Entanglement and the Bekenstein-Hawking entropy

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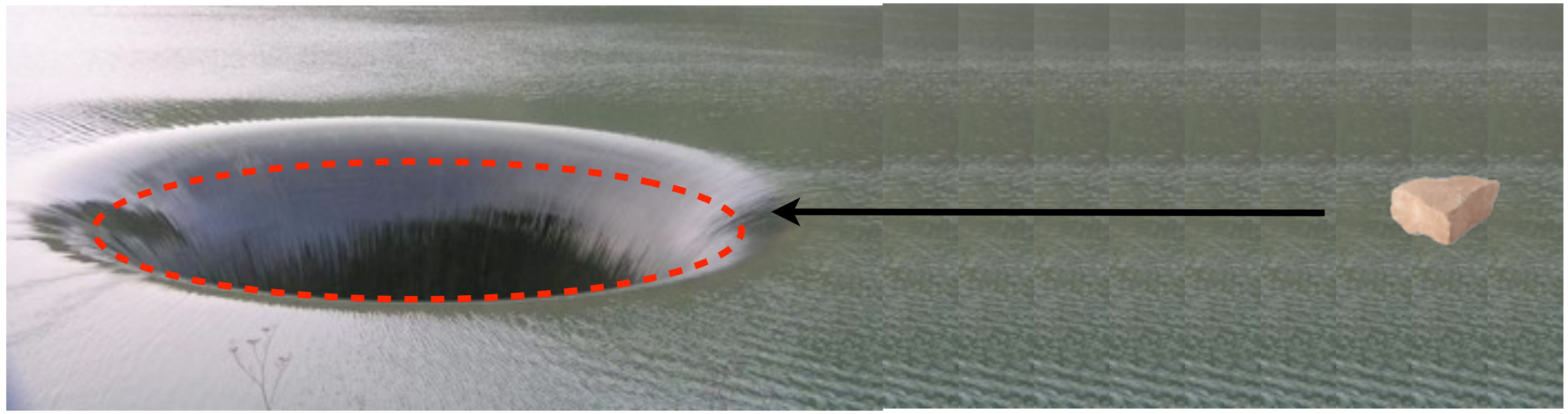
International Loop Quantum Gravity Seminar

Process:



matter falling in	M	A	δE
area increases	$M + \delta E$	$A + \delta A$	

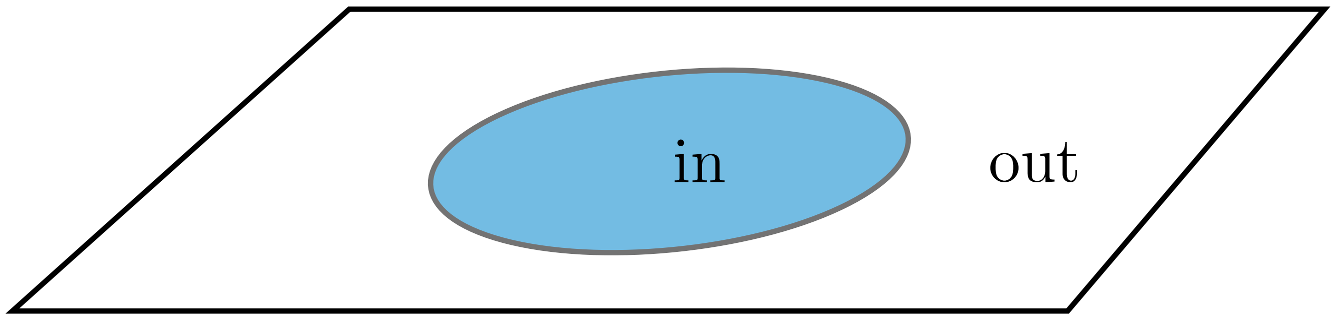
Entropy change δS_{BH}

$$\delta S_{BH} = \frac{\kappa\, c^3}{\hbar} \frac{\delta A}{4\, G}$$

Puzzle: the entropy of what?

Vacuum entanglement in QFT

Sorkin 1983



Vacuum state $|0\rangle$

on Minkowski space

- Reduced density matrix $\rho_0 = \mathrm{Tr}_{\mathrm{in}} |0\rangle\langle 0|$
- Entropy due to correlations at space-like separations

$$S_0 = -\mathrm{Tr}_{\mathrm{out}}(\rho_0 \log \rho_0) \quad = \ +\infty$$

- Divergent: introduce UV cut-off

$$S_0 \ = \ c_0\, A_0\, \Lambda^2 \ + \ c_1 \log \Lambda \ + \ c_2$$

Related to BH entropy? $\Lambda \sim \dfrac{1}{\sqrt{G}} \quad \Rightarrow \quad S_0 \sim c_0 \dfrac{A_0}{G}$

Problems:

