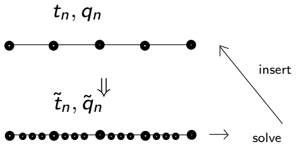
  where $q(s), t(s)$ are solutions of continuum dynamics with boundary data t_n, q_n, t_{n+1}, q_{n+1}.

- The Hessian $\frac{\partial^2 S_{\mathrm{perf}}}{\partial x_n \partial x_m}$ has N zero eigenvectors (exhibits *exact* gauge symmetries of continuum: half of the variables are gauge). Reproduces exact continuum dynamics.

Improving the action via coarse graining

- The perfect action can be constructed by coarse graining process (see e.g. BB, Dittrich 0907.4323 [gr-qc]): $S_{\rm disc}(q_0,t_0,\ldots,q_N,t_N) \to S_{\rm imp}(q_0,t_0,\ldots,q_N,t_N)$

$$S_{\rm imp} = \sum_{n=0}^{N} S^{(n)}_{\rm imp}(q_n,t_n,q_{n+1},t_{n+1})$$



where

$$S^{(n)}_{\rm imp}(q_n,t_n,q_{n+1},t_{n+1}) \;=\; S_{\rm discr}(\tilde{q}_0,\tilde{t}_0,\ldots,\tilde{q}_M,\tilde{t}_M)\Big|_{\frac{\partial S_{\rm discr}}{\partial \tilde{q}_n}=\frac{\partial S_{\rm discr}}{\partial \tilde{t}_n}=0,\; q_n=\tilde{q}_0,\ldots,t_{n+1}=\tilde{t}_M}$$

is the value of the discrete action $S_{\rm discr}$ on a solution of a further refinement of the interval $[t_n, t_{n+1}]$ into M pieces.

- The improved action is defined on the coarse lattice, but incorporates dynamics of fine lattice. $\Rightarrow \frac{\partial^2 S_{\rm imp}}{\partial x_n \partial x_m}$ has N small Eigenvectors.
- Perfect action arises as limit of infinite refinement:

$$\lim_{M\to\infty} S_{\rm imp} \;=\; S_{\rm perf}$$

Problems with discretisation:

- For naively discretised reparametrization-invariant theories, both t_n and q_n are fixed by dynamics, but in the continuum limit only $q_n(t_n)$ is predicted by theory $\Rightarrow$ Mismatch of degrees of freedom (e.g. correlation function between t_n and t_{n+1} becomes pure gauge in continuum).
- Too many variables integrated over in path integral (divergencies)!
- Different size of gauge orbits causes problems for perturbation theory Dittrich, Hoehn, 0912.1817 [gr-qc]
- Numerical issue: very small Eigenvalues of Hessian H (for large N) cause numerical problems.
- In lattice field theories: discretisation causes lattice artifacts which often range over several lattice sites. Bietenholz hep-lat/9709117

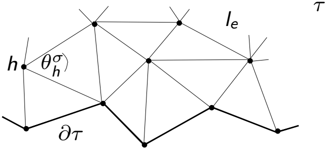
In lattice field theories, these issues are well-known, and dealt with by improving the action (replacing S_{discr} with another action S_{imp}, which captures gauge degrees of freedom better).

- For lattice field theories, improved (and perfect) actions are highly sought after. Symanzik Nucl.Phys.B226:187,1983 Gupta hep-lat/9807028, Bietenholz, Wiese hep-lat/9510026 They are constructed using renormalization group techniques.
- Improving the action reduces artifacts significantly.
- Perfect actions always exist for asymptotically free theories.

Recap: Regge gravity

- In Regge calculus Regge '61, one replaces smooth metric by piecewise-linear flat metrics living on a triangulation τ, which consists of D-dim. simplices σ.

Discretised variables: edge lengths l_e.
F_h: volume of $D-2$ subsimplex h ("hinge")
θ_h^σ: interior dihedral angle at h in σ
$\epsilon_h := 2\pi - \sum_{\sigma\supset h}\theta_h^\sigma$ deficit angle
$\psi_h := \pi - \sum_{\sigma\supset h}\theta_h^\sigma$.

