4-volume in spin foam models from knotted boundary graphs

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in collab with
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In this talk: Convex polytopes in $\mathbb{R}^4$

4d geometry from 2d bivectors
Motivation

Spin Foam models:

Based on simplicity constraints (GR as constrained BF theory)

Basis: spin network functions

\[ j_{ab}, \lambda_a \leftrightarrow j_{ab}, \vec{n}_{ab} + \text{closure constraint} \]

Reconstruction of 4d polytopes \( P \) from 3d boundary data:

\[ j_{ab}, \vec{n}_b \rightarrow B_{ab} \rightarrow P \]

\text{crit. stationary pt. in asymptotics}

Simplicity constraints: conditions on bivectors \( B_{ab} \) s.t. 4d polytope \( P \) exists

[Barrett, Crane '97, Livine, Speziale '07, Engle et al '07, Freidel, Krasnov '07, Barrett et al '08, Kaminski et al '09, Dittrich, Ryan '10, Bianchi et al '11]
I Motivation

Problem:

For general polytopes \( P \), reconstruction is unknown

→ simplicity constraints for general polytopes?

→ EPRL-FK-KKL model:
  underconstrained (volume simplicity not implemented)

In this talk: How to reconstruct the 4-volume \( V \) from face bivectors \( B_{ab} \)

→ General formula requires knotting information of graph
  (generalisation of Han’s simplex construction + proofs)

[Han ‘11, Dona et al ‘17, BB, Belov ‘17, BB, Rabuffo ‘18, BB, ‘18]
I  Motivation

II  Volume of a 4d polyhedron

III  Quantum amplitude and asymptotics

IV  Quadratic volume simplicity constraint

V  Summary and outlook
**Volume of a 4d polyhedron**

**Bivectors** \( B \in \mathbb{R}^4 \wedge \mathbb{R}^4 \simeq so(4) \)

**Oriented graphs** \( \Gamma \subset S^3 \)

**nodes** \( n \), **oriented links** \( \ell \)

A **bivector geometry**:

- **graph** \( \Gamma \)
- **bivectors to links** \( \{B_\ell\}_\ell \)
- **diagonal simplicity**: \( B_\ell \wedge B_\ell = 0 \)
- **cross-simplicity**: \( B_\ell \wedge B_{\ell'} = 0 \) for links \( \ell, \ell' \) adjacent to the same node:
- **closure**: \( \sum_{\ell \ni n} [n, \ell] B_\ell = 0 \) for all nodes \( n \)
II Volume of a 4d polyhedron

Projection onto the plane:

2d graph with crossings \( C \)

For one crossing \( C \), define:

\[
V_C := \sigma(C') \ast \left( B_{\ell_1} \wedge B_{\ell_2} \right)
\]

Hodge operator:

\[
* : \wedge^4 \mathbb{R}^4 \rightarrow \mathbb{R}
\]

For the whole graph \( \Gamma \), define the number:

\[
V_\Gamma := \frac{1}{6} \sum_C V_C
\]

\[\sigma(C) = -1\]  \[\sigma(C') = +1\]
II Volume of a 4d polyhedron

Claim: \( V_\Gamma = \frac{1}{6} \sum_C \sigma(C) \ast (B_{\ell_1} \wedge B_{\ell_2}) \) does not depend on 2d projection

Proof: Reidemeister moves:

1.)

2.)

3.)

4.)

5.)

\( B_\ell \wedge B_\ell = 0 \)

\( \sigma(C_1) = -\sigma(C_2) \)

\( B_\ell \wedge B_{\ell'} = 0 \)

\( \sum_{\ell \geq n} [n, \ell] B_\ell = 0 \)
Every 4-dim convex polytope $P \subset \mathbb{R}^4$ uniquely determines a bivector geometry.

