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Page theorem and the entropy of Bell-network states

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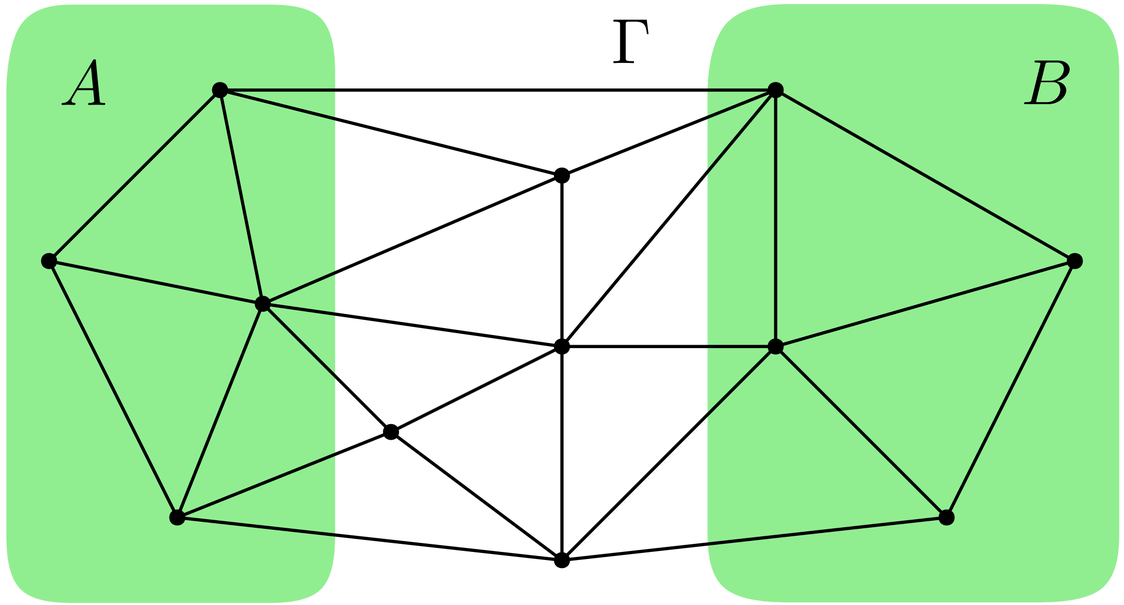
International Loop Quantum Gravity Seminar — Spring 2019

Correlation in loop quantum gravity

Setup:

- Hilbert space $\mathcal{H}_\Gamma$ (fixed graph)
- State $|\psi\rangle \in \mathcal{H}_\Gamma$
- Region $A \subset \Gamma$
- Subalgebra of observables $O_A \in \mathcal{A}_A$
- Correlations

$$\mathcal{C} = \langle\psi|\, O_A\, O_B |\psi\rangle \;-\; \langle\psi|\, O_A |\psi\rangle\, \langle\psi|\, O_B |\psi\rangle$$



This talk: Information-theoretic bounds on correlations

$$\frac{1}{2}\left(\frac{\mathcal{C}(O_A, O_B)}{\|O_A\|\|O_B\|}\right)^2 \;\leq\; S_A(|\psi\rangle) + S_B(|\psi\rangle) - S_{AB}(|\psi\rangle)$$

Plan:

- Typical entropy of a subsystem: Page curve and its variance

- Entropy and correlations in Bell-network states

Typical entropy of a subsystem

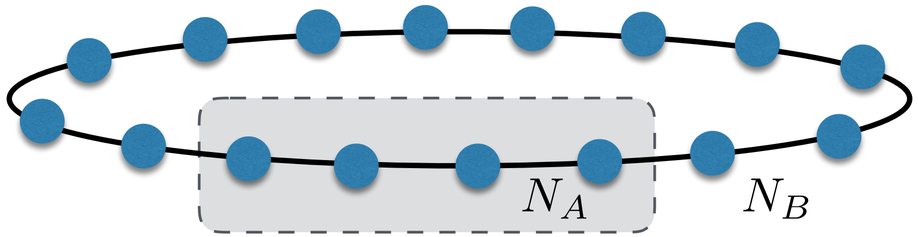
Example: spin chain

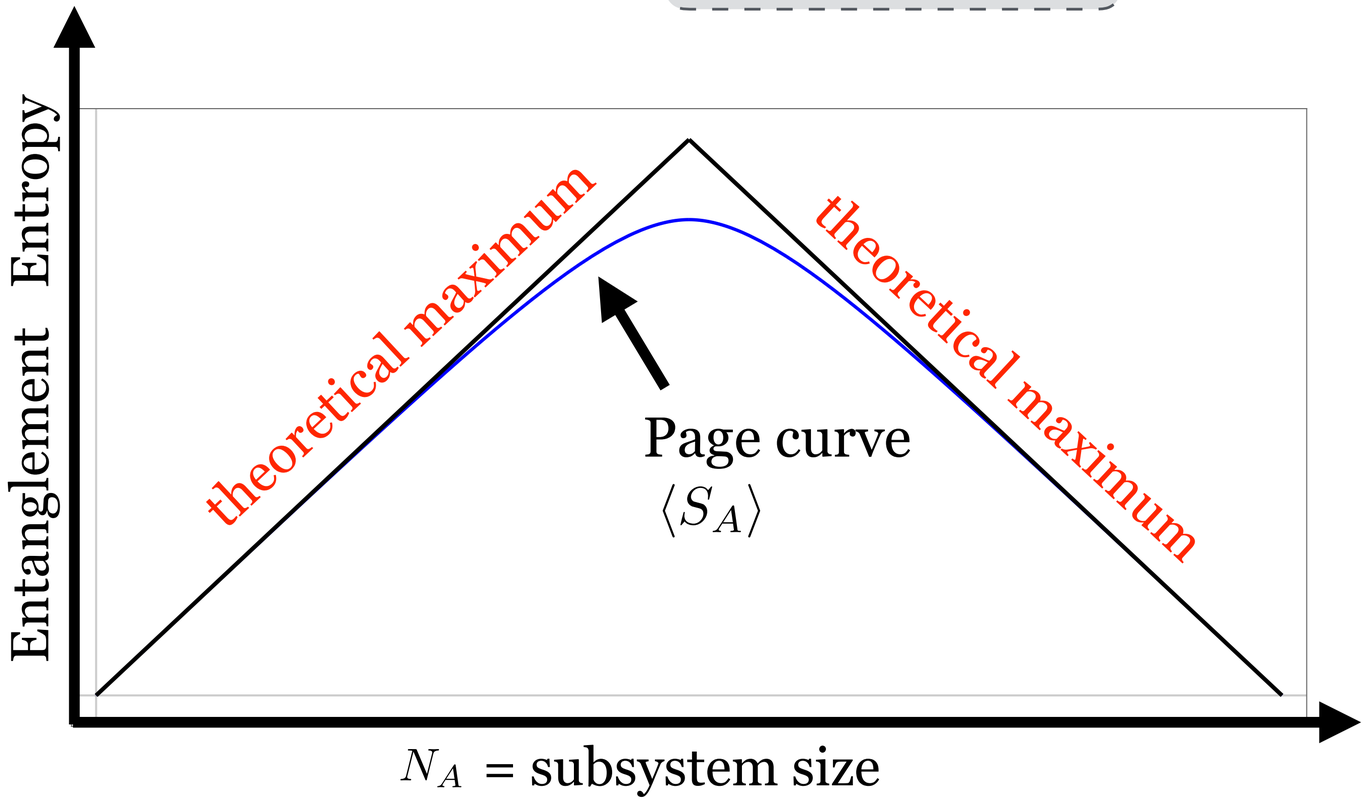
- Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$
- <u>Random</u> state $|\psi\rangle \in \mathcal{H}$
- Subsystem A, $d_A = \dim \mathcal{H}_A = 2^{N_A}$
- Restricted state $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$
- Entanglement entropy

$$S_A(|\psi\rangle) = -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

- Average over all states

$$\langle S_A \rangle = \int d\mu(|\psi\rangle)\ S_A(|\psi\rangle)$$

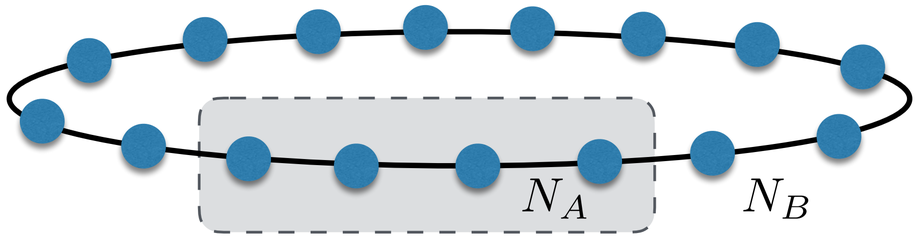




[Page *PRL*('95)] *The average entropy is close to maximal*

$$d_B \gg 1\,, \qquad \langle S_A \rangle \approx \log d_A - \frac{1}{d_A d_B}\,\frac{d_A^2 - 1}{2}$$