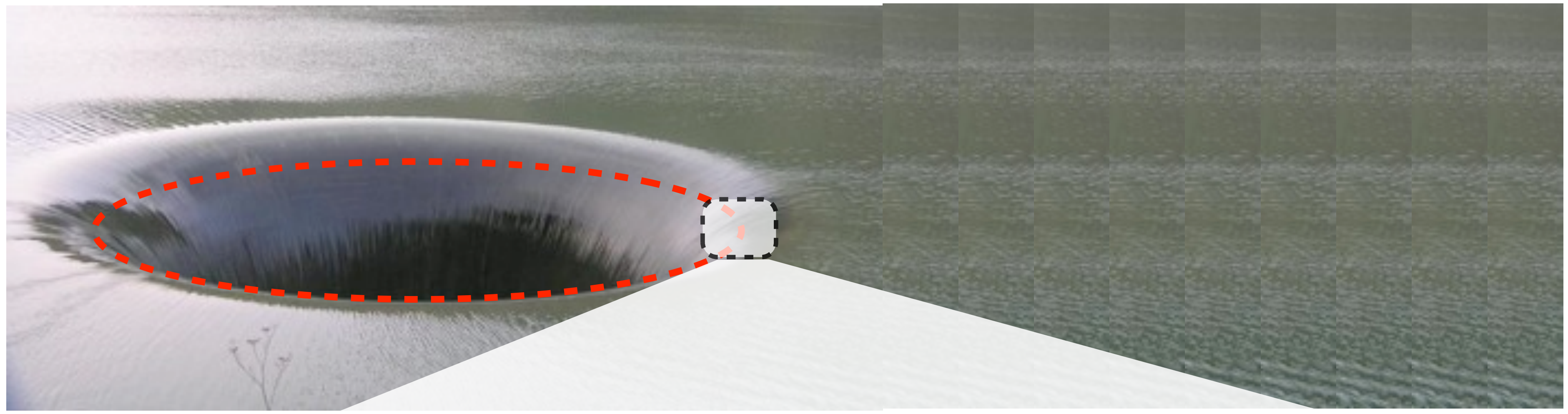
- $1/4$
- species

New result: arXiv:1211.0522

in low-energy processes

δS_{ent} = finite and universal = $\dfrac{\delta A}{4G}$

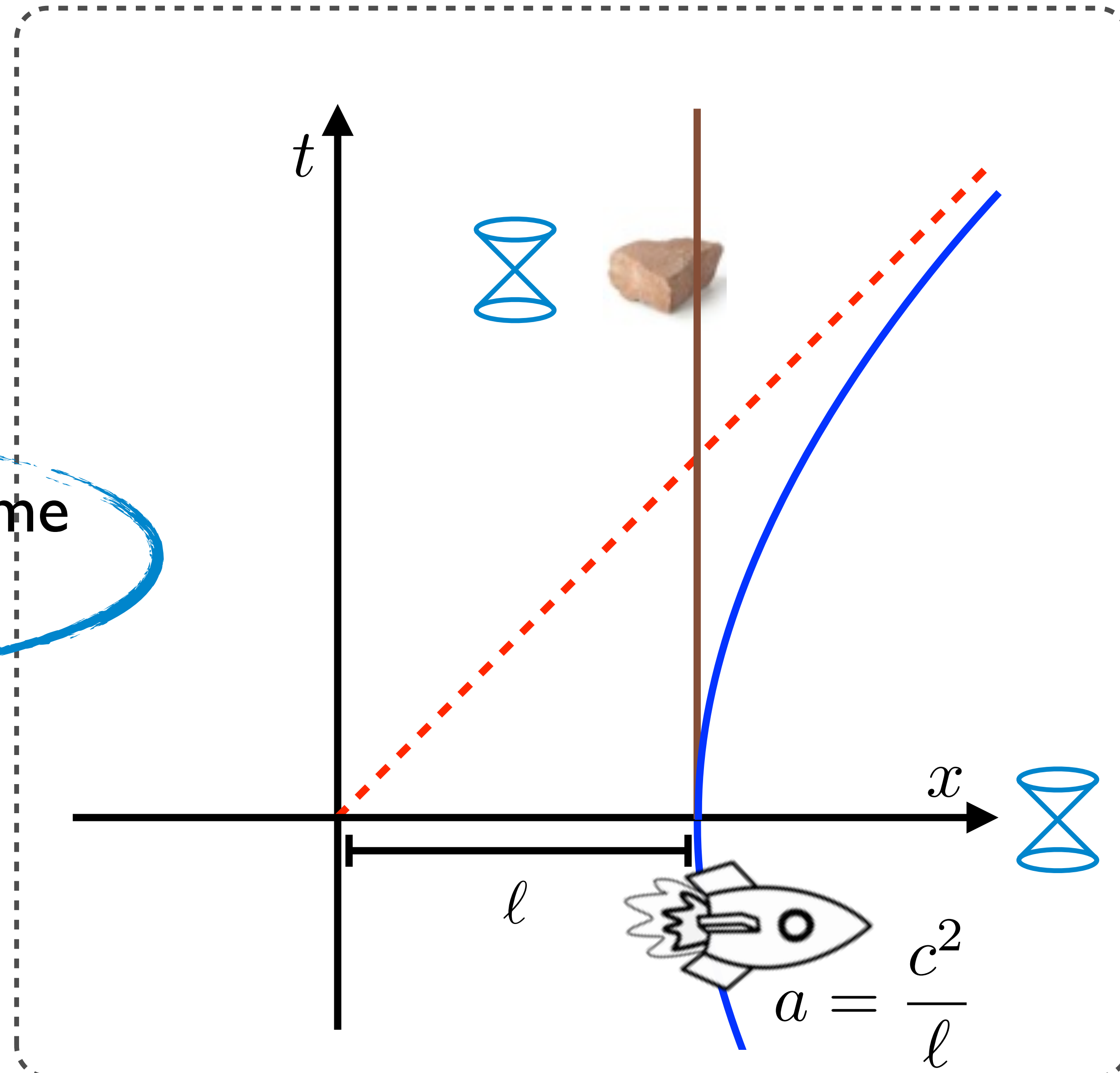
Causal horizon



Horizon radius:

$$R_H = \frac{2GM}{c^2}$$

Minkowski space-time
at scales $d \ll R_H$



Curvature at the horizon:

$$R^{\mu}{}_{\nu\rho\sigma} \sim \frac{1}{R_H^2}$$

Entanglement and the Bekenstein-Hawking entropy: arXiv:1211.0522 [gr-qc]

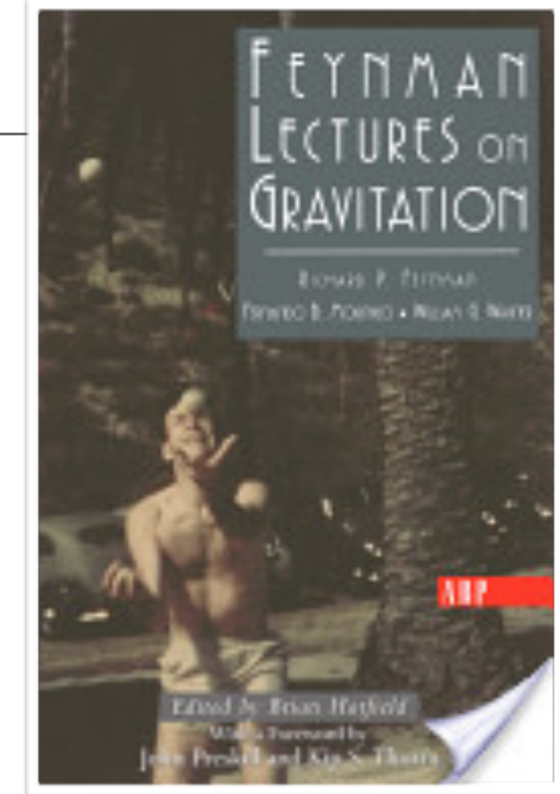
I. Perturbative quantum gravity

II. The perturbed Rindler horizon

III. Entanglement and thermality

IV. Area law for entanglement perturbations

I. Perturbative quantum gravity



- Minkowski space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}$$

- Graviton field $h_{\mu\nu}$

- Action: (Einstein-Hilbert + matter; perturbative)

$$I \;=\; I_{\rm grav}[h_{\mu\nu}] \;+\; I_{\rm matt}[\phi] \;+\; \sqrt{8\pi G} \int d^4x\, h_{\mu\nu} T^{\mu\nu}$$

E.O.M.

$$(*) \qquad \Box h_{\mu\nu} \;=\; -\sqrt{8\pi G}\,\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{\rho}_{\rho}\right)$$

(Harmonic gauge $\partial^{\mu} h_{\mu\nu} = \frac{1}{2}\partial_{\nu} h^{\mu}{}_{\mu}$)

Energy-momentum tensor

$$T_{\mu\nu} \;\sim\; \underbrace{(\partial h)^2}_{\text{gravitons}} \;+\; \underbrace{(\partial\phi)^2}_{\text{matter}} \;+ \sqrt{8\pi G}\,(\ldots)$$

Entanglement and gravity

I. Perturbative quantum gravity

$$(*) \qquad \Box h_{\mu\nu} = -\sqrt{8\pi G}\,(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\rho}_{\rho})$$