Polytope $P \rightarrow$ boundary graph $\Gamma \subset S^3$ (dual to 3d boundary polyhedron)

2d faces $f$ of $P$ $\leftrightarrow$ links $\ell$ of $\Gamma$

face+orientation $\rightarrow$ bivector

$$B_\ell = N \wedge M \quad N, M \in \mathbb{R}^4$$

$$\text{Area}(f) = |N||M| \sin \angle(N, M)$$

$\rightarrow$ bivector geometry (diagonal-, cross-simplicity + closure automatically satisfied)

Claim: For a 4d polytope $\text{Vol}(P) = V_\Gamma = \frac{1}{6} \sum_C \sigma(C) \ast (B_{\ell_1} \wedge B_{\ell_2})$

Sketch for proof:
1.) Show that it is true for 4-simplex
2.) Show how the invariant behaves under cutting of polytopes
3.) Show that every polytope can be cut successively into simplices
1.) True for a 4-simplex:

By direct calculation

Spanned by four vectors \( e_1, e_2, e_3, e_4 \in \mathbb{R}^4 \)

\[
B_1 = \frac{1}{2} e_1 \wedge e_2 \quad B_2 = \frac{1}{2} e_3 \wedge e_4
\]

Only one crossing in the boundary graph:

\[
V_\Gamma = \frac{1}{24} * (e_1 \wedge e_2 \wedge e_3 \wedge e_4) = V
\]

\( \rightarrow \) Claim proven for 4-simplices

[Han '11]
2.) Cutting/glueing of polytopes (graph surgery)

two graphs $\Gamma_1$, $\Gamma_2$

with mirrored nodes $n_1$, $n_2$

(identical bivectors, but reverse orientations)

graphs can be merged together, to one big graph

Easy to show:

under this procedure, $V_{\Gamma}$ is additive:

$$V_{\Gamma_1 \#_{(n_1,n_2)} \Gamma_2} = V_{\Gamma_1} + V_{\Gamma_2}$$
2.) Cutting/glueing of polytopes (graph surgery)

Cutting of one convex polytope with hyperplane into two polytopes (3d analogue of image):

Boundary graph gets split up.

\[ \Gamma_{P_1} \#_{(p,q)} \Gamma_{P_2} \sim \Gamma_P \]

Regain old bdy graph with two moves: (~ trivial subdivision of bdy polytopes)

1.)

\[ \text{Moves leave } V_{\Gamma} \text{ invariant } \rightarrow V_{\Gamma_P} = V_{\Gamma_{P_1}} + V_{\Gamma_{P_2}} \]
3.) Remains to show:

Every convex $n$-polytope can be successively cut into $n$-simplices via $(n - 1)$ hyperplanes.

Proof: induction over the dimension $n$:

a) Fix internal vertex $v$
b) Subdivide polytope by the hyperplane $H_f$ spanned by $v$ and $(n - 2)$ face $f$
c) → Pyramids over $(n - 1)$ polytopes → subdivide those

For $n = 2$ the process leads to 2-simplices (triangles):

→ done
II Volume of a 4d polyhedron

This finishes the proof:

For every convex 4d polytope $P$, the 4-volume can be computed by its bivectors:

$$V_P = \frac{1}{6} \sum C \sigma(C) \ast \left( B_{\ell_1} \wedge B_{\ell_2} \right)$$

→ Needs knotting information of the boundary graph!
I  Motivation

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III Quantum amplitude and asymptotics

EPRL-FK-KKL model:

Boundary states: $SU(2)$ -spin network functions $\psi \in \mathcal{H}_{SU(2)}$

\[ \mathcal{H}_{SU(2)} = \bigoplus_{k_\ell} \left( \bigotimes_{n} \text{Inv}_{SU(2)}(V_{k_1} \otimes \cdots \otimes V_{k_m}) \right) \]

\[ \mathcal{H}_{SU(2) \times SU(2)} = \bigoplus_{j_\ell^\pm} \left( \bigotimes_{n} \text{Inv}_{SU(2) \times SU(2)}(V_{j_1^\pm} \otimes \cdots \otimes V_{j_m^\pm}) \right) \]

Boosting map: $\beta : \mathcal{H}_{SU(2)} \rightarrow \mathcal{H}_{SU(2) \times SU(2)}$