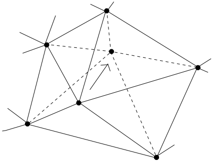


The discretised action is given by

$$S_{\mathrm{Regge}} = \sum_{h\in\tau^\circ} F_h\epsilon_h - \Lambda\sum_\sigma V_\sigma + \sum_{h\in\partial\tau} F_h\psi_h$$

Gauge symmetry for Regge in $3D$

- The equations of motion for $3D$ Regge gravity (for $\Lambda = 0$) are $\epsilon_e = 0$, so the solutions are locally flat metrics - coincides exactly with solutions of continuum dynamics! As a result, the discrete dynamics exhibits exact gauge symmetries (compare to the $V = 0$ case for $1D$ systems)
- Symmetry corresponds to displacement of vertices, position of which can be freely chosen (3 gauge directions per vertex). Symmetry can be derived from discrete Bianchi identities, and is even an off-shell symmetry. Freidel, Louapre gr-qc/0212001



- Same kind of symmetry exists for $4D$ Regge and $\Lambda = 0$, whenever boundary conditions are chosen which lead to flat solutions $\epsilon_\Delta = 0$ in the interior. $\Rightarrow$ again, matches continuum solution $R^\mu{}_{\nu\sigma\rho} = 0$, again vertex displacement symmetry is present (4 gauge directions per vertex). Can (in linearised theory) be shown to correspond to action of 4-diffeomorphisms in continuum. Roček, Williams Phys.Lett.B 104 (1981)

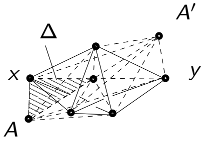
Breaking of gauge symmetries in Regge

In the continuum limit of $4D$ Regge the vertex displacement symmetry is restored. However, whenever discrete solutions do not reproduce continuum dynamics exactly, one has breaking of gauge symmetries. Examples:

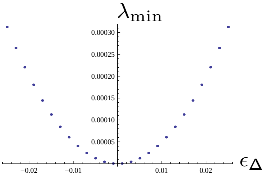
- $3D$ Regge with $\Lambda \neq 0$. The equations of motion are

$$\epsilon_e - \Lambda \sum_{\sigma \supset e} \frac{\partial V_\sigma}{\partial l_e} = 0$$

  $\Rightarrow$ Unique solution for fixed boundary data: No gauge symmetries.



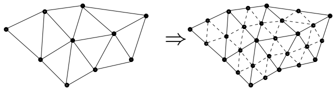
- $4D$ Regge with $\Lambda = 0$: BB, Dittrich: 0905.1670 [gr-qc] Consider triangulation resulting from tent moves $A \to A'$ (one inner vertex). The boundary conditions determine the deficit angle ϵ_Δ at interior triangle Δ. Whenever the angle ϵ_Δ is non-zero (solution with curvature), then the minimal eigenvalue $\lambda_{\min}$ of the Hessian does not vanish $\Rightarrow$ no gauge symmetries! In fact: $\lambda_{\min} \sim \epsilon_\Delta^2$.



Improving the action for $3D$ Regge with $\Lambda \neq 0$

BB, Dittrich: 0907.4323 [gr-qc]

- For this case the coarse graining procedure can be applied to improve the Regge action. In the limit of infinite refinement, the improved action converges to the Regge action for simplices of constant curvature $\kappa = \Lambda$.



$$S_{\text{Regge}} = \sum_e l_e \epsilon_e - \Lambda \sum_\sigma V_\sigma \qquad \Rightarrow \qquad S^{(\kappa)}_{\text{Regge}} = \sum_e l_e \epsilon_e^{(\kappa)} + 2\kappa \sum_\sigma V_\sigma^{(\kappa)}$$

- The perfect action in fact $S^{(\kappa)}_{\text{Regge}}$ possesses exact gauge symmetries (vertex displacement symmetry: 3 gauge dof per vertex).
- Equations of motion are $\epsilon_e^{(\kappa)} = 0$: Solutions are metric of constant curvature κ. Regge action with flat simplices can be seen as linear approximation: $S^{(\kappa)}_{\text{Regge}} = S_{\text{Regge}} + O(\kappa^2)$.

Improving the action for $4D$ Regge:

- In general: (improved and) perfect actions are non-local, but (in case of Regge) hopefully triangulation-independent. BB, Dittrich, He, work in progress
- In $4D$, one can only hope to compute perfect action approximately. Perturb Regge around flat solution (regular $4D$ hypercubes) and improve order by order:

$$S = \epsilon^2 \frac{1}{2} S_{ee'} \, l_e^{(1)} l_{e'}^{(1)} + \epsilon^3 \left(S_{ee'} l_e^{(1)} l_{e'}^{(2)} + \frac{1}{3!} S_{ee'e''} l_e^{(1)} l_{e'}^{(1)} l_{e''}^{(1)} \right) + \dots$$