II. The perturbed Rindler horizon

III. Entanglement and thermality of the vacuum

IV. Area law for entanglement perturbations

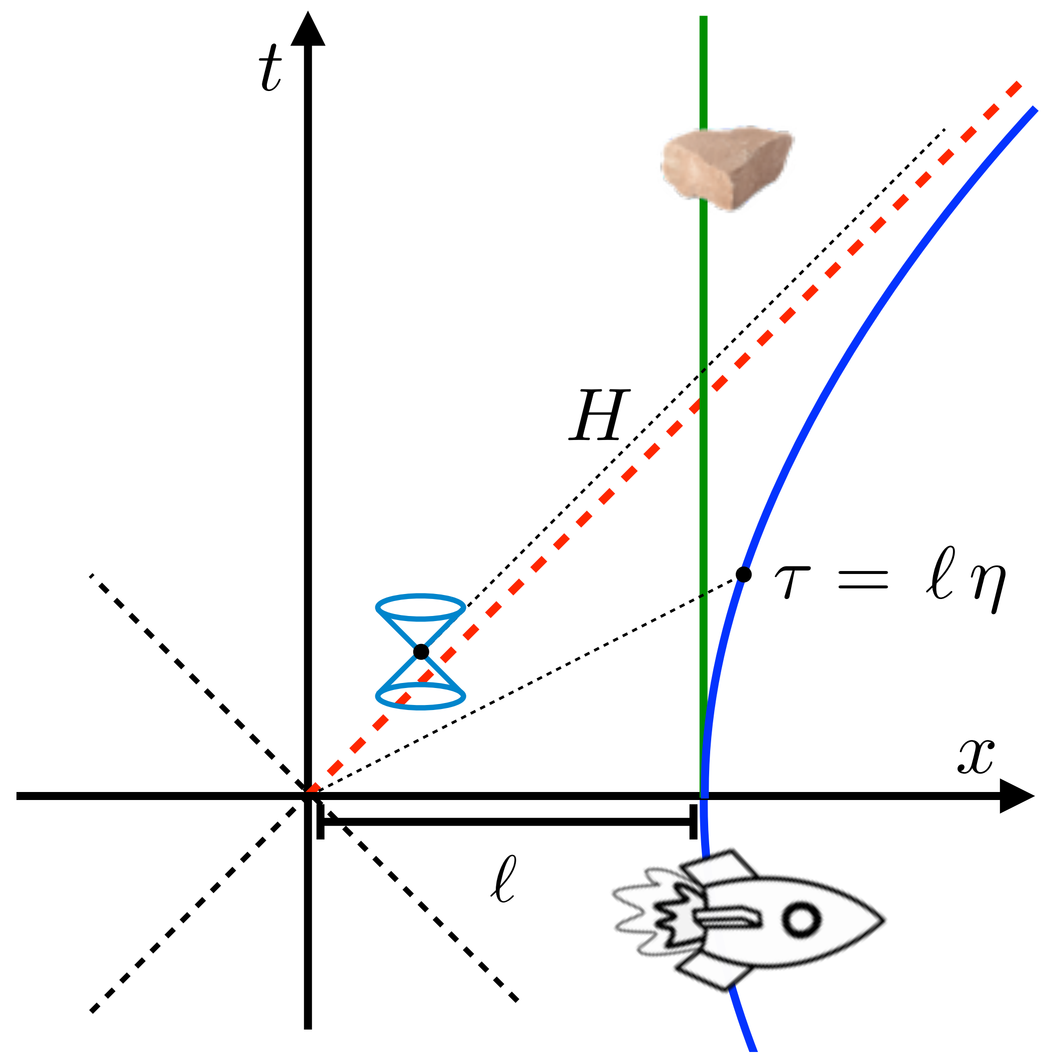
II. The Rindler horizon

Minkowski space-time

Cartesian coord. $x^\mu = (t, x, y, z)$

Rindler coord. (η, ρ, y, z)

$$\begin{cases} t = \rho \sinh \eta \\ x = \rho \cosh \eta \end{cases}$$



Metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= -\rho^2\, d\eta^2 + d\rho^2 + dy^2 + dz^2$$

Accelerated observer at $\rho = \ell$: proper time $\tau = \ell\,\eta$, acceleration $a = \dfrac{1}{\ell}$

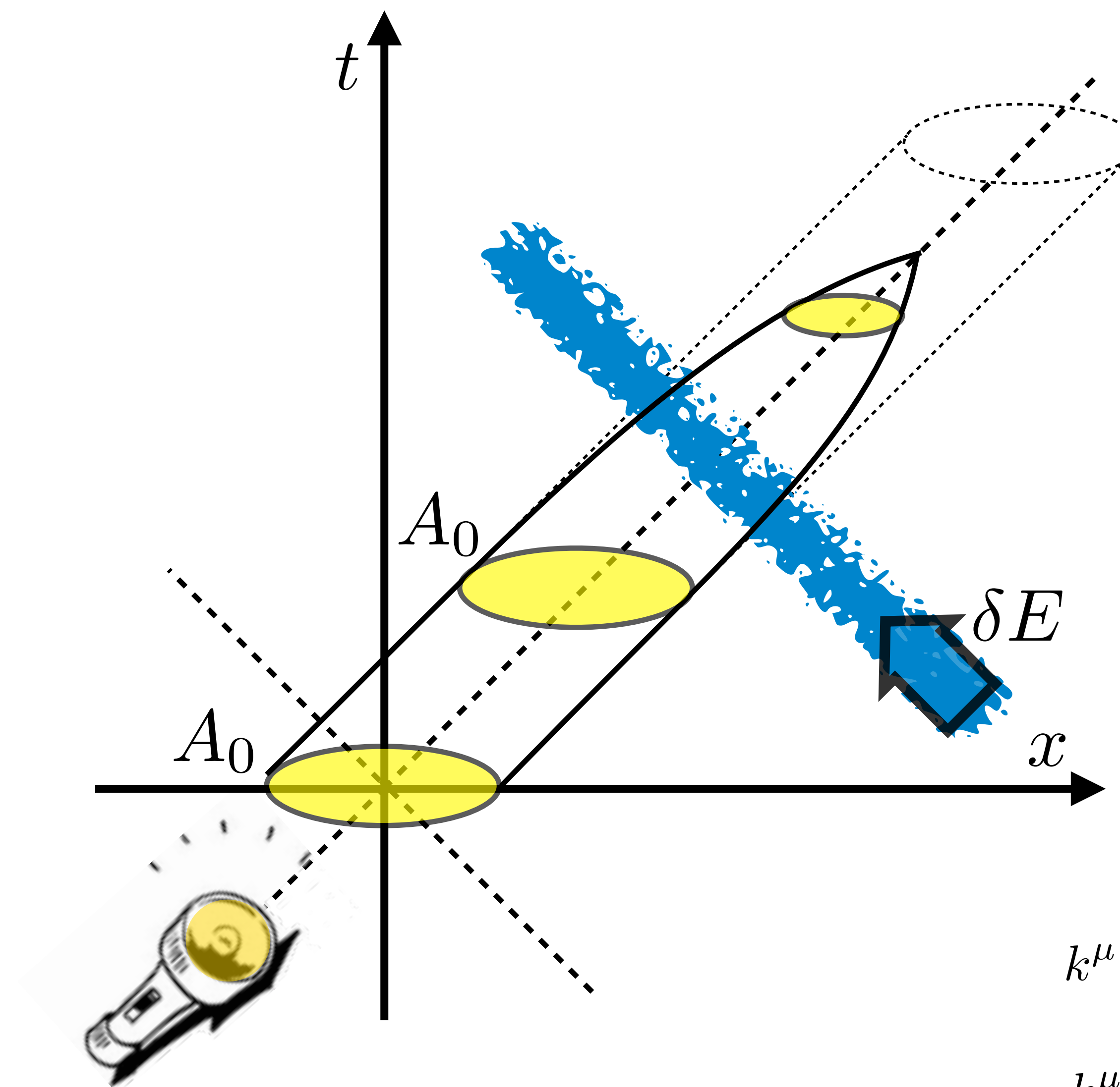
Rindler Horizon H: light-like surface (v, y, z) with $v = t + x$

II. The perturbed Rindler horizon

- Collimated beam of light rays

- Energy crossing the beam

Gravity is attractive: it focuses light rays



$$k^\mu = \frac{\partial x^\mu}{\partial v}$$

$$k^\mu \nabla_\mu k^\nu = 0$$

Perturbatively:

interaction $h_{\mu\nu}$ with light rays

In GR:

curvature ⇨ geodesic deviation

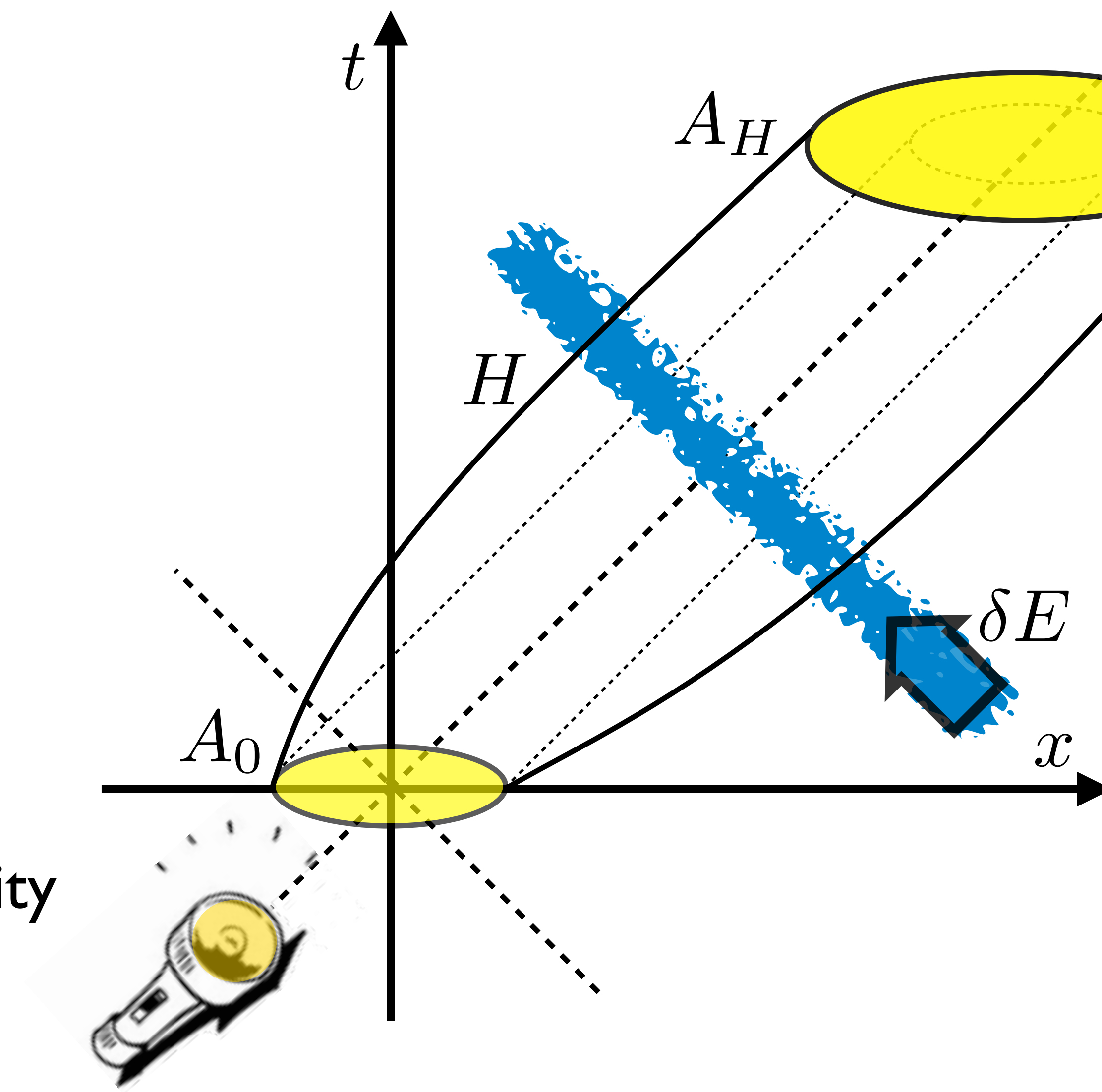
expansion, Raychaudhuri eq.

II. The perturbed Rindler horizon

Event horizon:

Light rays finely balanced between → out of sight

→ reach infinity



Gravity is attractive: it focuses light rays

⇨ diverging beam so that collimated at infinity

Area increase:

$$A_H = A_0 + \sqrt{8\pi G} \int_H (-\Box h_{\mu\nu} k^\mu k^\nu)\, v dv\, dy dz$$

Entanglement and gravity

I. Perturbative quantum gravity

(*) $$\Box h_{\mu\nu} = -\sqrt{8\pi G}\,(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\rho}_{\rho})$$

II. The perturbed Rindler horizon

(**) $$A_H = A_0 + \sqrt{8\pi G}\int_H (-\Box h_{\mu\nu}k^{\mu}k^{\nu})\,vdv\,dydz$$

III. Entanglement and thermality of the vacuum

IV. Area law for entanglement perturbations

II. Rindler energy

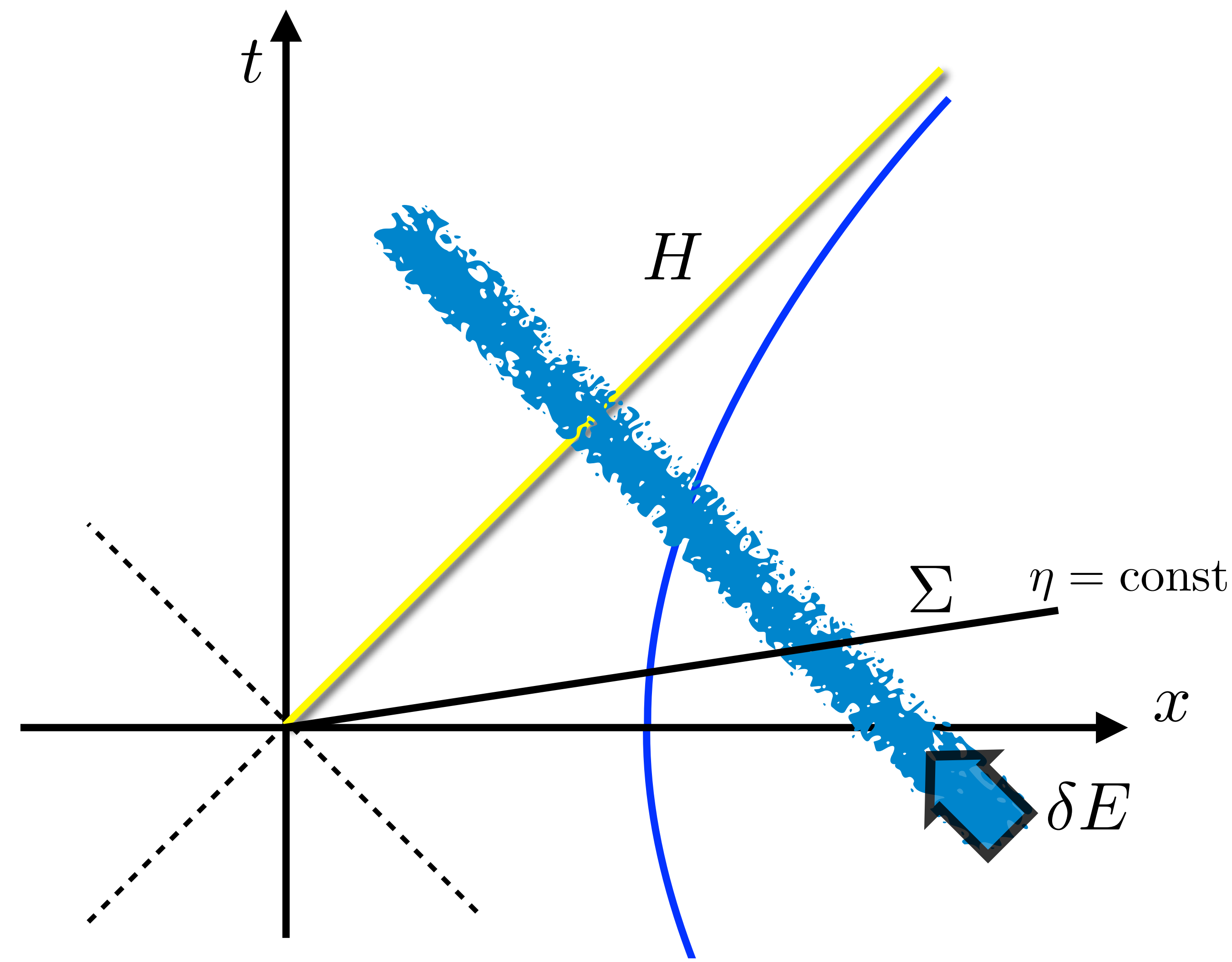
Rindler energy: χ^μ = boost Killing vector

$$K = \int T_{\mu\nu}\chi^\mu d\Sigma^\nu$$

$$= \int_H T_{\mu\nu}\, v\, k^\mu\, dH^\nu$$

$$= \int_H T_{\mu\nu}\, k^\mu k^\nu\, v dv\, dy dz$$

(**)