(insert into highest / lowest weight)

\[ j_\ell^\pm = \frac{1}{2} |1 \pm \gamma| k_\ell \]

Amplitude: inner product between boosted boundary state and $BF$ vacuum state

\[ \mathcal{A}(\psi) := \langle \Psi_0 | \beta \psi \rangle \]

Amplitude: inner product between boosted boundary state and $BF$ vacuum state

\[ \Psi_0 \sim \prod_{\ell} \delta^{m_\ell^\pm}_{n_\ell^\pm} \]

Isomorphism $\mathbb{R}^4 \wedge \mathbb{R}^4 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2) \rightarrow \hat{B}$ derivative operators on $\mathcal{H}_{SU(2) \times SU(2)}$

[Engle, Pereita, Rovelli, Livine ‘07, Freidel, Krasnov ‘07, Kaminski, Kisielowski, Lewandowski ‘09]
Deformed amplitude:

\[ \hat{V} := \frac{1}{6} \sum_C \sigma(C) * (\hat{B}_1 \wedge \hat{B}_2) \quad B \sim (\vec{b}^+, \vec{b}^-) \]

\[ = \frac{1}{6} \sum_C \sigma(C) \delta_{I,J} \left( \hat{b}_1^+, \hat{b}_2^+, J - \hat{b}_1^-, \hat{b}_2^-, J \right) \]

\[ = \frac{1}{6} \sum_C \sum_{\epsilon=\pm} \frac{\epsilon 4 \gamma^2}{(1 \epsilon \gamma)^2} \sum_{I=1}^3 D_{(j_L^\epsilon)} (X_I^\epsilon) \otimes D_{(j^\epsilon, J)} (X_I^\epsilon) \]

Deformation parameter \( \omega \)

\[ \mathcal{A}^\omega (\psi) := \left\langle \Psi_0 \right| \exp \left( i \omega \hat{V} \right) \beta \psi \right\rangle = \mathcal{A}^{\omega,+} \mathcal{A}^{\omega,-} \]

Deformed amplitude factorizes (Euclidean signature, \( \gamma < 1 \))

Note: usually, cosmological constant is incorporated via quantum groups (state sum, boundary Hilbert space) \( \rightarrow \) Here we stay with classical groups

[Han ‘11, Haggard et al ‘15, Dittrich, Geiller ‘16, BB, Rabuffo ‘18]
Claim: Large $j$ asymptotics of $A^{\omega}(\psi)$: same critical & stationary points as the one for normal amplitude $A(\psi)$, and Hessian matrix is also the same!

Sketch of proof: First we expand the exponential into a sum, then we make the assumption that we can in actuality exchange the sum and the asymptotic limit.

Calculation can be performed for $\pm$ sectors separately

Livine-Speziale coherent states on boundary: $\vert j, \vec{n}\rangle = g_{\vec{n}} \vert j, j\rangle$

Undeformed amplitude (one sector), for links $(ab)$:

$$A(\psi) = \int_{SU(2)^{N_T}} dg_a \prod_{b \rightarrow a} \langle j_{ab}, n_{ab} \vert (g_a)^{-1} g_b \vert j_{ab}, n_{ba} \rangle$$

Deformed amplitude contains, for each crossing (e.g. plus-sector):

$$\langle \Psi | = \langle j_{ab}, n_{ab} \vert (g_a)^{-1} \otimes \langle j_{a'b'}, n_{a'b'} \vert (g_{a'})^{-1} \langle \Psi | \exp \left( \frac{4i\omega}{(1 + \gamma)^2} \sum_{I=1}^{3} X_I \otimes X_I \right) \vert \Phi \rangle$$

$$\vert \Phi \rangle = g_b \vert j_{ab}, n_{ba} \rangle \otimes g_{b'} \vert j_{a'b'}, n_{b'a'} \rangle$$
Expansion of the exponential:

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{4i\omega}{(1 + \gamma)^2} \right)^n \sum_{I_1, I_2, \ldots, I_n = 1}^{3} \langle j_{ab}, n_{ab} | (g_a)^{-1} X_{I_1} X_{I_2} \cdots X_{I_n} g_{b} | j_{ab}, n_{ba} \rangle \\
\times \langle j_{a'b'}, n_{a'b'} | (g_{a'})^{-1} X_{I_1} X_{I_2} \cdots X_{I_n} g_{b'} | j_{a'b'}, n_{b'a'} \rangle
\]

Insert resolution of identity \( n - 1 \) times: \((2j + 1) \int_{S^2} d^2 n |j, n\rangle \langle j, n| = 1_{V_j}\)

using \( \langle j, n|X_I |j, n'\rangle = j \langle n|\sigma_I |n'\rangle \langle n|n'\rangle^{2j-1}\)

we get:

\[
\langle j_{ab}, n_{ab} | (g_a)^{-1} X_{I_1} X_{I_2} \cdots X_{I_n} g_{b} | j_{ab}, n_{ba} \rangle = \int_{(S^2)^{n-1}} d^2 n_i \ a(n_i, g_a, g_b) \ e^{S(n_i, g_a, g_b)}
\]

with

\[
a(n_i, g_a, g_b) = (2j + 1)^{n-1} j^n \frac{\langle n_{ab} | (g_a)^{-1} \sigma_I | n_1 \rangle \langle n_1 | \sigma_I | n_2 \rangle \cdots \langle n_{n-1} | \sigma_I | g_b | n_{ba} \rangle}{\langle n_{ab} | (g_a)^{-1} | n_1 \rangle \langle n_1 | n_2 \rangle \cdots \langle n_{n-1} | g_b | n_{ba} \rangle}
\]

\[
S(n_i, g_a, g_b) = 2j \left( \ln \langle n_{ab} | (g_a)^{-1} | n_1 \rangle + \ln \langle n_1 | n_2 \rangle + \cdots + \ln \langle n_{n-1} | g_b | n_{ba} \rangle \right)
\]

\( \rightarrow \) Here we exchange asymptotic limit and infinite sum
Integration variables: \( g_a = e^{ix_a^I\sigma_I} g_a^{(c)}(\phi_i, \theta_i), n_i(\xi_i, \chi_i) \)

critical stationary points: \( \text{Re}(S) = 0 \quad \partial S = 0 \)

\[
\begin{align*}
g_a n_{ab} &= g_b n_{ba} \quad &n_i &= g_b n_{ba}, \quad n_i' &= g_{b'} n_{b'a'} \text{ for all } i. \\
\text{Hessian matrix:} & \quad \hat{H}^c_{d,IJ} := \frac{\partial^2 S}{\partial x_c^I \partial x_d^J} \quad &\text{same matrix as undeformed case}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 S}{\partial \phi_i^2} &= \frac{\partial^2 S}{\partial \theta_i^2} = -j_{ab} \\
\frac{\partial^2 S}{\partial \phi_i \partial \phi_{i+1}} &= \frac{\partial^2 S}{\partial \phi_i \partial \phi_{i+1}} = j_{ab} \frac{\partial^2 S}{\partial \phi_i \partial \theta_i} = 0 \\
\frac{\partial^2 S}{\partial \phi_i \partial \theta_{i+1}} &= i \frac{j_{ab}}{2}, \quad \frac{\partial^2 S}{\partial \phi_{i+1} \partial \theta_i} = -i \frac{j_{ab}}{2} \\
\frac{\partial^2 S}{\partial x_b^I \partial \phi_1} &= -\frac{\partial^2 S}{\partial x_a^I \partial \phi_1} = j_{ab} \left( iV_2^I - V_1^I \right) \\
\frac{\partial^2 S}{\partial x_b^I \partial \theta_1} &= -\frac{\partial^2 S}{\partial x_a^I \partial \theta_1} = j_{ab} \left( iV_1^I + V_2^I \right)
\end{align*}
\]

\( G \sigma J G^{-1} = V_J^I \sigma_I. \)