$\downarrow$ improve

$$S = \epsilon^2 \frac{1}{2} S_{ee'}^{\mathrm{imp}} \, l_e^{(1)} l_{e'}^{(1)} + \epsilon^3 \left(S_{ee'}^{\mathrm{imp}} l_e^{(1)} l_{e'}^{(2)} + \frac{1}{3!} S_{ee'e''} l_e^{(1)} l_{e'}^{(1)} l_{e''}^{(1)} \right) + \dots$$

$\downarrow$ improve

$$S = \epsilon^2 \frac{1}{2} S_{ee'}^{\mathrm{imp}} \, l_e^{(1)} l_{e'}^{(1)} + \epsilon^3 \left(S_{ee'}^{\mathrm{imp}} l_e^{(1)} l_{e'}^{(2)} + \frac{1}{3!} S_{ee'e''}^{\mathrm{imp}} l_e^{(1)} l_{e'}^{(1)} l_{e''}^{(1)} \right) + \dots$$

Consistency conditions for perturbation

$$l_e = l_e^{(0)} + \epsilon l_e^{(1)} + \epsilon^2 l_e^{(2)} + \ldots$$

$$S = \epsilon^2 \frac{1}{2} S_{ee'} l_e^{(1)} l_{e'}^{(1)} + \epsilon^3 \left(S_{ee'} l_e^{(1)} l_{e'}^{(2)} + \frac{1}{3!} S_{ee'e''} l_e^{(1)} l_{e'}^{(1)} l_{e''}^{(1)} \right) + \ldots$$

- Because $l_e^{(0)}$ describes flat solutions, the Hessian $S_{ee'}$ possesses zero vectors y_e^g (gauge symmetries corresponding to vertex translations Roček, Williams Phys.Lett.B 104 (1981)). To linear order:

$$S_{ee'} y_{e'}^g = 0$$

$$l_e^{(n)} = l_p^{(n)} y_e^p + l_g^{(n)} y_e^g \quad \text{(Separation into physical and gauge modes)}$$

$$\Rightarrow \quad l_p^{(1)} \text{ determined, } l_g^{(0)},\ l_g^{(1)} \text{ free}$$

- To higher order, however:

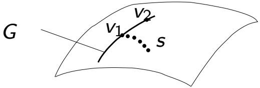
$$\Rightarrow \quad l_p^{(2)} \text{ determined, } l_g^{(1)},\ l_g^{(2)} \text{ free}$$

plus: condition on background gauge variables $l_g^{(0)}$!

Consistency conditions for perturbation

- Since higher order terms break gauge symmetry, solving for higher order determines background gauge! $\Rightarrow$ Due to fact that the solution we perturb around has higher symmetry than generically. Perturbative solutions s (with curvature) have smaller gauge orbit than flat solution (which has 4 gauge dof per vertex).

G = gauge orbit of flat solution. v_1 and v_2 gauge equivalent, but perturbation around v_1 is consistent, around v_2 is not!



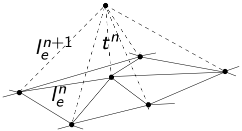
- One can show (Dittrich, Hoehn 0912.1817 [gr-qc]): The eom fixing background gauge parameters are equivalent to

$$\left(\frac{\partial S}{\partial l_e} y_e^g\right)\Big|_{2^{\text{nd}}\text{ order}} = y_e^g \left(\frac{\partial S_{HJ}}{\partial l_e^{(0)}}\right)\Big|_{2^{\text{nd}}\text{ order}}$$

The same issue in canonical language:

In canonical formulation, gauge symmetries of the action arise as constraints. Situation in Regge?

- In BB, Dittrich 0905.1670 [gr-qc] a Hamiltonian framework was presented, which matches exactly presence (or breaking) of gauge-symmetries of action. Based on tent moves (Sorkin et al gr-qc/9411008).



- Tent moves leave combinatorics of $3D$ hypersurface invariant. Regge action for part between n-th and $n+1$-st hypersurface: S_n. Generating function for canonical transformation $l_e^n \to l_e^{n+1}$, $p_e^n \to p_e^{n+1}$

$$p_t^n := -\frac{\partial S_n}{\partial t^n} \qquad p_e^n := -\frac{\partial S_n}{\partial l_e^n}$$

$$p_t^{n+1} := -\frac{\partial S_n}{\partial t^{n+1}} \qquad p_e^{n+1} := -\frac{\partial S_n}{\partial l_e^{n+1}}$$

- Equations of motion: $p_t^n = p_t^{n+1} = 0$, $p_e^n = -p_e^{n+1}$.

