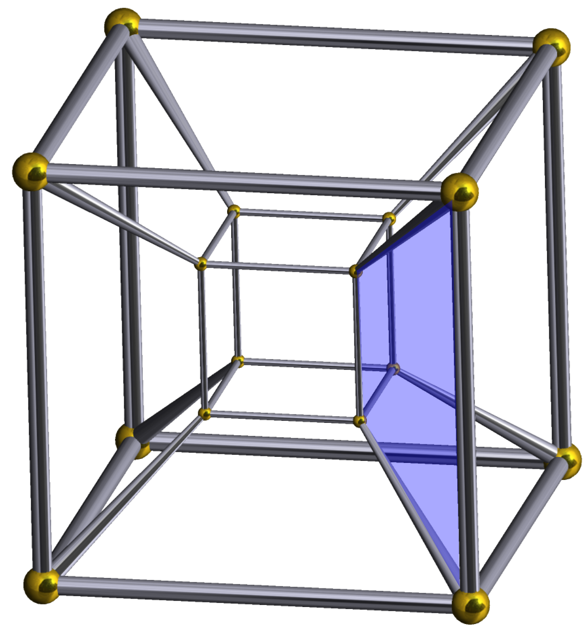
4-volume in spin foam models from knotted boundary graphs

Benjamin Bahr
II. Institute for Theoretical Physics
University of Hamburg
Luruper Chaussee 149
22761 Hamburg

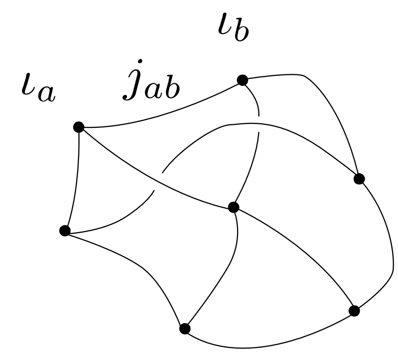
International Loop Quantum Gravity Seminar
25th September 2018

in collab with
Vadim Belov, Giovanni Rabuffo

In this talk: Convex polytopes in $\mathbb{R}^4$



4d geometry from 2d bivectors

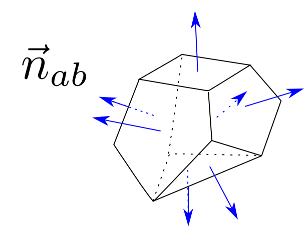


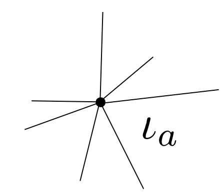
Spin Foam models:

Based on simplicity constraints (GR as constrained BF theory)

Basis: spin network functions

$j_{ab}, \iota_a \leftrightarrow j_{ab}, \vec{n}_{ab}$ + closure constraint





Reconstruction of 4d polytopes P from 3d boundary data:

$$j_{ab}, \vec{n}_b \longrightarrow B_{ab} \overset{?}{\longrightarrow} P$$

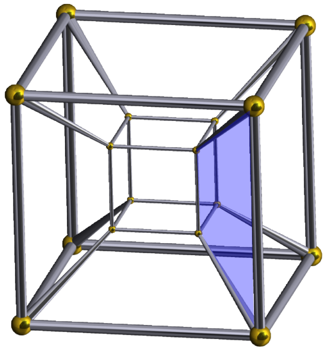
crit. stationary pt. in asymptotics

Simplicity constraints: conditions on bivectors B_{ab} s.t. 4d polytope P exists

[Barrett, Crane '97, Livine, Speziale '07, Engle et al '07, Freidel, Krasnov '07, Barrett et al '08, Kaminski et al '09, Dittrich, Ryan '10, Bianchi et al '11]

Problem:

For general polytopes P, reconstruction is unknown



$\rightarrow$ simplicity constraints for general polytopes?

$\rightarrow$ EPRL-FK-KKL model:
underconstrained (volume simplicity not implemented)

In this talk: How to reconstruct the 4-volume V from face bivectors B_{ab}

$\rightarrow$ General formula requires knotting information of graph
(generalisation of Han's simplex construction + proofs)

[Han '11, Dona et al '17, BB, Belov '17, BB, Rabuffo '18, BB, '18]

I Motivation

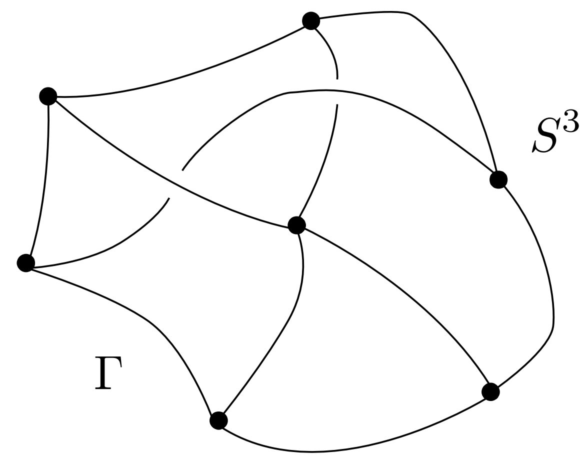
II <u>Volume of a 4d polyhedron</u>

III Quantum amplitude and asymptotics

IV Quadratic volume simplicity constraint

V Summary and outlook

II Volume of a 4d polyhedron

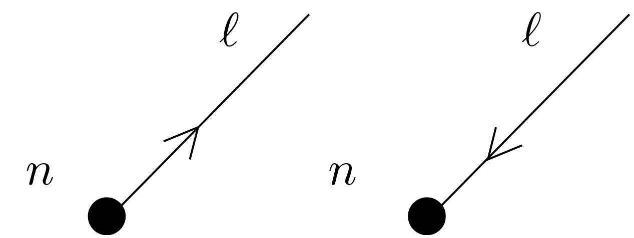


Bivectors $B \in \mathbb{R}^4 \wedge \mathbb{R}^4 \simeq \mathfrak{so}(4)$

Oriented graphs $\Gamma \subset S^3$

nodes n, oriented links ℓ

A <u>bivector geometry</u>:



$[n,\ell] = +1$ $\qquad$ $[n,\ell] = -1$

graph Γ

bivectors to links $\{B_\ell\}_\ell$

diagonal simplicity: $B_\ell \wedge B_\ell = 0$

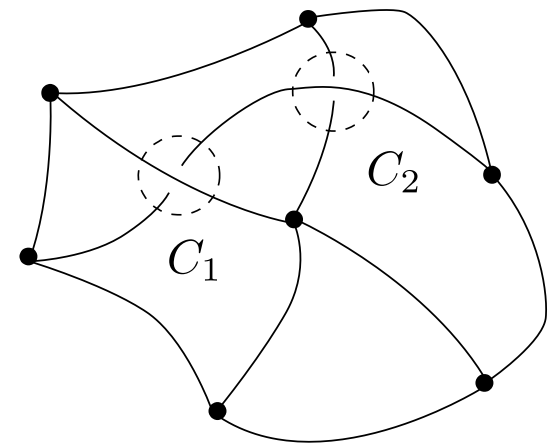
cross-simplicity: $B_\ell \wedge B_{\ell'} = 0$ for links ℓ, ℓ' adjacent to the same node:

closure: $\sum_{\ell \supset n} [n,\ell]\, B_\ell \;=\; 0$ for all nodes n

II Volume of a 4d polyhedron

Projection onto the plane:

2d graph with crossings C

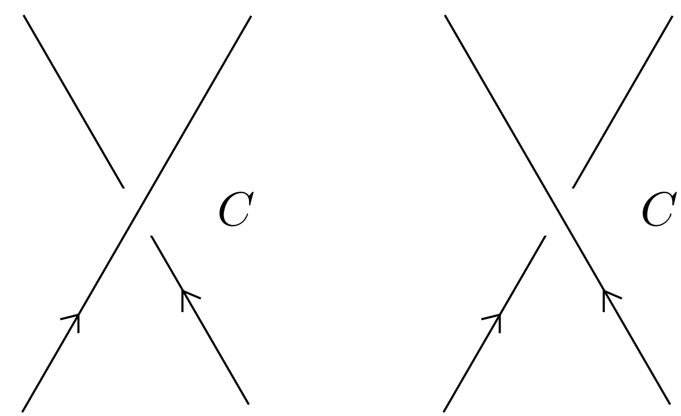


For one crossing C , define:

$$V_C \;:=\; \sigma(C) \,*\, \Big(B_{\ell_1} \wedge B_{\ell_2}\Big)$$

Hodge operator:

$$* \;:\; \wedge^4 \mathbb{R}^4 \longrightarrow \mathbb{R}$$



$\sigma(C) = -1$ $\quad$ $\sigma(C) = +1$

For the whole graph Γ , define the number:

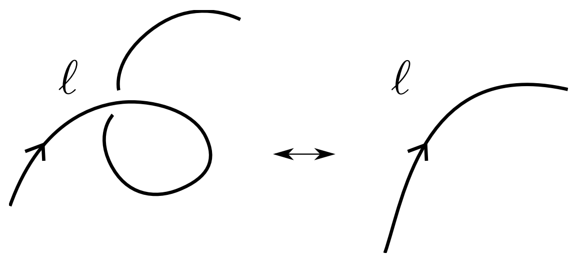
$$V_\Gamma \;:=\; \frac{1}{6}\sum_C V_C$$

II Volume of a 4d polyhedron

Claim: $V_\Gamma \;=\; \frac{1}{6}\sum_C \sigma(C)\, *\,(B_{\ell_1}\wedge B_{\ell_2})$ does not depend on 2d projection

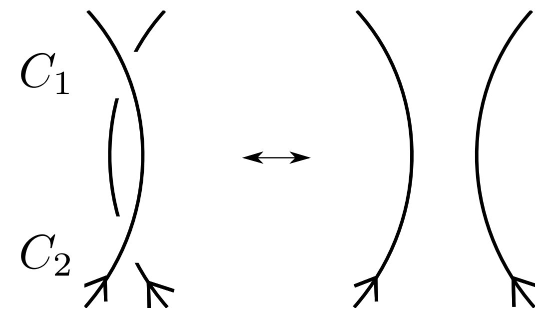
Proof: Reidemeister moves:

1.)



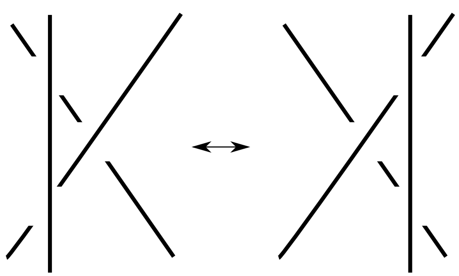
$B_\ell \wedge B_\ell = 0$

2.)



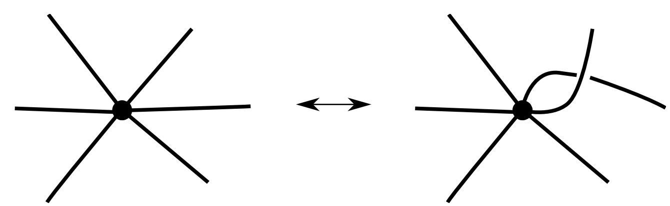
$\sigma(C_1) \;=\; -\sigma(C_2)$

3.



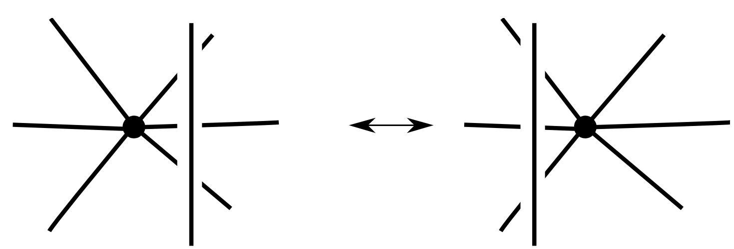
trivial

4.)



$B_\ell \wedge B_{\ell'} = 0$

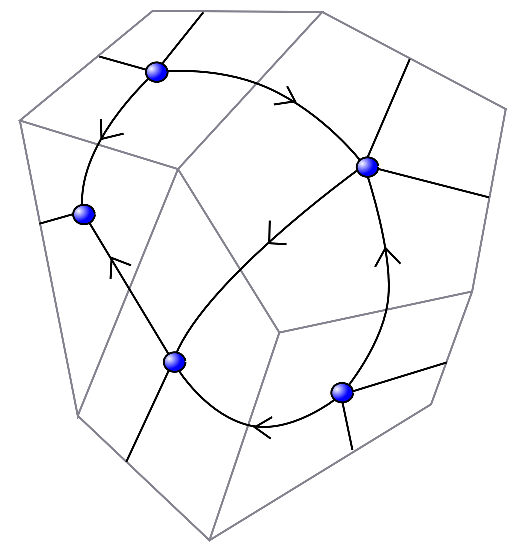
5.)



$$\sum_{\ell \supset n} [n,\ell] B_\ell = 0$$

II Volume of a 4d polyhedron

Every 4-dim convex polytope $P \subset \mathbb{R}^4$ uniquely determines a bivector geometry.



polytope P $\rightarrow$ boundary graph $\Gamma \subset S^3$
(dual to 3d boundary polyhedron)