Energy flux through the horizon, $u^\mu = a\chi^\mu$

$$\delta E = \int_H T_{\mu\nu} u^\mu dH^\nu$$

$$= a\, K$$

$$\chi^\mu = \frac{\partial x^\mu}{\partial \eta} \qquad d\Sigma^\mu = \chi^\mu \rho^{-1} d\rho dy dz$$

$$k^\mu = \frac{\partial x^\mu}{\partial v} \qquad dH^\mu = k^\mu dv dy dz$$

$$\chi^\mu \overset{H}{=} v k^\mu \qquad d\Sigma^\mu \overset{H}{=} dH^\mu$$

Entanglement and gravity

I. Perturbative quantum gravity

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II. The perturbed Rindler horizon

(**) $$A_H = A_0 + \sqrt{8\pi G}\int_H (-\Box h_{\mu\nu}k^{\mu}k^{\nu})\, v dv\, dydz$$

(***) $$K = \int_H T_{\mu\nu}\, k^{\mu}k^{\nu}\, v dv\, dydz$$

III. Entanglement and thermality of the vacuum

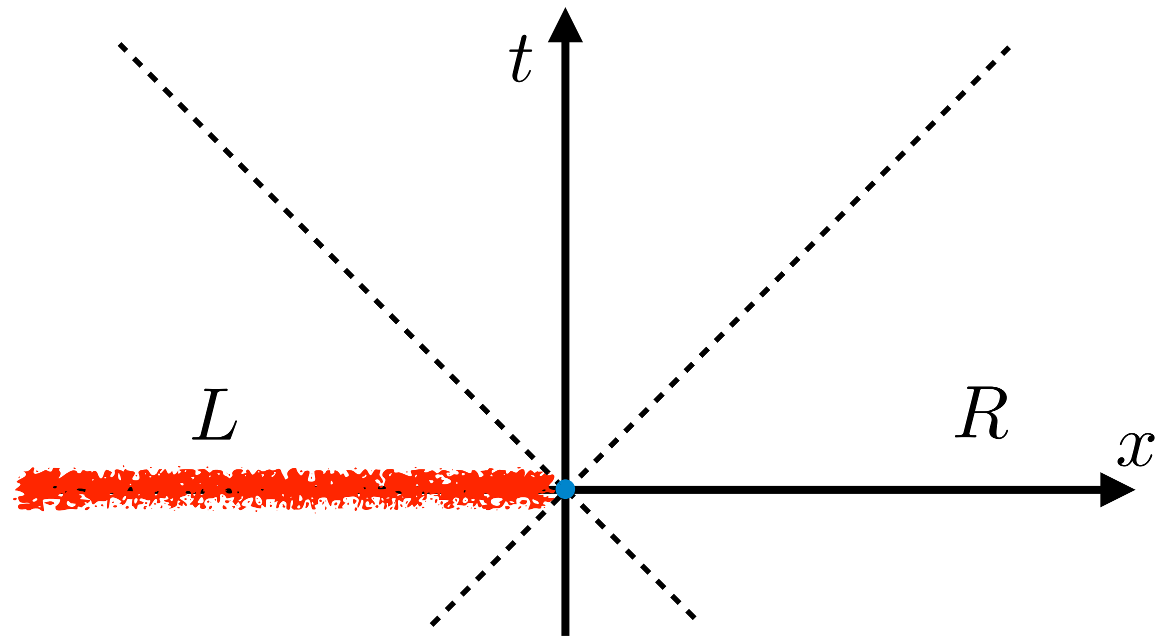
IV. Area law for entanglement perturbations

III. Entanglement and thermality of the vacuum

Free field, Minkowski vacuum

$$|0\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\pi \epsilon_n} \, |\bar{n}_L\rangle \otimes |n_R\rangle$$

$$K|n\rangle = \epsilon_n |n\rangle$$



Minkowski vacuum, interacting QFT, $\varphi_L = (h^L_{\mu\nu}, \phi_L)$

$$|\Omega\rangle = \frac{1}{\sqrt{Z}} \int D\varphi_L \, D\varphi_R \left(\langle \varphi_L | e^{-\pi K} | \varphi_R \rangle \right) |\varphi_L\rangle \otimes |\varphi_R\rangle$$

Reduced density matrix: Gibbs state at temperature $T_{\text{geom}} = \frac{1}{2\pi}$

$$\rho_0 = \mathrm{Tr}_L |\Omega\rangle\langle\Omega|$$

$$= \int D\varphi_L \, |\langle \varphi_L | \Omega \rangle|^2 = \frac{e^{-2\pi K}}{Z}$$

Entanglement and gravity

I. Perturbative quantum gravity

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III. Entanglement and thermality of the vacuum

(****) $$\rho_0 = \frac{e^{-2\pi K}}{Z}$$

IV. Area law for entanglement perturbations

Entanglement entropy of the vacuum

$$S_{\mathrm{ent}}(|\Omega\rangle) = -\mathrm{Tr}(\rho_0 \log \rho_0)$$

If you can't change anything,
you are not doing thermodynamics

IV. Low-energy perturbation of the vacuum

Excited state $|\mathcal{E}\rangle$, $\qquad \rho_1 = \mathrm{Tr}_L |\mathcal{E}\rangle\langle\mathcal{E}| \qquad S_{\rm ent}(|\mathcal{E}\rangle) = -\mathrm{Tr}(\rho_1 \log \rho_1)$

Entanglement entropy perturbation $\qquad \delta S_{\rm ent} = S_{\rm ent}(|\mathcal{E}\rangle) - S_{\rm ent}(|\Omega\rangle)$

$\|\delta\rho\| \ll 1$, $\qquad \delta\rho = \rho_1 - \rho_0$

$$\delta S_{\rm ent} = -\delta \mathrm{Tr}(\rho \log \rho) = -\mathrm{Tr}(\delta\rho \, \log \rho_0) - \cancel{\mathrm{Tr}(\rho_0 \frac{1}{\rho_0} \delta\rho)}$$

$\mathrm{Tr}(\delta\rho) = 0$

$$\overset{(***)}{=} 2\pi \mathrm{Tr}(K\delta\rho) + \log Z \, \cancel{\mathrm{Tr}(\delta\rho)}$$

$$\overset{(***)}{=} 2\pi \, \mathrm{Tr}\Big(\int_H T_{\mu\nu} \, k^\mu k^\nu \, v dv \, dy dy \; \delta\rho\Big)$$

$$\overset{(*)}{=} 2\pi \, \frac{1}{\sqrt{8\pi G}} \mathrm{Tr}\Big(\int_H -\Box h_{\mu\nu} \, k^\mu k^\nu \, v dv \, dy dy \; \delta\rho\Big)$$

$$\overset{(**)}{=} 2\pi \, \frac{1}{8\pi G} \mathrm{Tr}\big(A_H \; \delta\rho\big) = \frac{\delta A}{4G}$$