Typical entropy of a subsystem



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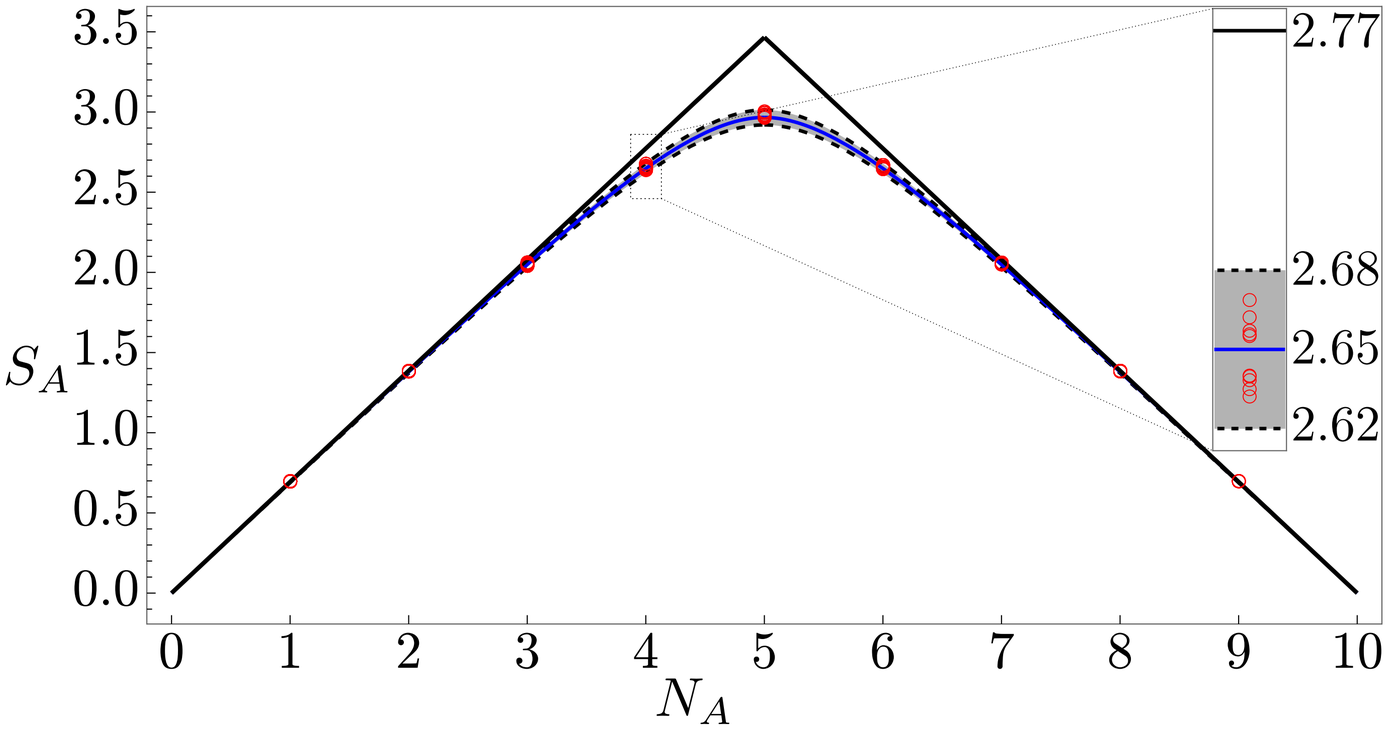

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[E.B. & P.Donà ('19)] *The variance is small*

$$\Delta S_A \approx \frac{1}{d_A d_B}\sqrt{\frac{d_A^2 - 1}{2}}$$

Average entropy of a subsystem [Page *PRL*('95)]

System: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Step 1. $$\langle \mathrm{Tr}\rho_A^r \rangle = \int d\mu(|\psi\rangle)\, \mathrm{Tr}\rho_A^r \;=\; \int \left(\sum_{a=1}^{d_A} \lambda_a^r \right) \mu\left(\lambda_1, \ldots, \lambda_{d_A}\right) \prod_{k=1}^{d_A} d\lambda_k \,.$$