2d faces f of P $\leftrightarrow$ links ℓ of Γ

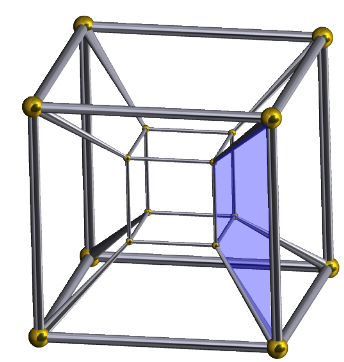
face+orientation $\rightarrow$ bivector

$$B_\ell = N \wedge M \qquad N, M \in \mathbb{R}^4$$

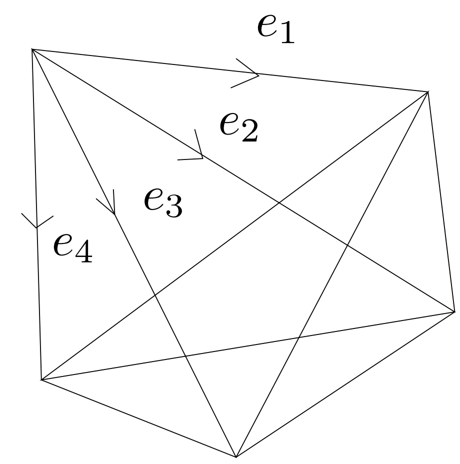
$$\mathrm{Area}(f) = |N|\,|M|\,\sin\sphericalangle(N, M)$$

$\rightarrow$ <u>bivector geometry</u> (diagonal-, cross-simplicity + closure automatically satisfied)

Claim: For a 4d polytope $\mathrm{Vol}(P) = V_\Gamma = \frac{1}{6}\sum_C \sigma(C) * (B_{\ell_1} \wedge B_{\ell_2})$



Sketch for proof:
1.) Show that it is true for 4-simplex
2.) Show how the invariant behaves under cutting of polytopes
3.) Show that every polytope can be cut successively into simplices



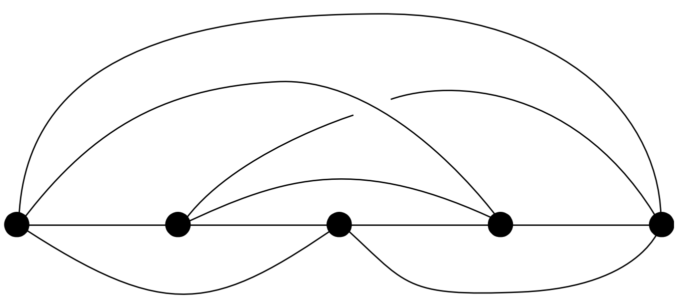
1.) True for a 4-simplex:

By direct calculation

Spanned by four vectors $e_1, e_2, e_3, e_4 \in \mathbb{R}^4$

$$B_1 = \frac{1}{2} e_1 \wedge e_2 \qquad B_2 = \frac{1}{2} e_3 \wedge e_4$$

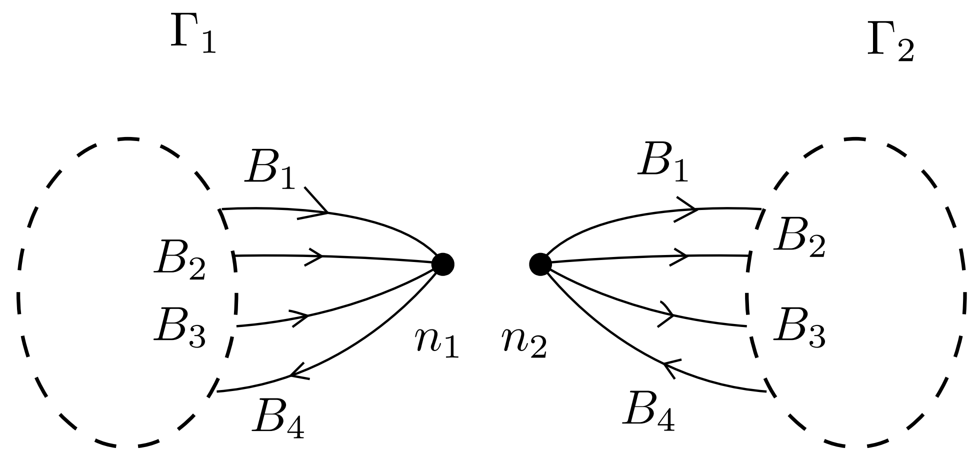
Only one crossing in the boundary graph:



$$V_\Gamma = \frac{1}{24} * (e_1 \wedge e_2 \wedge e_3 \wedge e_4) = V$$

→ Claim proven for 4-simplices

2.) Cutting/glueing of polytopes (graph surgery)

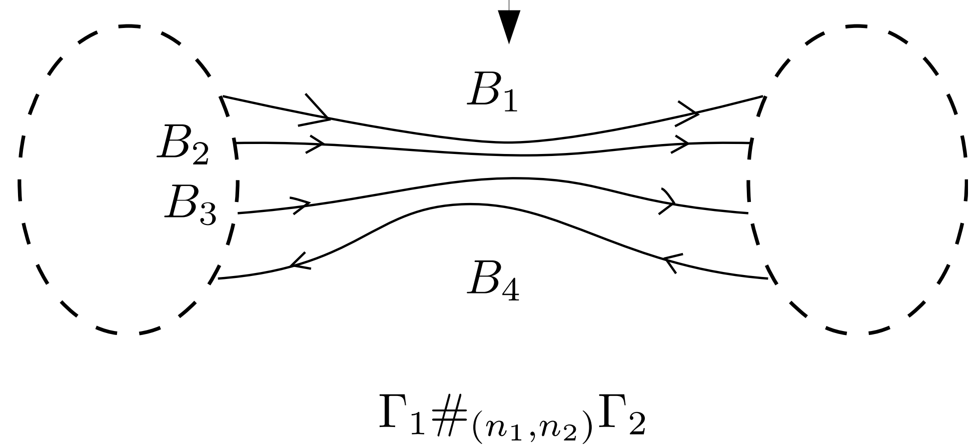


two graphs $\Gamma_1,\ \Gamma_2$

with <u>mirrored nodes</u> n_1, n_2

(identical bivectors, but
reverse orientations)

graphs can be merged together, to one big graph



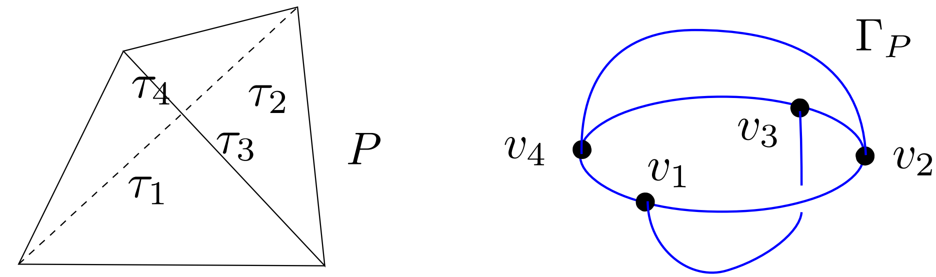
Easy to show:

under this procedure, V_Γ is additive:

$$V_{\Gamma_1\#_{(n_1,n_2)}\Gamma_2} = V_{\Gamma_1} + V_{\Gamma_2}$$

2.) Cutting/glueing of polytopes (graph surgery)