Entanglement and gravity

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II. The perturbed Rindler horizon

$$(**) \qquad A_H = A_0 + \sqrt{8\pi G}\int_H (-\Box h_{\mu\nu}k^\mu k^\nu)\, v dv\, dy dz$$

$$(*{*}*) \qquad K = \int_H T_{\mu\nu}\, k^\mu k^\nu\, v dv\, dy dz$$

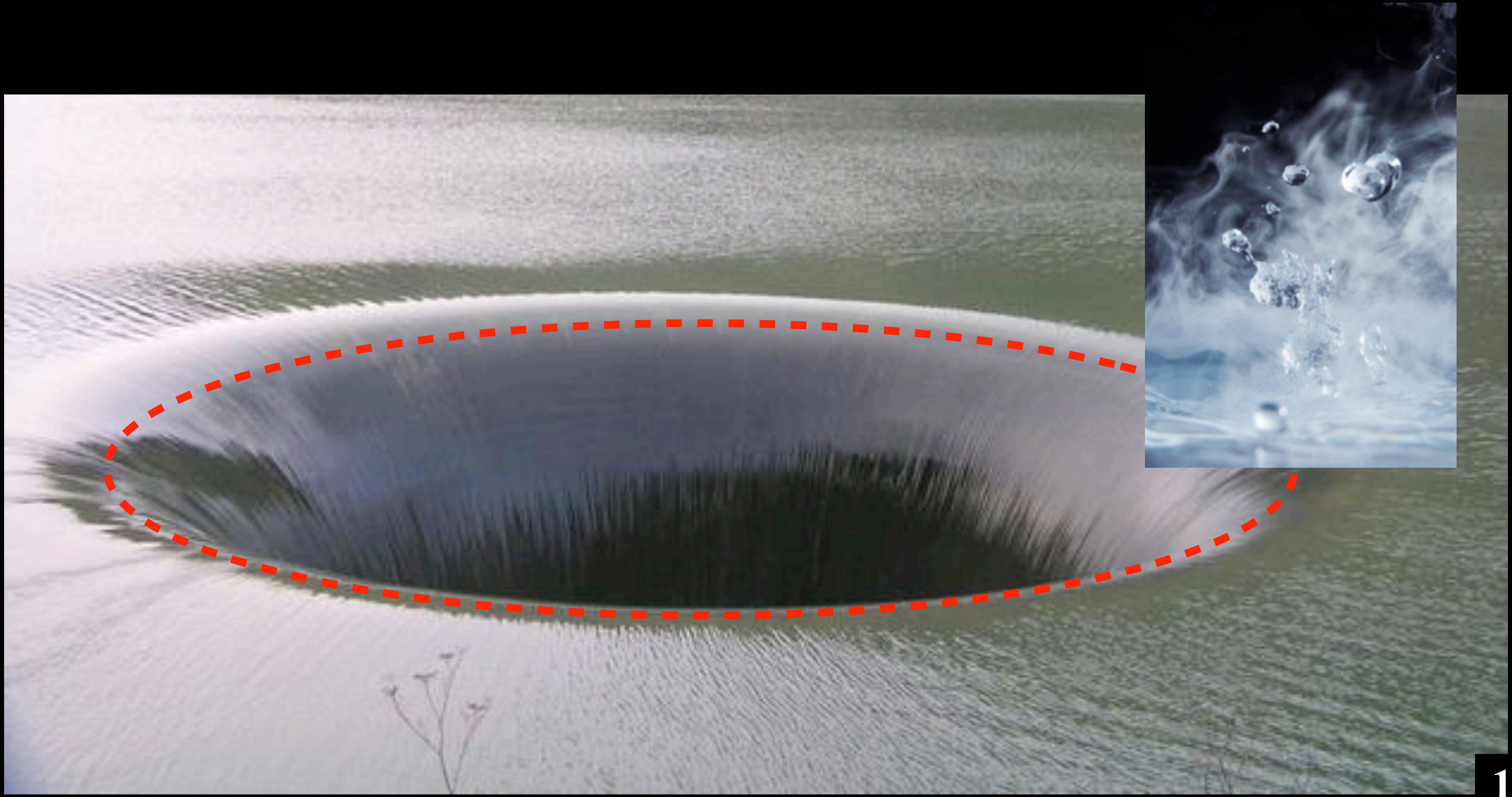
III. Entanglement and thermality of the vacuum

$$(****) \qquad \rho_0 = \frac{e^{-2\pi K}}{Z}$$

IV. Area law for entanglement perturbations

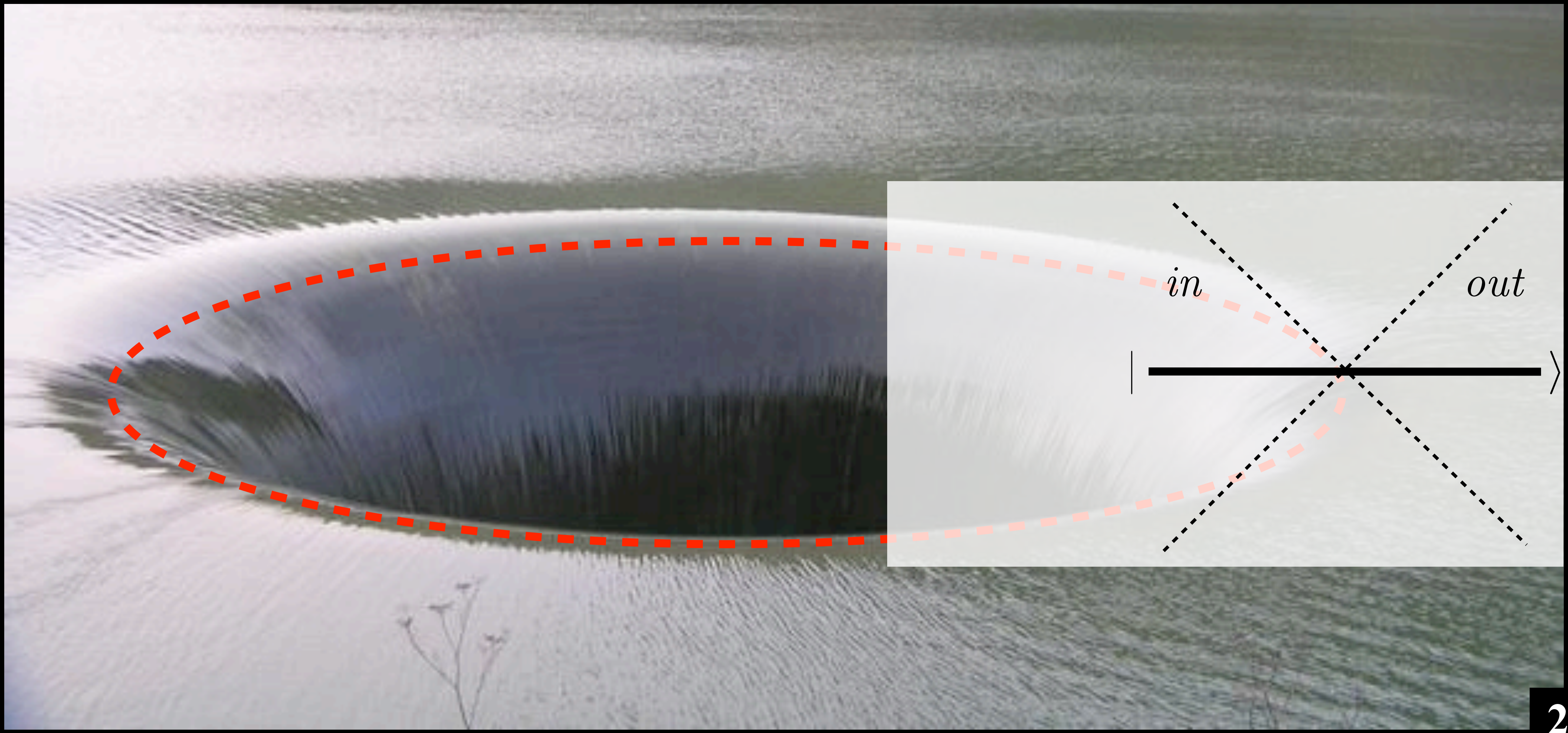
$$\Rightarrow \qquad \delta S_{\text{ent}} = S_{\text{ent}}(|\mathcal{E}\rangle) - S_{\text{ent}}(|\Omega\rangle) = \frac{\delta A}{4G}$$

Why are black holes hot ?