Eigenvalues of the density matrix and induced integration measure

[Lloyd & Pagels, Ann. Phys. ('88)]

Step 2. $$\langle S_A \rangle = -\langle \mathrm{Tr}\rho_A \log \rho_A \rangle = -\lim_{r\to 1} \partial_r \langle \mathrm{Tr}\rho_A^r \rangle$$

Exact result: $$\langle S_A \rangle = \psi\left(d_A d_B + 1\right) - \psi\left(d_B + 1\right) - \frac{d_A - 1}{2 d_B}$$

diGamma function

$\psi(x) = \Gamma'(x)/\Gamma(x)$

Asymptotics: $$\langle S_A \rangle = \log d_A - \frac{1}{d_A d_B} \frac{d_A^2 - 1}{2} \;+\; O(1/d_B^2)$$

Variance of the entropy of a subsystem [E.B. & P.Donà ('19)]

System: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Step 1. $$\langle \mathrm{Tr}(\rho_A^{r_1})\,\mathrm{Tr}(\rho_A^{r_2})\rangle = \int \left(\sum_{a=1}^{d_A} \lambda_a^{r_1}\right)\left(\sum_{b=1}^{d_A} \lambda_b^{r_2}\right) \mu\left(\lambda_1, \ldots, \lambda_{d_A}\right) \prod_{k=1}^{d_A} d\lambda_k \,.$$

Eigenvalues of the density matrix and induced integration measure

[Lloyd & Pagels, Ann. Phys. ('88)]

Step 2. $$\langle S_A^2\rangle = \langle (\mathrm{Tr}\rho_A \log \rho_A)^2\rangle = \lim_{\substack{r_1\to 1\\ r_2\to 1}} \partial_{r_1}\partial_{r_2}\langle \mathrm{Tr}\rho_A^{r_1}\,\mathrm{Tr}\rho_A^{r_2}\rangle$$

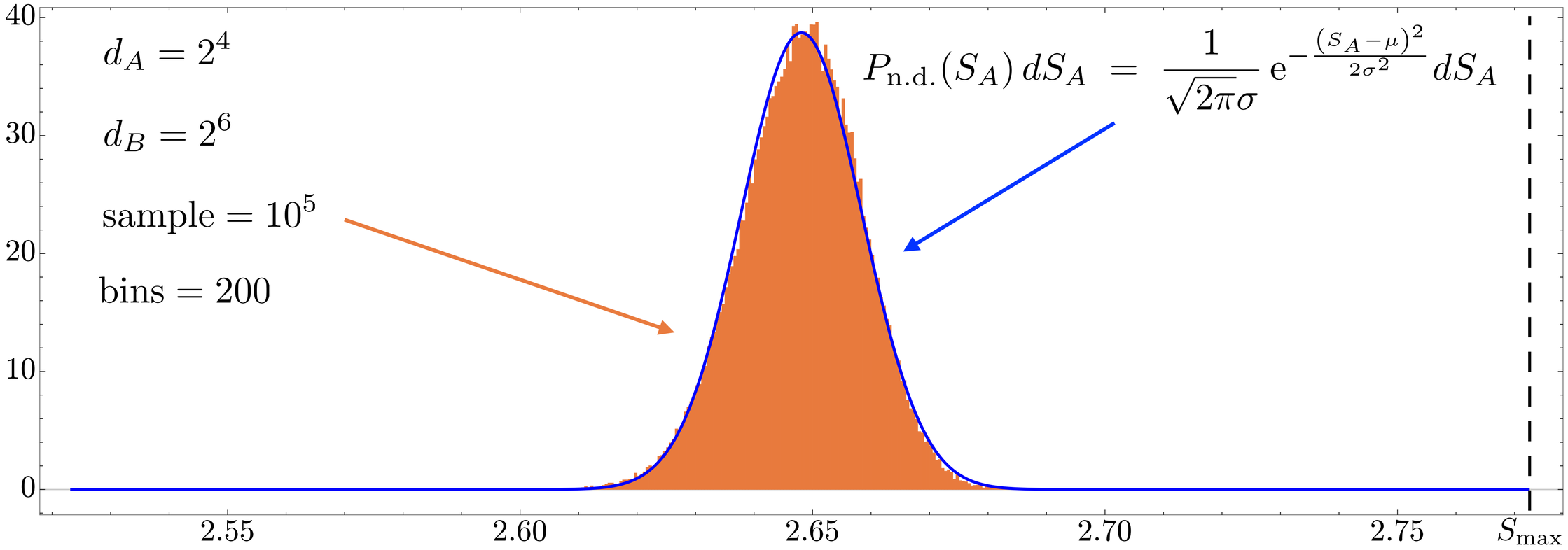
Exact result: $$(\Delta S_A)^2 = \langle S_A^2\rangle - \langle S_A\rangle^2 = -\frac{(d_A-1)(d_A+2d_B-1)}{4d_B^2(d_A d_B+1)} + \frac{d_A+d_B}{d_A d_B+1}\psi'\left(d_B+1\right) - \psi'\left(d_A d_B+1\right) \,.$$