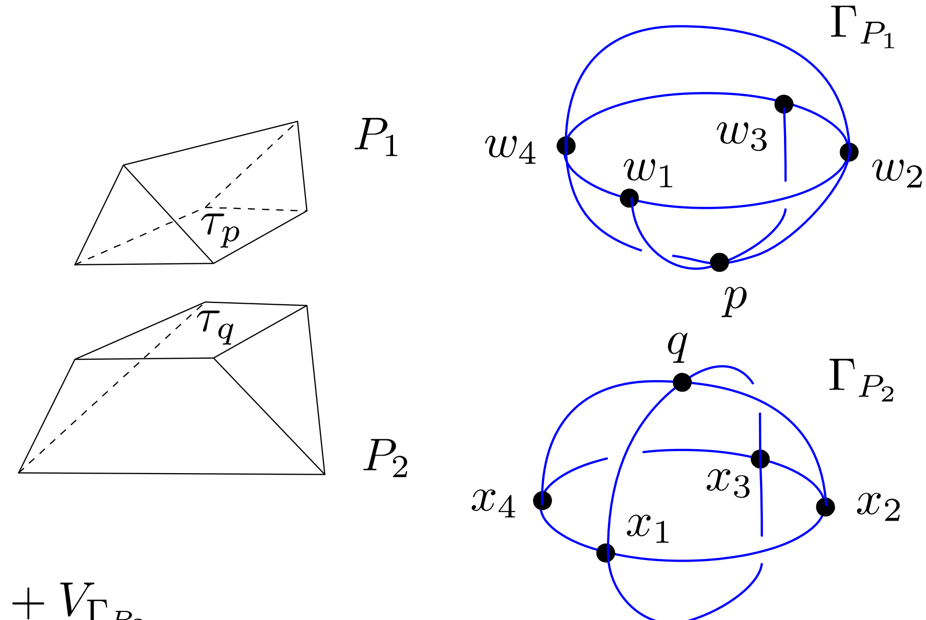
Cutting of one convex polytope
with hyperplane into two polytopes
(3d analogue of image):



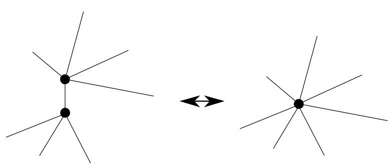
Boundary graph gets split up.

$$\Gamma_{P_1} \#_{(p,q)} \Gamma_{P_2} \; \sim \; \Gamma_P$$

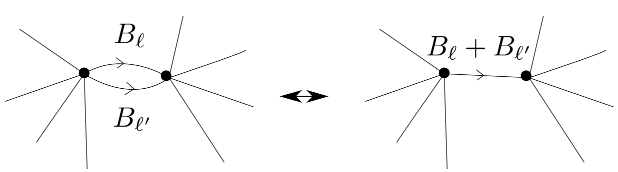
Regain old bdy graph with two moves:
(~ trivial subdivision of bdy polytopes)



1.)



2.)



Moves leave V_Γ invariant $\rightarrow \; V_{\Gamma_P} \; = \; V_{\Gamma_{P_1}} + V_{\Gamma_{P_2}}$

3.) Remains to show:

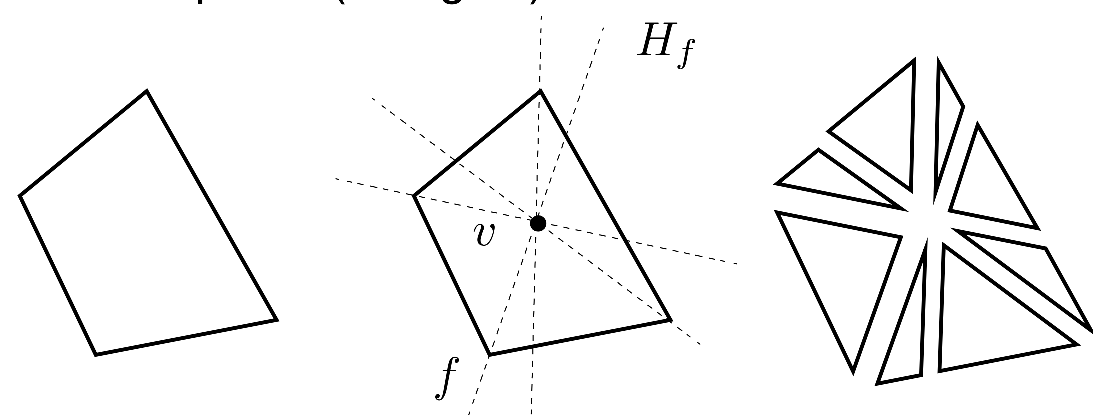
Every convex n-polytope can be successively cut into n-simplices via $(n-1)$ hyperplanes.

Proof: induction over the dimension n :

a) Fix internal vertex v
b) Subdivide polytope by the hyperplane H_f spanned by v and $(n-2)$ face f
c) → Pyramids over $(n-1)$ polytopes → subdivide those

For $n=2$ the process leads to 2-simplices (triangles):

→ done



This finishes the proof:

For every convex 4d polytope P , the 4-volume can be computed by its bivectors:

$$V_P \; = \; \frac{1}{6}\sum_C \sigma(C) * \Big(B_{\ell_1} \wedge B_{\ell_2}\Big)$$

$\rightarrow$ Needs knotting information of the boundary graph!

EPRL-FK-KKL model:

Boundary states: $SU(2)$ -spin network functions $\psi \in \mathcal{H}_{\mathrm{SU}(2)}$

$$\mathcal{H}_{\mathrm{SU}(2)} = \bigoplus_{k_\ell} \left(\bigotimes_n \mathrm{Inv}_{SU(2)}(V_{k_1} \otimes \cdots V_{k_m}) \right)$$

$$\mathcal{H}_{SU(2)\times SU(2)} = \bigoplus_{j_\ell^\pm} \left(\bigotimes_n \mathrm{Inv}_{SU(2)\times SU(2)}(V_{j_1^\pm} \otimes \cdots V_{j_m^\pm}) \right)$$

Boosting map: $\beta : \mathcal{H}_{\mathrm{SU}(2)} \longrightarrow \mathcal{H}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}$
(insert into highest / lowest weight)

$$j_\ell \pm = \frac{1}{2}|1 \pm \gamma| k_\ell$$

$$\mathcal{A}(\psi) := \langle \Psi_0 \,|\, \beta\psi \rangle$$

Amplitude: inner product between
boosted boundary state and BF vacuum state $\quad \Psi_0 \sim \prod_\ell \delta^{m_\ell^\pm}{}_{n_\ell^\pm}$

Isomorphism $\mathbb{R}^4 \wedge \mathbb{R}^4 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2) \quad \rightarrow \hat{B}$ derivative operators on $\mathcal{H}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}$

[Engle, Pereita, Rovelli, Livine ‘07, Freidel, Krasnov ‘07, Kaminski, Kisielowski, Lewandowski ‘09]