Why are black holes hot ?

because of entanglement across the horizon



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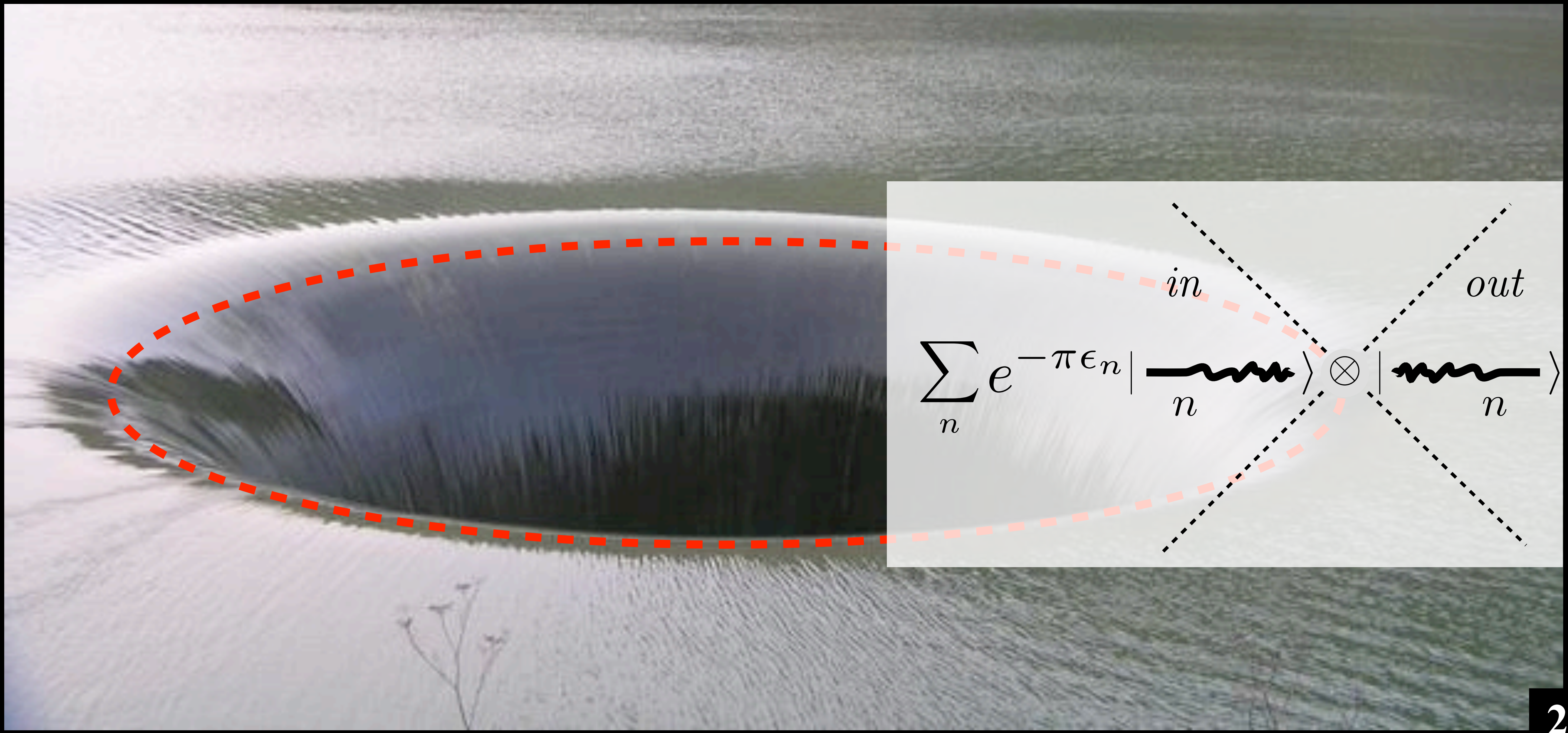
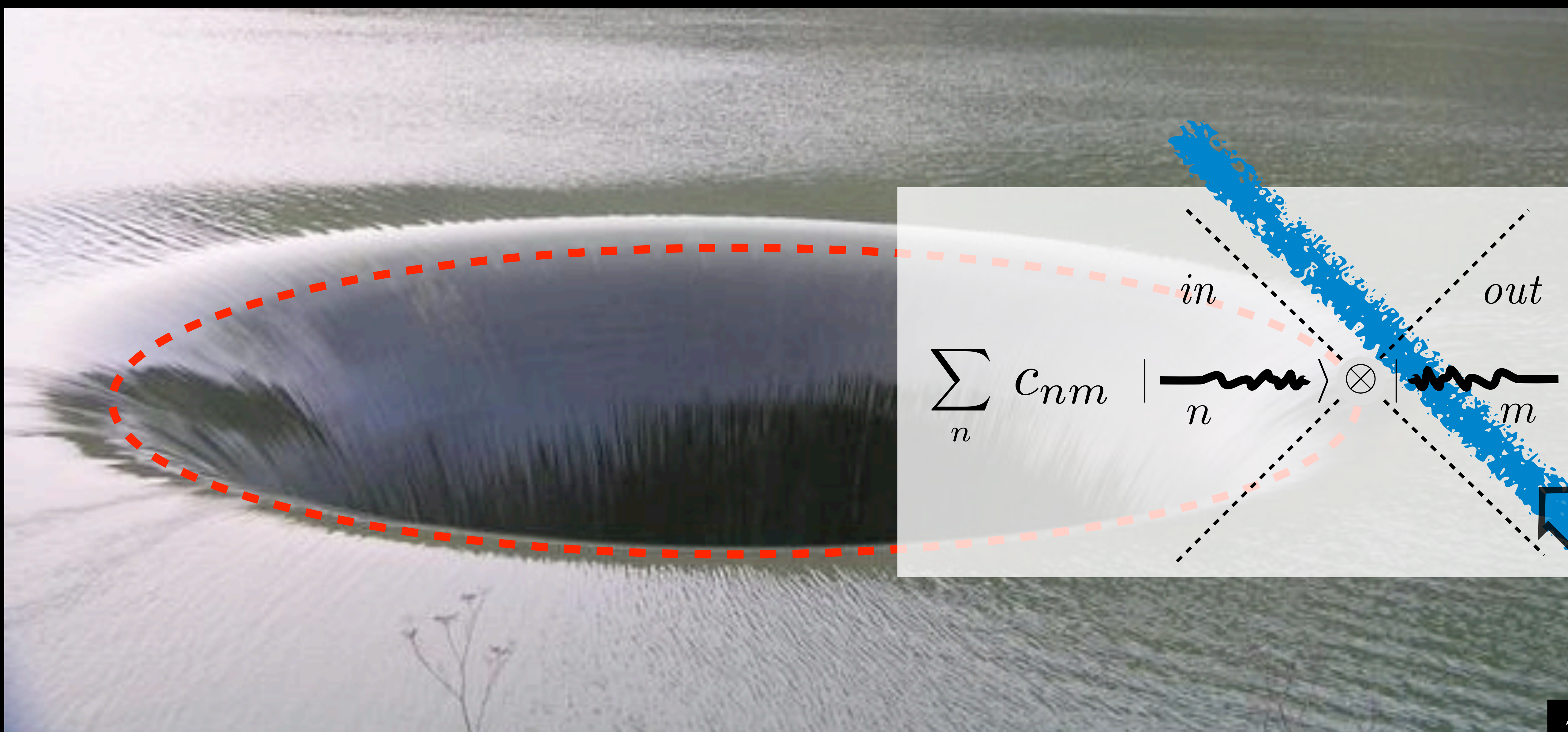


Why are black holes hot ?

because of entanglement across the horizon

$$\delta S_{\rm ent} \;=\; S_{\rm ent}(|\mathcal{E}\rangle) \;-\; S_{\rm ent}(|\Omega\rangle) \;=\; \frac{\delta A}{4G}$$