diGamma function

$\psi(x) = \Gamma'(x)/\Gamma(x)$

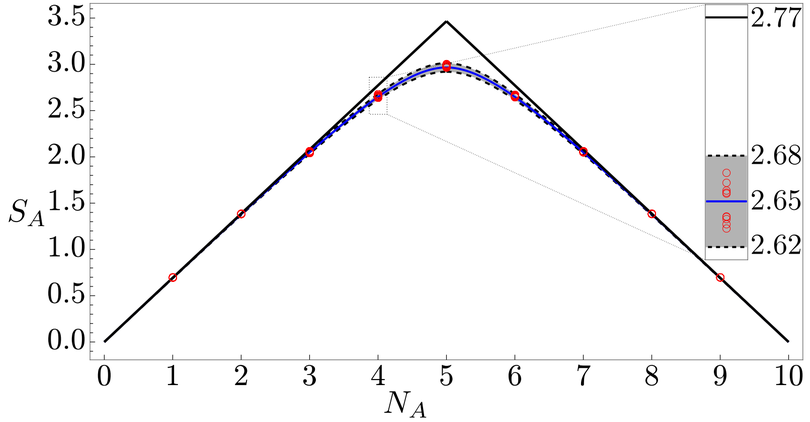
Asymptotics: $$(\Delta S_A)^2 = \frac{d_A^2-1}{2d_A^2 d_B^2} + O(1/d_B^3)$$

Numerical data vs Normal distribution with given mean and variance [E.B. & P.Donà ('19)]