\( G := (g_b g_{nba})^{-1} \)
Finally, the total Hessian matrix:

\[
H = \begin{pmatrix}
A & B \\
B^T & \tilde{H}
\end{pmatrix}
\]

where

\[
A \in \mathbb{C}^{2(n-1) \times 2(n-1)} \\
B \in \mathbb{C}^{4(n-1) \times 3(N-1)} \\
\tilde{H} \in \mathbb{C}^{3(N-1) \times 3(N-1)}
\]

It follows:

\[
\det(H) = \det(A) \det(\tilde{H} - B^T A^{-1} B)
\]

\[
= (j_{ab} j_{a'b'})^{2(n-1)} = 0
\]

same matrix as undeformed case

undеformed Hessian

\[
\det(H) = (j_{ab} j_{a'b'})^{2(n-1)} \det(\tilde{H}).
\]
Asymptotics: \( j \rightarrow \lambda j, \quad \omega \rightarrow \lambda^{-2}\omega \)

(for + -sector, one critical stationary point)

\[
\mathcal{A}^\omega \rightarrow \mathcal{A} \sum_{n=0}^\infty \frac{1}{n!} \left( \frac{4i\omega \lambda^{-2}}{(1+\gamma)^2} \right)^n \left( \frac{1}{4\pi} \right)^{2(n-1)} \left( \frac{2\pi}{\lambda} \right)^{2(n-1)} \\
\times \sum_{I_1, I_2, \ldots, I_n = 1}^3 4^{n-1} \frac{(\lambda j_{ab})^{2n-1}(\lambda j_{a'b'})^{2n-1}}{\sqrt{(j_{ab} j_{a'b'})^{2(n-1)}}} \prod_{i=1}^n (\tilde{n}_{ba})^{I_i} (\tilde{n}_{b'a'})^{I_i} 
\]

\[
= \mathcal{A} \sum_{n=0}^\infty \frac{\lambda^{2n}}{n!} (j_{ab} j_{a'b'})^n \left( \frac{4i\omega \lambda^{-2}}{(1+\gamma)^2} \right)^n \left( \sum_{I=1}^3 (\tilde{n}_{ba})^I (\tilde{n}_{b'a'})^I \right)^n 
\]

\[
= \mathcal{A} e^{i\omega \tilde{X}_{a'b'} \cdot \tilde{Y}_{a'b'}} 
\]

\[\tilde{X}_{ab} = k_{ab} \tilde{n}_{ab}, \quad \tilde{Y}_{a'b'} = k_{a'b'} \tilde{n}_{a'b'}\]

[Barrett et al '08, Conrady, Freidel '08, Han '11, BB, Rabuffo '18]
Assume that boundary data allows for two distinct solutions $g_a^{(c)} = g_a^\pm$

(e.g. “Regge boundary data”, allowing only one 4d-polyhedron of volume $V$)

→ mixed terms give volume term, same-sign terms cancel

→ For the full amplitude:

$$\mathcal{A} \longrightarrow \frac{1}{W} + \frac{1}{W^*} + \frac{1}{|D|} \cos(S_{\text{Regge}})$$

$$\mathcal{A}^\omega \longrightarrow \frac{1}{W} + \frac{1}{W^*} + \frac{1}{|D|} \cos(S_{\text{Regge}} - \omega V)$$

Careful: In general more solutions, which all appear in the asymptotics
(same 3d boundary data can allow for several different 4d polyhedra)
I Motivation
II Volume of a 4d polyhedron
III Quantum amplitude and asymptotics
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V Summary and outlook
IV Quadratic volume simplicity constraint

For general 4d-polytopes, the EPRL-FK-KKL model is underconstrained.

For a 4-simplex, there is no problem:

\[ B_f \wedge B_f = 0 \quad \text{diagonal simplicity} \]
\[ B_f \wedge B_{f'} = 0 \quad \text{cross simplicity} \]
\[ B_f \wedge B_{f'} = V \quad \text{volume simplicity} \]

EPRL-FK-KKL-model construction only rests on
diagonal-, cross-simplicity constraint + closure constraint (gauge invariance)

→ Implemented on boundary Hilbert space

For a 4-simplex, these are enough:

diagonal- + cross-simplicity + closure imply volume-simplicity
(in particular: reconstruction of 4-simplex form bivectors)

But: For general 4d polytopes: volume-simplicity constraint is missing!