III Quantum amplitude and asymptotics

Deformed amplitude:

$$\hat{V} \;:=\; \frac{1}{6}\sum_{C}\sigma(C) * \left(\hat{B}_1 \wedge \hat{B}_2\right) \qquad\qquad B \;\sim\; \left(\vec{b}^{+}, \vec{b}^{-}\right)$$

$$= \frac{1}{6}\sum_{C}\sigma(C)\delta_{IJ}\left(\hat{b}_1^{+,I}\hat{b}_2^{+,J} - \hat{b}_1^{-,I}\hat{b}_2^{-,J}\right)$$

$$= \frac{1}{6}\sum_{C}\sum_{\epsilon=\pm}\frac{\epsilon 4\gamma^2}{(1\,\epsilon\,\gamma)^2}\sum_{I=1}^{3} D_{(j_L^\epsilon)}(X_I^\epsilon) \otimes D_{(j_{L'}^\epsilon)}(X_I^\epsilon)$$

Deformation parameter ω

$$\mathcal{A}^{\omega}(\psi) \;:=\; \left\langle \Psi_0 \right| \exp\left(i\omega\hat{V}\right)\beta\psi \Big\rangle \;=\; \mathcal{A}^{\omega,+}\mathcal{A}^{\omega,-}$$

Deformed amplitude factorizes (Euclidean signature, $\gamma < 1$)

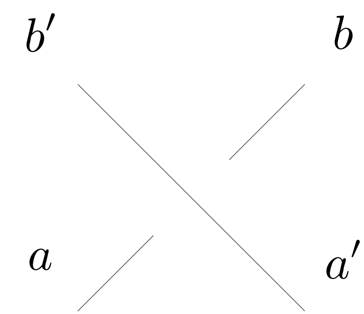
Note: usually, cosmological constant is incorporated via quantum groups (state sum, boundary Hilbert space) → Here we stay with classical groups

[Han ‘11, Haggard et al ‘15, Dittrich, Geiller ‘16, BB, Rabuffo ‘18]

Claim: Large j asymptotics of $\mathcal{A}^{\omega}(\psi)$: same critical & stationary points as the one for normal amplitude $\mathcal{A}(\psi)$, and Hessian matrix is also the same!

Sketch of proof: First we expand the exponential into a sum, then we make the assumption that we can in actuality exchange the sum and the asymptotic limit.

Calculation can be performed for $\pm$ sectors separately



Livine-Speziale coherent states on boundary: $|j,\, \vec{n}\rangle \;=\; g_{\vec{n}}|j,\, j\rangle$

Undeformed amplitude (one sector), for links (ab) :

$$\mathcal{A}(\psi) \;=\; \int_{SU(2)^{N_\Gamma}} dg_a \prod_{b \to a} \langle j_{ab},\, n_{ab}|\; (g_a)^{-1}\, g_b\; |j_{ab},\, n_{ba}\rangle$$

Deformed amplitude contains, for each crossing (e.g. plus-sector):

$$\langle\Psi| \;=\; \langle j_{ab},\, n_{ab}|(g_a)^{-1} \otimes \langle j_{a'b'},\, n_{a'b'}|(g_{a'})^{-1}$$
$$|\Phi\rangle \;=\; g_b\; |j_{ab},\, n_{ba}\rangle \otimes g_{b'}\; |j_{a'b'},\, n_{b'a'}\rangle$$

$$\langle\Psi|\; \exp\left(\frac{4i\omega}{(1+\gamma)^2}\sum_{I=1}^{3} X_I \otimes X_I\right)\; |\Phi\rangle$$

Expansion of the exponential:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{4i\omega}{(1+\gamma)^2} \right)^n \sum_{I_1, I_2, \ldots, I_n = 1}^{3} \langle j_{ab},\, n_{ab}|\, (g_a)^{-1}\, X_{I_1} X_{I_2} \cdots X_{I_n}\, g_b\, |j_{ab},\, n_{ba}\rangle$$
$$\times \langle j_{a'b'},\, n_{a'b'}|\, (g_{a'})^{-1}\, X_{I_1} X_{I_2} \cdots X_{I_n}\, g_{b'}\, |j_{a'b'},\, n_{b'a'}\rangle$$

Insert resolution of identity $n-1$ times: $\quad (2j+1) \int_{S^2} d^2 n\, |j,\, n\rangle\langle j,\, n| \;=\; 1_{V_j}$

using $\quad \langle j,\, n|X_I|j,\, n'\rangle \;=\; j\langle n|\sigma_I|n'\rangle\; \langle n|n'\rangle^{2j-1}$

we get:

$$\langle j_{ab},\, n_{ab}|\, (g_a)^{-1}\, X_{I_1} X_{I_2} \cdots X_{I_n}\, g_b\, |j_{ab},\, n_{ba}\rangle = \int_{(S^2)^{n-1}} d^2 n_i\; a(n_i, g_a, g_b)\; e^{S(n_i, g_a, g_b)}$$

with

$$a(n_i, g_a, g_b) \;=\; (2j+1)^{n-1} j^n \frac{\langle n_{ab}|(g_a)^{-1}\sigma_{I_1}|n_1\rangle}{\langle n_{ab}|(g_a)^{-1}|n_1\rangle} \frac{\langle n_1|\sigma_{I_2}|n_2\rangle}{\langle n_1|n_2\rangle} \cdots \frac{\langle n_{n-1}|\sigma_{I_n} g_b|n_{ba}\rangle}{\langle n_{n-1}|g_b|n_{ba}\rangle}$$

$$S(n_i, g_a, g_b) \;=\; 2j\Big(\ln\langle n_{ab}|(g_a)^{-1}|n_1\rangle + \ln\langle n_1|n_2\rangle + \cdots + \ln\langle n_{n-1}|g_b|n_{ba}\rangle \Big)$$

→ Here we exchange asymptotic limit and infinite sum

Integration variables: $g_a = e^{ix_a^I \sigma_I} g_a^{(c)}$, $n_i(\phi_i, \theta_i)$, $n'_i(\xi_i, \chi_i)$

critical stationary points: $\mathrm{Re}\,(S) \;=\; 0 \qquad \partial S \;=\; 0$

$$g_a n_{ab} \;=\; g_b n_{ba} \qquad\qquad n_i = g_b n_{ba}, \qquad n'_i = g_{b'} n_{b'a'} \text{ for all } i.$$