Dynamics and constraints:

Dittrich, Hoehn 0912.1817 [gr-qc]

- For four-valent vertex being evolved by tent move: Only flat solution $\epsilon_\Delta = 0$!

$$C_e \;=\; p^n_e \;-\; \sum_{\Delta\supset e} \frac{\partial a_\Delta}{\partial l^n_e}\,\psi_\Delta(l^n_e) \;=\; 0$$

  Equation only between variables l^n_e, p^n_e at time step n. $\Rightarrow$ constraint

- Higher-valent vertex being evolved by tent move: More than flat solutions.

$$C_e \;=\; p^n_e \;-\; \sum_{\Delta\supset e} \frac{\partial a_\Delta}{\partial l^n_e}\,\psi_\Delta(l^n_e, l^{n+1}_e) \;=\; 0$$

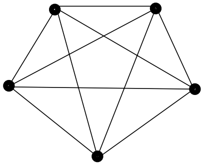
  Equation involves variables l^n_e, l^{n+1}_e at different time steps. Momenta depend (weakly) on l^{n+1}_e $\Rightarrow$ "pseudo-constraints"

"Dynamics" for one simplex

Dittrich, Ryan 0807.2806 [gr-qc], Dittrich, Hoehn 0912.1817 [gr-qc]

- Example: Triangulation consisting of one four-simplex.
- Canonical transformation:

$$A_\Delta = A_\Delta(l_e)$$
$$p_\Delta = \frac{\partial l_e}{\partial A_\Delta} p_e$$



- Constraints $C_e \to C_\Delta$:

$$C_\Delta = p_\Delta + \psi_\Delta$$

- $\Rightarrow$ Constraints fix momenta to be exterior angles (as computed by the lengths).
- Are fist class!
- Generate deformations of (boundary) hypersurface.
- Ten lengths (+ momenta), ten constraints: $\Rightarrow$ no dof left! (flat interior)

Linearised constraints for $4D$ linearised Regge

Dittrich, Hoehn 0912.1817 [gr-qc]

- Linearised $4D$ Regge around flat sector: $l_e = l_e^{(0)} + \epsilon l_e^{(1)} + \ldots$, $p_e = p_e^{(0)} + \epsilon p_e^{(1)} + \ldots$. Since Hessian has null vectors $\frac{\partial^2 S_{\text{Regge}}}{\partial l_e \partial l_{e'}} y^g_{e'} = 0$, one also has (linearised) constraints:

$$C_g = p_e^{n,(1)} y_e^g + y_{e'}^g \left(\frac{\partial}{\partial l_{e'}} \sum_{\Delta \supset e} \frac{\partial a_\Delta}{\partial l_e} \psi_\Delta \right)\Bigg|_{l_e = l_e^{(0)}} l_e^{n,(1)}$$

- Constraints give relation between intrinsic and extrinsic geometry
- Are first class! (despite very complicated form of dihedral angles)
- Generate linearized deformation of hypersurface (via vertex translations): Hamiltonian and diffeomorphism constraints
- Are preserved by linearized tent move dynamics

Independence of action on triangulation?

Since the perfect action should exhibit exact gauge symmetries, it should reproduce the continuum dynamics. Therefore, it should, if computed w.r.t. different triangulations with same boundary, yield the same result on solutions (be "triangulation-independent").

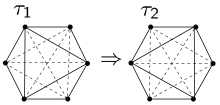
- Classification of four-manifolds (Pfeiffer gr-qc/0404088): Path integral for GR should be Diff-invariant of four-manifolds. Theorem by Pachner:

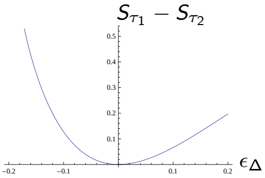
  Smooth 4-manifolds / diffeos $\Leftrightarrow$ piecewise linear 4-manifolds / Pachner moves

- Here Triangulation is *not* seen as approximation or cut-off, but to contain whole information of smooth manifold!
- Triangulation-independence not because of "topological theory", but rather because of action on renormalization-group fixed point ("quantum perfect action").

Triangulation-(in-)dependence in Regge

- In $3D$ Regge with $\Lambda = 0$. ✓
- In $3D$ Regge with $\Lambda \neq 0$ *if* one uses perfect action $S^{(\kappa)}_{\text{Regge}}$. ✓
- In $4D$, whenever there is some part of the interior which is flat (includes all $1-5$ moves of simplices). ✓
- In $4D$, the Regge action is *not* invariant under the $3-3$ move. The difference of S_{Regge} before/after $3-3$ move is again quadratic in ϵ_Δ for (either) interior triangle Δ!