[Barrett, Crane ‘97, Engle et al ‘07, Freidel, Krasnov ‘07, BB, Steinhaus ‘15]
IV Quadratic volume simplicity constraint

Note: There is an additional under-constrained-ness due to twisted geometries

→ e.g. area-angle constraints, ensure face-matching in the 4-simplex case
→ these “twisted” degrees of freedom are suppressed in the large-$j$ asymptotics

These are not related to the volume-simplicity constraint!

For more general 4d-polytopes, the volume-simplicity problem adds even more non-metric degrees of freedom.

These also manifest in non-face-matching
But: cannot be removed via area-angle-constraints (“conformal d.o.f.”)

[Dittrich, Speziale ‘08, Freidel, Speziale ‘10, Freidel, Ziprick ‘13, BB, Steinhaus ‘15, Dona et al ‘17, BB, Belov ‘17]
IV Quadratic volume simplicity constraint

Example: 4d-hypercubic graph: $\Gamma$

Consider a certain bivector geometry on that boundary graph cuboidal: each polyhedron is a 3d cube in $\mathbb{R}^4$. All bivectors along great circles coincide. Six great circles $\rightarrow$ six areas

$$a_1, \ldots , a_6$$

However: hypercuboid is only specified by four numbers.

$\rightarrow$ There are more bivector geometries than hypercuboids.

$$B_1 = a_1 \, (e_y \wedge e_z), \quad B_2 = a_2 \, (e_z \wedge e_x), \quad B_3 = a_3 \, (e_x \wedge e_y),$$

$$B_4 = a_4 \, (e_z \wedge e_t), \quad B_5 = a_5 \, (e_t \wedge e_y), \quad B_6 = a_6 \, (e_x \wedge e_t),$$

[BB, Steinhaus ‘15]
Consider the three Hopf links $H_1$, $H_2$, $H_3$ of the hypercuboidal boundary graph.

Hopf link: Only crossings with each other! $C \not\subset H$

Define Hopf-link volume:

$$V_C = \sigma(C) \ast (B_{\ell_1} \wedge B_{\ell_2})$$

$$V_H := \frac{1}{6} \sum_{C \not\subset H} V_C$$

For the hypercuboidal bivector geometry, one gets:

$$V_{H_1} = \frac{1}{3} a_1 a_6 \quad V_{H_2} = \frac{1}{3} a_2 a_5 \quad V_{H_3} = \frac{1}{3} a_3 a_4$$

Total volume is the sum of these: $$V_{\text{tot}} = \frac{1}{3} (a_1 a_6 + a_2 a_5 + a_3 a_4)$$

For an actual hypercuboid $\rightarrow$ all equal $\rightarrow$ Volume constraint on Hopf links

$$V_{H_1} = V_{H_2} = V_{H_3}$$

[BB, Belov '17]
This implies: discretisation of simplicity constraints:

- **diagonal simplicity** discretised on: links
  \[ B_\ell \land B_\ell = 0 \]

- **cross simplicity** discretised on: nodes
  \[ B_\ell \land B_{\ell'} = 0 \]

- **volume simplicity** discretised on: Hopf-links
  \[ \sum_{C \in H_1} V_C = \sum_{C \in H_2} V_C \]

The Hopf-link-volume has to agree for each Hopf-link in the boundary graph.

→ General polytopes?
IV Quadratic volume simplicity constraint

How to impose Hopf link volume simplicity constraint on the quantum level?

a) problem: dynamics vs kinematics:

diagonal-simplicity and cross-simplicity constraints are *kinematical*: can be formulated within one vertex (intertwiner).

volume simplicity constraint is dynamical: need information about 4d shape → need information about (flat) dynamics: amplitude.

\[ V_\Psi := \mathcal{A}(\hat{V}\Psi), \quad V_{H,\Psi} := \mathcal{A}(\hat{V}_H\Psi), \]

b) problem: cosine problem

Amplitude \( \mathcal{A} \) contains contributions from both orientations of \( \mathbb{R}^4 \) → volume counts positive and negative. → all \( V_{H,\Psi} = \mathcal{A}(\hat{V}_H\Psi) \) are zero in asymptotic limit.