Hessian matrix: $\tilde{H}^{cd}_{IJ} \;:=\; \dfrac{\partial^2 S}{\partial x_c^I \partial x_d^J}$

same matrix as undeformed case

$$\frac{\partial^2 S}{\partial \phi_i^2} \;=\; \frac{\partial^2 S}{\partial \theta_i^2} \;=\; -j_{ab} \qquad\qquad \frac{\partial^2 S}{\partial \xi_i^2} \;=\; \frac{\partial^2 S}{\partial \chi_i^2} \;=\; -j_{a'b'}$$

$$\frac{\partial^2 S}{\partial \theta_i \partial \theta_{i+1}} \;=\; \frac{\partial^2 S}{\partial \phi_i \partial \phi_{i+1}} \;=\; \frac{j_{ab}}{2}, \qquad \frac{\partial^2 S}{\partial \phi_i \partial \theta_i} \;=\; 0$$

$$\frac{\partial^2 S}{\partial \phi_i \partial \theta_{i+1}} \;=\; i\frac{j_{ab}}{2}, \qquad \frac{\partial^2 S}{\partial \phi_{i+1} \partial \theta_i} \;=\; -i\frac{j_{ab}}{2}$$

$$\frac{\partial^2 S}{\partial x_b^I \partial \phi_1} \;=\; -\frac{\partial^2 S}{\partial x_a^I \partial \phi_1} \;=\; j_{ab}\Big(iV_2^I \;-\; V_1^I\Big)$$

$$\frac{\partial^2 S}{\partial x_b^I \partial \theta_1} \;=\; -\frac{\partial^2 S}{\partial x_a^I \partial \theta_1} \;=\; j_{ab}\Big(iV_1^I + V_2^I\Big)$$

$$G \sigma_J G^{-1} \;=\; V_J^I \sigma_I. \qquad G := (g_b g_{n_{ba}})^{-1}$$

Finally, the total Hessian matrix: $H = \begin{pmatrix} A & B \\ B^T & \tilde{H} \end{pmatrix}$

$A \in \mathbb{C}^{2(n-1)\times 2(n-1)}$

$B \in \mathbb{C}^{4(n-1)\times 3(N-1)}$

$\tilde{H} \in \mathbb{C}^{3(N-1)\times 3(N-1)}$

same matrix as undeformed case

$$\det(H) = \det(A)\det(\tilde{H} - B^T A^{-1} B)$$

$= (j_{ab} j_{a'b'})^{2(n-1)}$

$= 0$

It follows:

undeformed Hessian

$$\det(H) = (j_{ab} j_{a'b'})^{2(n-1)} \det(\tilde{H}).$$

Asymptotics: $j \to \lambda j, \qquad \omega \to \lambda^{-2}\omega$

(for +-sector, one critical stationary point)

$$\mathcal{A}^{\omega} \to \mathcal{A} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{4i\omega\lambda^{-2}}{(1+\gamma)^2} \right)^n \left(\frac{1}{4\pi} \right)^{2(n-1)} \left(\frac{2\pi}{\lambda} \right)^{2(n-1)}$$

$$\times \sum_{I_1, I_2, \ldots, I_n = 1}^{3} 4^{n-1} \frac{(\lambda j_{ab})^{2n-1} (\lambda j_{a'b'})^{2n-1}}{\sqrt{(j_{ab} j_{a'b'})^{2(n-1)}}} \prod_{i=1}^{n} (\tilde{n}_{ba})^{I_i} \, (\tilde{n}_{b'a'})^{I_i}$$

$$= \mathcal{A} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} (j_{ab} j_{a'b'})^n \left(\frac{4i\omega\lambda^{-2}}{(1+\gamma)^2} \right)^n \left(\sum_{I=1}^{3} (\tilde{n}_{ba})^I \, (\tilde{n}_{b'a'})^I \right)^n$$

$$= \mathcal{A} e^{i\omega \vec{X}_{ab} \cdot \vec{Y}_{a'b'}}$$

$$\vec{X}_{ab} = k_{ab} \, \tilde{n}_{ab}, \; \vec{Y}_{a'b'} = k_{a'b'} \, \tilde{n}_{a'b'}$$

undeformed amplitude

[Barrett et al ‘08, Conrady, Freidel ‘08, Han ‘11, BB, Rabuffo ‘18]

Assume that boundary data allows for two distinct solutions $g_a^{(c)} = g_a^{\pm}$

(e.g. “Regge boundary data”, allowing only one 4d-polyhedron of volume V)

→ mixed terms give volume term, same-sign terms cancel

→ For the <u>full</u> amplitude:

$$\mathcal{A} \longrightarrow \frac{1}{W} + \frac{1}{W^*} + \frac{1}{|D|}\cos(S_{\text{Regge}})$$

$$\mathcal{A}^{\omega} \longrightarrow \frac{1}{W} + \frac{1}{W^*} + \frac{1}{|D|}\cos(S_{\text{Regge}} - \omega V)$$

Careful: In general more solutions, which all appear in the asymptotics
(same 3d boundary data can allow for several different 4d polyhedra)

I Motivation

II Volume of a 4d polyhedron

III Quantum amplitude and asymptotics

IV <u>Quadratic volume simplicity constraint</u>

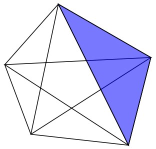
V Summary and outlook

IV Quadratic volume simplicity constraint

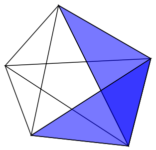
For general 4d-polytopes, the EPRL-FK-KKL model is underconstrained.

For a 4-simplex, there is no problem:

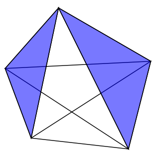
$B_f \wedge B_f \;=\; 0$ diagonal simplicity



$B_f \wedge B_{f'} = 0$ cross simplicity



$B_f \wedge B_{f'} \;=\; V$ volume simplicity



EPRL-FK-KKL-model construction only rests on
diagonal-, cross-simplicity constraint + closure constraint (gauge invariance)

$\rightarrow$ Implemented on boundary Hilbert space

For a 4-simplex, these are enough:

diagonal- + cross-simplicity + closure imply volume-simplicty
(in particular: reconstruction of 4-simplex form bivectors)

But: For general 4d polytopes: volume-simplicity constraint is missing!

[Barrett, Crane '97, Engle et al '07, Freidel, Krasnov '07, BB, Steinhaus '15]

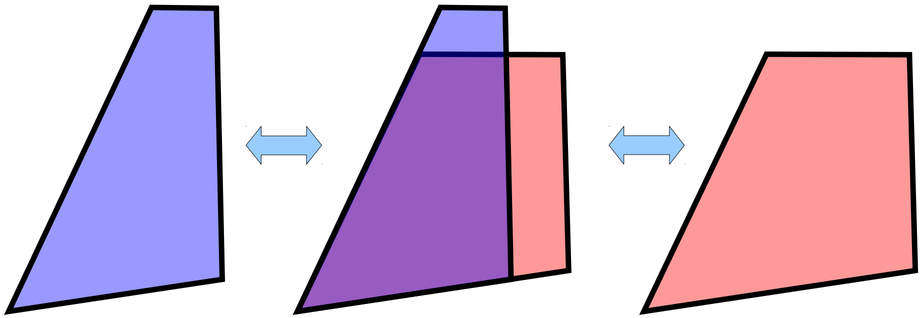
Note: There is an additional under-constrained-ness due to twisted geometries

→ e.g. area-angle constraints, ensure face-matching in the 4-simplex case
→ these "twisted" degrees of freedom are suppressed in the large- j asymptotics

These are not related to the volume-simplicity constraint!