$\Rightarrow$ Triangulation-independence in Regge is realised to the same extent in which action is invariant under vertex translations.

Summary

- Diffeomorphism-symmetry crucial for General Relativity $\Rightarrow$ fate upon discretisation important for quantisation!
- Generically, upon discretisation, reparametrization-invariance ($\Leftrightarrow$ Diffeomorphism-symmetry) is lost, whenever discrete dynamics exactly reproduces continuum dynamics.
- In particular, gauge symmetries are broken in Regge calculus, except for $3D$ with $\Lambda = 0$ and $4D$ flat sector (many different piecewise linear flat metric approximate the same (diffeomorphic) smooth metric).
- Either corresponds to anomaly $\Rightarrow$ cure! Or: many more microscopic dof than macroscopic dof, and diff-symmetry is only realized in continuum limit.

Improved and perfect actions

- Presence (or absence) of gauge symmetry is result of action (and of way in which it is discretised). Restore symmetries by "improving" action (coarse-graining procedure: incorporate dynamics on finer triangulation into coarser triangulation).

$$\begin{array}{ccc} S_{\mathrm{discr}} & \xrightarrow{\text{quantize}} & \int \mathcal{D}g_{\Delta}\, e^{iS_{\mathrm{discr}}} \\ \text{coarse grain} \downarrow & & \downarrow \text{renormalize} \\ S_{\mathrm{perf}} & \xrightarrow{\text{quantize}} & \int \mathcal{D}[g_{\mu\nu}]\, e^{iS_{\mathrm{perf}}} \end{array}$$

- Investigation of diff-invariant discrete actions worthwhile: Conditions for path integral? (compare LOST-thm in LQG)
- Complementary analysis to "sum over triangulations" (GFT approach!): Effectively include sum over (some) finer triangulations in S_{imp}.

- Improving action restores gauge symmetries:
  - $1D$ discrete rep-invariant mechanics: it works!
  - $3D$ Regge with $\Lambda \neq 0$: it works!
  - $3D$ Regge with scalar field? Dittrich, Banisch, work in progress
  - $4D$ Regge with $\Lambda \neq 0$: it works (for constantly curved sector)!
- For Regge, there is a discrete canonical analysis which exactly matches discrete covariant formulation, using tent moves:

$$\text{gauge symmetries present} \quad \Leftrightarrow \quad \text{exact } 1^{\text{st}}\text{-class constraints}$$

$$\text{gauge symmetries broken} \quad \Leftrightarrow \quad \text{'pseudo constraints'}$$

  (crf "consistent discretisation" approach Gambini-Pullin '03-'05)
- Furthermore: Regge is triangulation (in-)dependent to the same extent in which it is invariant under vertex translations. ($3-3$ move: $\Delta S \sim \epsilon^2$)

$\Rightarrow$ Hints that the following three problems are in fact related:

Find discrete action with exact gauge symmetry $\longleftrightarrow$ Find an anomaly-free discrete canonical constraint algebra

Find a discrete action which is independent of triangulation

Outlook

- Improve linearised $4D$ Regge: BB, Dittrich, He: work in progress Subtle: higher orders fix background gauge parameters!
- For general quadratic actions of variables x_i:

$$S_{\text{discr}} = \frac{1}{2} x_i M_{ij} x_j$$

Gauge directions: $M_{ij} y_i^g = 0$. Coarse grain variables: $X_I = b_{Ii} x_i$

$$S_{\text{imp}} = \frac{1}{2} X_I M_{IJ} X_J$$

with

$$M_{IJ} = b_{Ii} y_i^p (\tilde{M}^{-1})_{pq} y_j^q b_{Jj}, \qquad \text{where } \tilde{M}_{pq} := y_p^i M_{ij} y_q^j$$

- $M_{IJ} b_{Ii} y_i^g = 0 \Rightarrow$ Gauge directions are preserved under coarse graining.
- Restoration of triangulation independence in perfect limit?
- Locality of perfect action?

Outlook:

- In quantum theory: coarse graining (renormalization group flow) should lead to path integral which is exact "projector" onto physical Hilbert space.
- $\Rightarrow$ Works so far for $1D$ discretised quantum mechanics for (an-)harmonic oscillator (perturbatively and non-perturbatively) BB, Dittrich, Steinhaus, work in progress
- Quantise perfect action $S^{(\kappa)}_{\mathrm{Regge}}$ for $3D$ Regge with $\Lambda \neq 0$: Relation to Turaev-Viro?
- Turaev-Viro as renormalization group fixed point of "Ponzano-Regge $+\Lambda V$"?