→ possible solution: proper vertex?

→ our solution: use even function of volume

\[ V_{H,\Psi}^2 := \mathcal{A}(\hat{V}_H^2\Psi), \]

[Barrett et al ’08, Engle ’13, Engle, Zipfel ’15]
Boundary state: can be chosen to represent quantum version of hypercuboid \( \gamma \in (0, 1) \)
satisfies (quantum versions of) linear simplicity constraint

\[
\psi^{\pm}(h_e^{\pm}) = \langle \otimes_e D_{j_e^{\pm}}(h_e^{\pm}), \otimes_v t_v^{\pm} \rangle \\
\gamma_j^{\pm} = \frac{1 \pm \gamma}{2} j
\]

Livine-Speziale Intertwiners:

spins \leftrightarrow\text{ areas} \quad a \sim \gamma j

intertwiner \leftrightarrow 3d \text{ shapes}

\[
\psi^{\pm}_{j_1 \ldots j_6} = \int_{SU(2)} dg\ G \left[ \bigotimes_{i=1}^{3} |j_{a_i}^{\pm}, e_i\rangle \langle j_{a_i}^{\pm}, e_i| \right]
\]

depends on six spins: \( j_1 \ldots, j_6 \)

[Livine, Speziale ’07, Bianchi ’08, Donà, Speziale ’11, BB, Steinhaus ’15]
IV Quadratic volume simplicity constraint

Large j asymptotics: hypercuboid

\[ \mathcal{A}(\Psi) \sim \left( \frac{1}{D} + \frac{2}{|D|} + \frac{1}{D^*} \right), \]

Here \( D \) is a polynomial in the \( j_1 \ldots j_6 \) of degree 21

Large j asymptotics: Hopf link volumes

\[ \mathcal{A}(\hat{\Gamma}_{I_{H_1}}^2 \Psi) \sim \gamma^4 (j_1 j_6)^2 \left( \frac{1}{D} + \frac{1}{D^*} \right), \]
\[ \mathcal{A}(\hat{\Gamma}_{I_{H_2}}^2 \Psi) \sim \gamma^4 (j_2 j_5)^2 \left( \frac{1}{D} + \frac{1}{D^*} \right), \]
\[ \mathcal{A}(\hat{\Gamma}_{I_{H_3}}^2 \Psi) \sim \gamma^4 (j_3 j_4)^2 \left( \frac{1}{D} + \frac{1}{D^*} \right), \]

→ Demand that Hopf link volumes agree: linear condition → subspace of all spin network functions satisfying linear simplicity.

\[ j_1 j_6 = j_2 j_5 = j_3 j_4 \]

In the large j limit, hypercuboids with geometricity appear to satisfy this constraint → eliminated non-geometric degrees of freedom!

[BB, Steinhaus '15]
Summary:

* Proof of a formula for volume of 4-polytope in terms of its bivectors and crossings in its boundary graph

* Can be used to define a deformation of the EPRL-FK-KKL-amplitude with cosmological constant term
  Asymptotics: → weird terms & Hessian matrix unchanged!

* EPRL-KF-KKL-model underconstrained: no (quadratic) volume simplicity
  → Constraint can be discretized over Hopf links in bdy graph
  works in examples
V Summary and Outlook

Outlook:

* So far volume formula only for convex polytopes
  → proof easy to generalise to non-convex case
  → Non-convex polytopes appear in asymptotics of EPRL-FK-KKL model!
    (linear volume-simplicity constraint?)

* Connection to Haggard et al: Chern-Simons theory?

* Deformed amplitude: sensitive to graph knotting → physical IP!

* Connection to quantum groups?

* Hopf-link-volume simplicity constraint: general polytopes?

[BB '11, Haggard, Han, Kaminski, Riello '15, Dittrich, Geiller '16, Dona et al '17]