For more general 4d-polytopes, the volume-simplicity problem adds even more non-metric degrees of freedom.

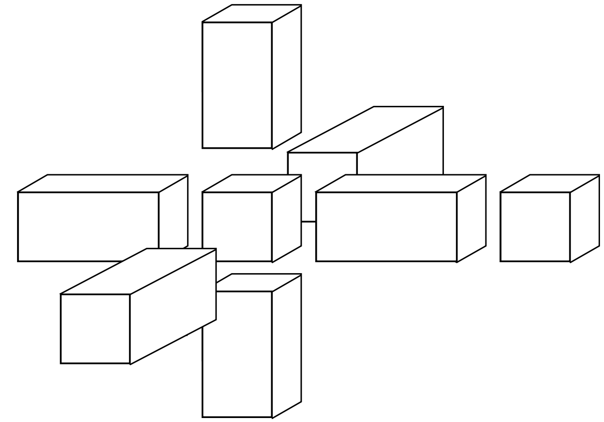
These also manifest in non-face-matching
But: cannot be removed via area-angle-constraints ("conformal d.o.f.")



[Dittrich, Speziale '08, Freidel, Speziale '10, Freidel, Ziprick '13, BB, Steinhaus '15, Dona et al '17, BB, Belov '17]

IV Quadratic volume simplicity constraint

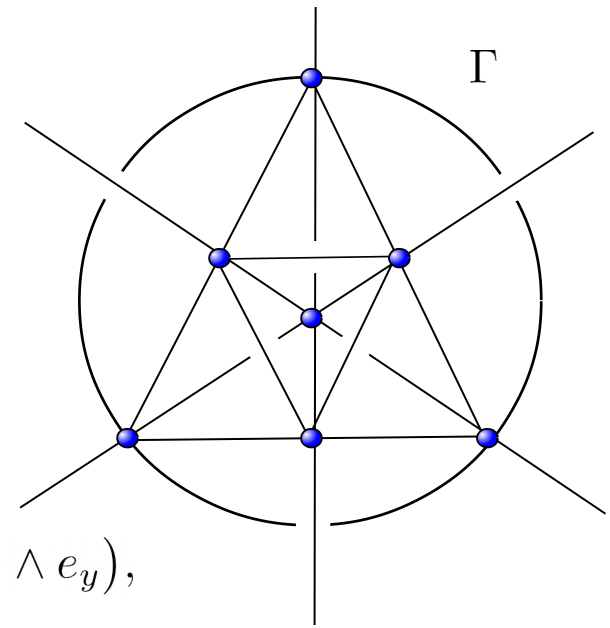
Example: 4d-hypercubic graph: Γ



Consider a certain bivector geometry on that boundary graph

cuboidal: each polyhedron is a 3d cube in $\mathbb{R}^4$. All bivectors

along great circles coincide. Six great circles → six areas



$$a_1, \ldots, a_6$$

However: hypercuboid is only specified by four numbers.

→ There are more bivector geometries than hypercuboids.

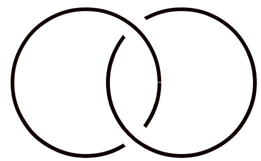
$$B_1 = a_1 (e_y \wedge e_z), \quad B_2 = a_2 (e_z \wedge e_x), \quad B_3 = a_3 (e_x \wedge e_y),$$
$$B_4 = a_4 (e_z \wedge e_t), \quad B_5 = a_5 (e_t \wedge e_y), \quad B_6 = a_6 (e_x \wedge e_t),$$

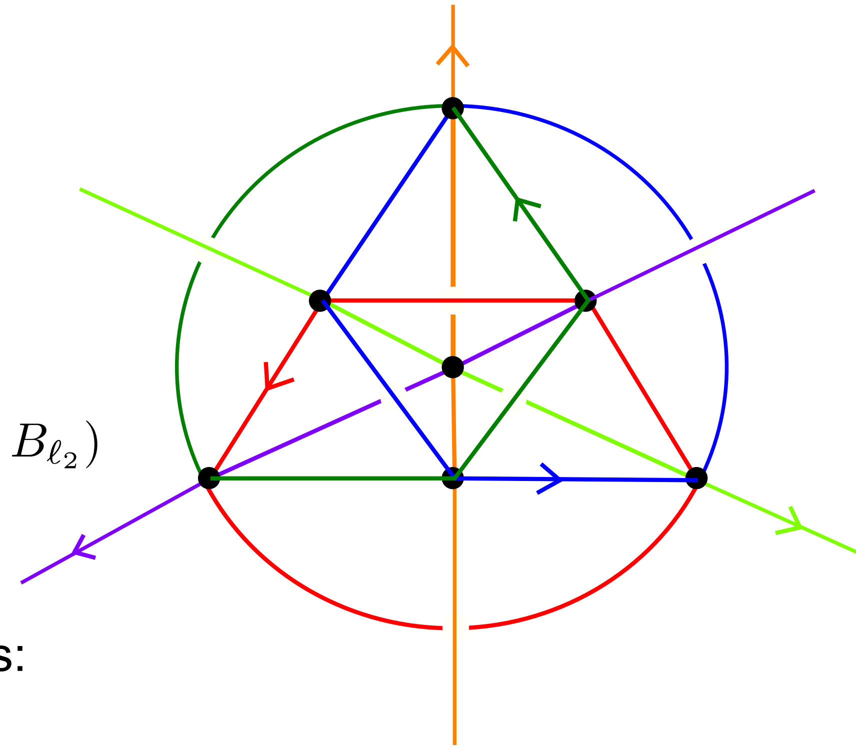
[BB, Steinhaus ‘15]

IV Quadratic volume simplicity constraint

Consider the three Hopf links $H_1,\, H_2,\, H_3$
of the hypercuboidal boundary graph.

Hopf link: Only crossings
with each other! $C \between H$





Define Hopf-link volume:

$$V_C \;=\; \sigma(C) * (B_{\ell_1} \wedge B_{\ell_2})$$

$$V_H \;:=\; \frac{1}{6} \sum_{C \between H} V_C$$

For the hypercuboidal bivector geometry, one gets:

$$V_{H_1} \;=\; \frac{1}{3} a_1 a_6 \qquad V_{H_2} \;=\; \frac{1}{3} a_2 a_5 \qquad V_{H_3} \;=\; \frac{1}{3} a_3 a_4$$

Total volume is the sum of these: $V_{\text{tot}} \;=\; \frac{1}{3}\big(a_1 a_6 + a_2 a_5 + a_3 a_4\big)$

For an actual hypercuboid → all equal → Volume constraint on Hopf links

$$\boxed{V_{H_1} = V_{H_2} = V_{H_3}}$$

[BB, Belov '17]

IV Quadratic volume simplicity constraint

This implies: discretisation of simplicity constraints:

constraint:		discretised on:	
diagonal simplicity	$\longrightarrow$	links	$B_\ell \wedge B_\ell \;=\; 0$
cross simplicity	$\longrightarrow$	nodes	$B_\ell \wedge B_{\ell'} \;=\; 0$
volume simplicity	$\longrightarrow$	Hopf-links	$\displaystyle\sum_{C \between H_1} V_C \;=\; \sum_{C \between H_2} V_C$

The Hopf-link-volume has to agree for each Hopf-link in the boundary graph.