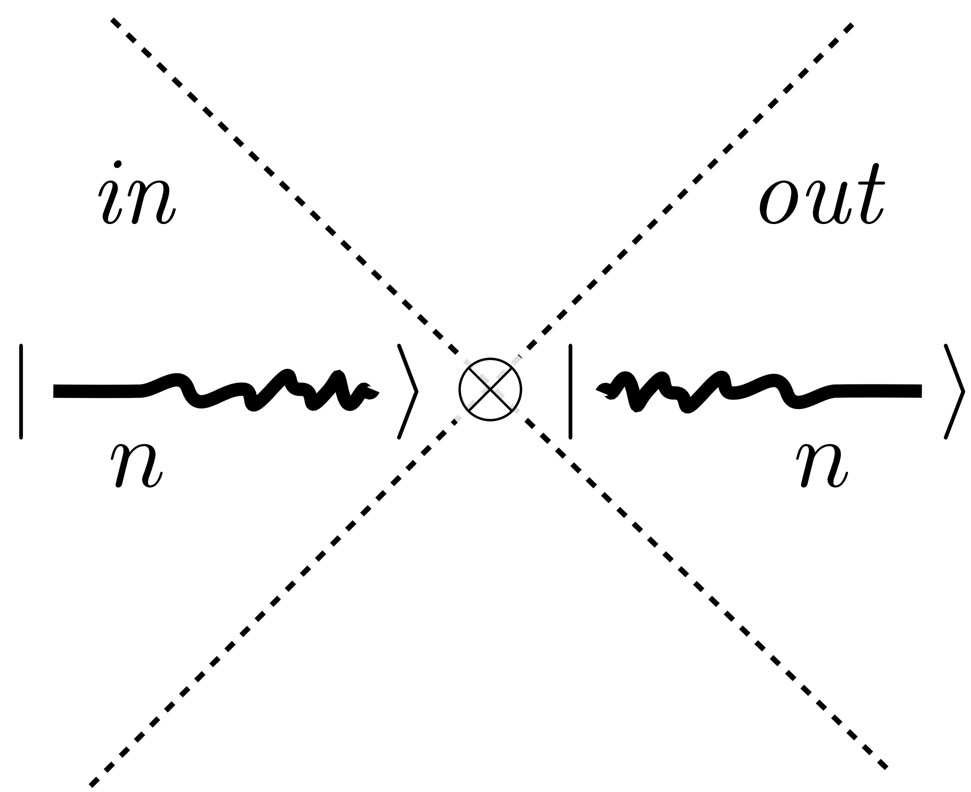


Relevance to Loop Quantum Gravity: I

In the spin-foam calculation of black-hole entropy   arXiv:1204.5122

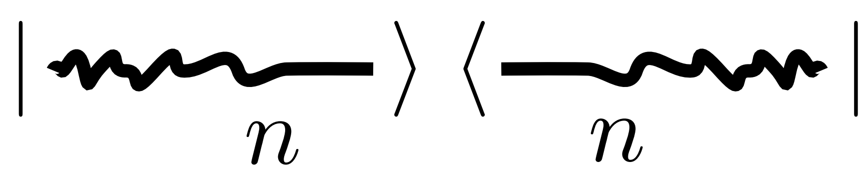
- entanglement and the boost Hamiltonian   ilqgs/bianchi101612.pdf

$$|HH\rangle = \sum_n e^{-\pi\epsilon_n}$$



- relation to the thermal ensemble picture   (ABCK, ENP, ilqgs/perez032712.pdf)

$$\rho = \sum_n e^{-2\pi\epsilon_n}$$

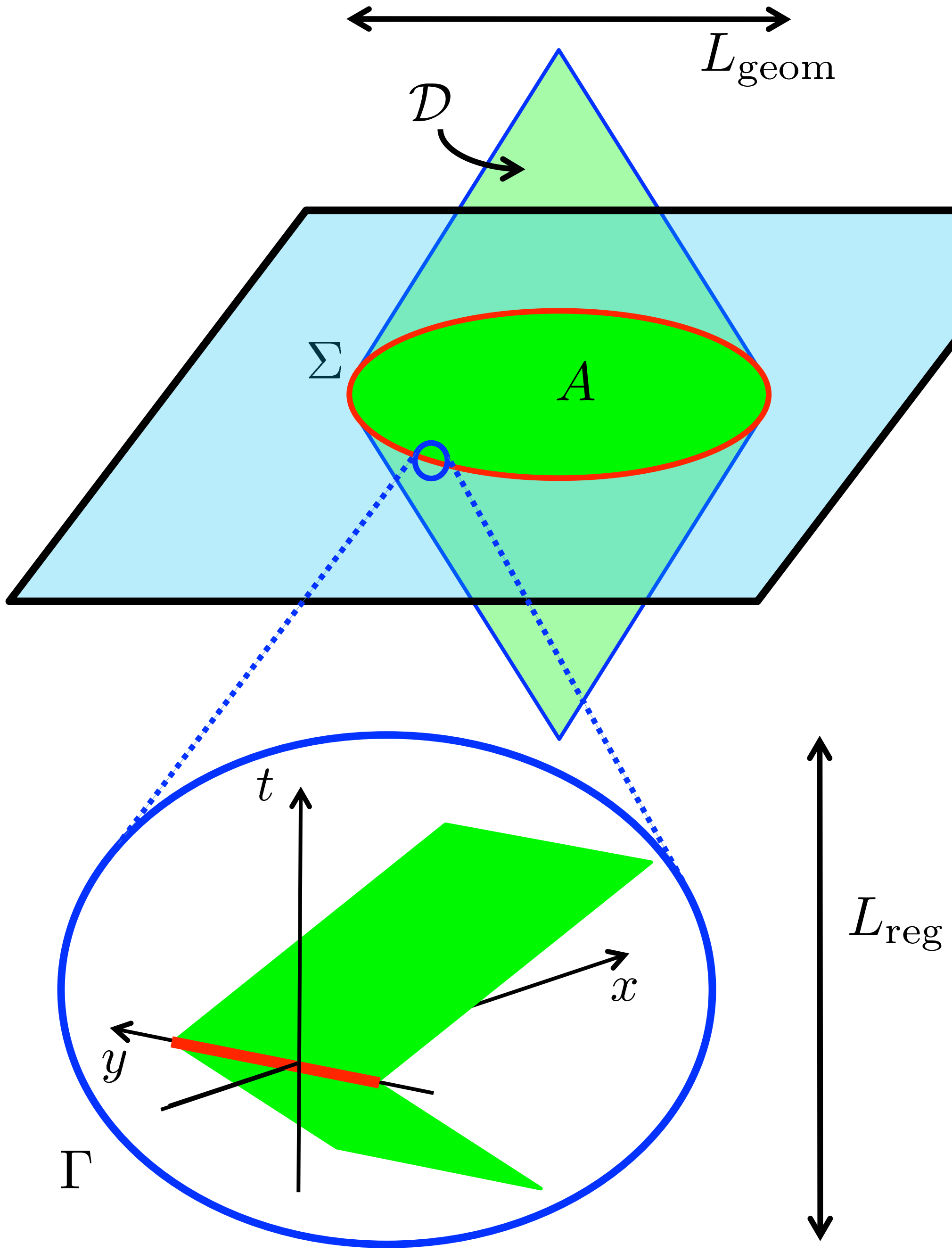


Entanglement entropy of spin-network states (W. Donnelly, 0802.0880, 1109.0036)

On the architecture of spacetime geometry

1212.5183 [hep-th]

with R. Myers



Conjecture:

In a theory of quantum gravity,
- for every large region
- and smooth spacetime

$$S_{\rm ent} = 2\pi \frac{Area}{\ell_P^2} + \ \ldots$$

Entanglement entropy of a spin-network as a probe of semiclassicality