→ General polytopes?

How to impose Hopf link volume simplicity constraint on the quantum level?

a) problem: dynamics vs kinematics:

diagonal-simplicity and cross-simplicity constraints are *kinematical*: can be formulated within one vertex (intertwiner).

volume simplicity constraint is dynamical: need information about 4d shape
$\rightarrow$ need information about (flat) dynamics: amplitude.

$$V_\Psi := \mathcal{A}(\hat{V}\Psi), \qquad V_{H,\Psi} := \mathcal{A}(\hat{V}_H\Psi),$$

b) problem: cosine problem

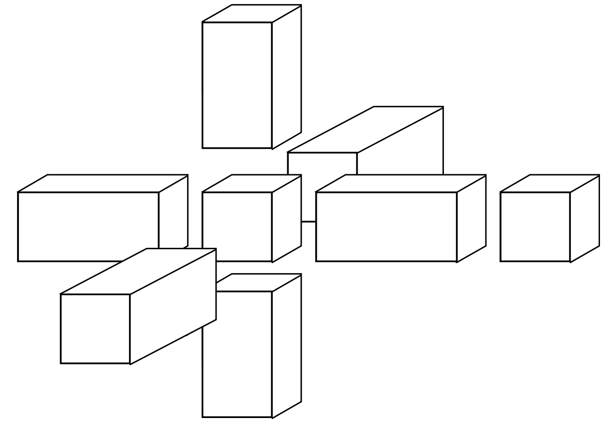
Amplitude $\mathcal{A}$ contains contributions from both orientations of $\mathbb{R}^4$ $\rightarrow$ volume counts positive and negative. $\rightarrow$ all $V_{H,\Psi} = \mathcal{A}(\hat{V}_H\Psi)$ are zero in asymptotic limit.
$\rightarrow$ possible solution: proper vertex?

$\rightarrow$ our solution: use even function of volume $\quad V^2_{H,\Psi} := \mathcal{A}(\hat{V}^2_H\Psi),$

[Barrett et al ‘08, Engle ‘13, Engle, Zipfel ‘15]

IV Quadratic volume simplicity constraint

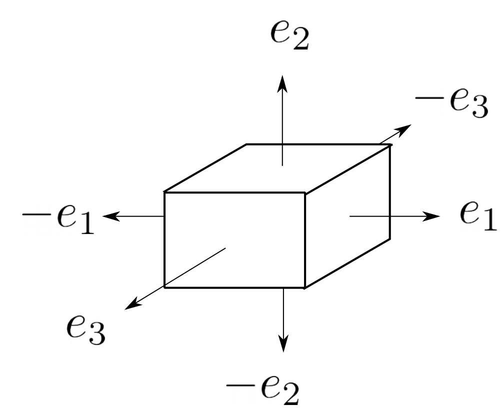
Boundary state: can be chosen to represent
quantum version of hypercuboid $\gamma \in (0,1)$



satisfies (quantum versions of) linear simplicity constraint

$$\psi^{\pm}(h_e^{\pm}) = \langle \otimes_e D_{j_e^{\pm}}(h_e^{\pm}),\ \otimes_v \iota_v^{\pm} \rangle \qquad j^{\pm} = \frac{1 \pm \gamma}{2} j$$

Livine-Speziale Intertwiners:



spins $\leftrightarrow$ areas $\quad a \sim \gamma j$

intertwiner $\leftrightarrow$ 3d shapes

$$\iota^{\pm}_{j_{a_1} j_{a_2} j_{a_3}} = \int_{SU(2)} dg\ g \triangleright \left[\bigotimes_{i=1}^{3} |j^{\pm}_{a_i}, e_i\rangle \langle j^{\pm}_{a_i}, e_i| \right]$$

depends on six spins: $\quad j_1 \ \ldots, j_6$

[Livine, Speziale ‘07, Bianchi ‘08, Doná, Speziale ‘11, BB, Steinhaus ‘15]

IV Quadratic volume simplicity constraint

Large j asymptotics: hypercuboid

$$\mathcal{A}(\Psi) \sim \left(\frac{1}{D} + \frac{2}{|D|} + \frac{1}{D^*}\right),$$

Here D is a polynomical in the $j_1 \ldots, j_6$ of degree 21

Large j asymptotics: Hopf link volumes

$$\mathcal{A}(\hat{V}_{H_1}^2 \Psi) \sim \gamma^4 (j_1 j_6)^2 \left(\frac{1}{D} + \frac{1}{D^*}\right),$$

$$\mathcal{A}(\hat{V}_{H_2}^2 \Psi) \sim \gamma^4 (j_2 j_5)^2 \left(\frac{1}{D} + \frac{1}{D^*}\right),$$

$$\mathcal{A}(\hat{V}_{H_3}^2 \Psi) \sim \gamma^4 (j_3 j_4)^2 \left(\frac{1}{D} + \frac{1}{D^*}\right),$$

→ Demand that Hopf link volumes agree: linear condition → subspace of all spin network functions satisfying linear simplicity.

$$j_1 j_6 = j_2 j_5 = j_3 j_4$$

In the large j limit, hypercuboids with geometricity appear to satisfy this constraint → eliminated non-geometric degrees of freedom!

[BB, Steinhaus ‘15]

I Motivation

II Volume of a 4d polyhedron

III Quantum amplitude and asymptotics

IV Quadratic volume simplicity constraint

V Summary and outlook

Summary:

* Proof of a formula for volume of 4-polytope in terms of its bivectors and crossings in its boundary graph

* Can be used to define a deformation of the EPRL-FK-KKL-amplitude with cosmological constant term
  Asymptotics: → weird terms & Hessian matrix unchanged!

* EPRL-KF-KKL-model underconstrained: no (quadratic) volume simplicity

  → Constraint can be discretized over Hopf links in bdy graph

  works in examples

Outlook:

- So far volume formula only for convex polytopes
  → proof easy to generalise to non-convex case
  → Non-convex polytopes appear in asymptotics of EPRL-FK-KKL model!
  (linear volume-simplicity constraint?)

- Connection to Haggard et al: Chern-Simons theory?

- Deformed amplitude: sensitive to graph knotting → physical IP!

- Connection to quantum groups?

- Hopf-link-volume simplicity constraint: general polytopes?

[BB ‘11, Haggard, Han, Kaminski, Riello ‘15, Dittrich, Geiller ‘16, Dona et al ‘17]