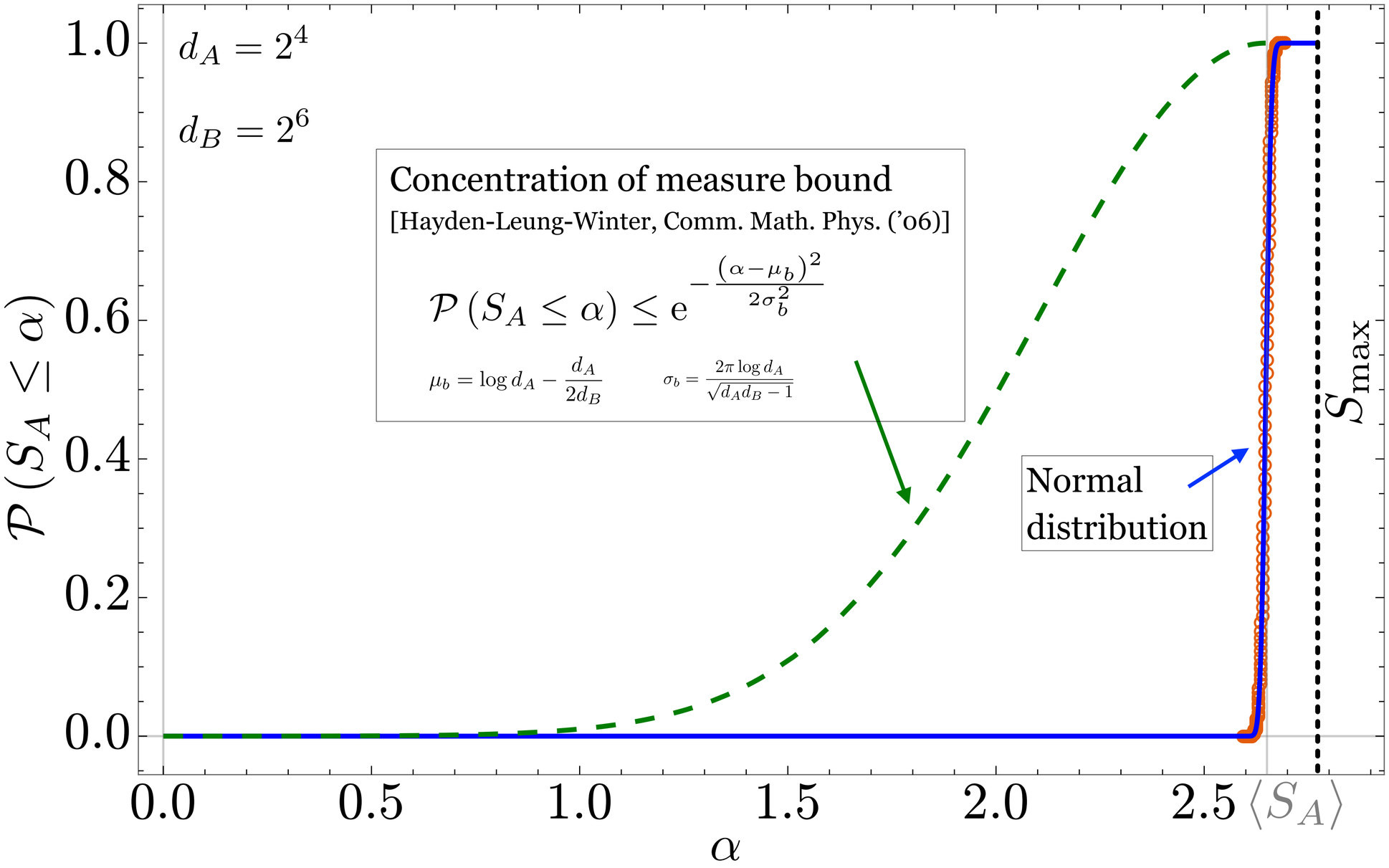


Narrow distribution for $1 \ll N_A \leq N_B$

$$\frac{\Delta S_A}{S_{\max} - \langle S_A \rangle} \approx \sqrt{2}\, \mathrm{e}^{-N_A \log 2}$$

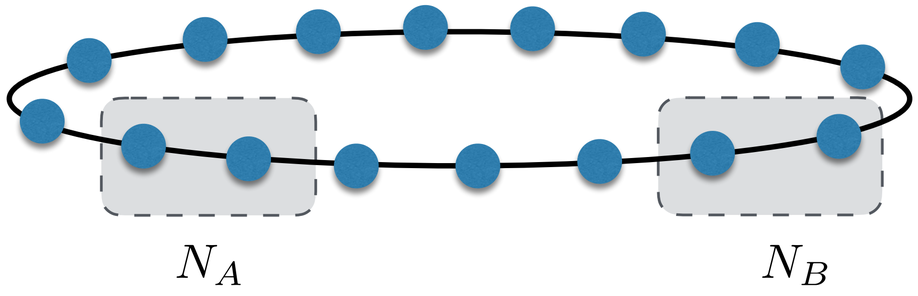


Comparison to previous bounds on the support of the distribution [E.B. & P.Donà ('19)]



Information theoretic bound on correlations

[Wolf, Verstraete, Hastings and Cirac, *PRL* ('08)]



$$\mathcal{C} = \langle\psi|\, O_A\, O_B|\psi\rangle \;-\; \langle\psi|\, O_A|\psi\rangle\, \langle\psi|\, O_B|\psi\rangle$$

$$N = N_A + N_B + N_C$$

$$N_C \gg 1$$

$$\frac{1}{2}\left(\frac{\mathcal{C}(O_A, O_B)}{\|O_A\|\|O_B\|}\right)^2 \;\leq\; S_A(|\psi\rangle) + S_B(|\psi\rangle) - S_{AB}(|\psi\rangle)$$

For a random state:

$$\left\langle \frac{1}{2}\left(\frac{\mathcal{C}(O_A, O_B)}{\|O_A\|\|O_B\|}\right)^2 \right\rangle \;\leq\; \left(\cancel{\log} d_A - \frac{1}{d_A d_B d_C}\,\frac{d_A^2-1}{2}\right) + \left(\cancel{\log} d_B - \frac{1}{d_A d_B d_C}\,\frac{d_B^2-1}{2}\right) - \left(\cancel{\log}(d_A d_B) - \frac{1}{d_A d_B d_C}\,\frac{d_A^2 d_B^2-1}{2}\right)$$

$$= O(2^{-N_C})$$

Correlations are exponentially suppressed in a random state

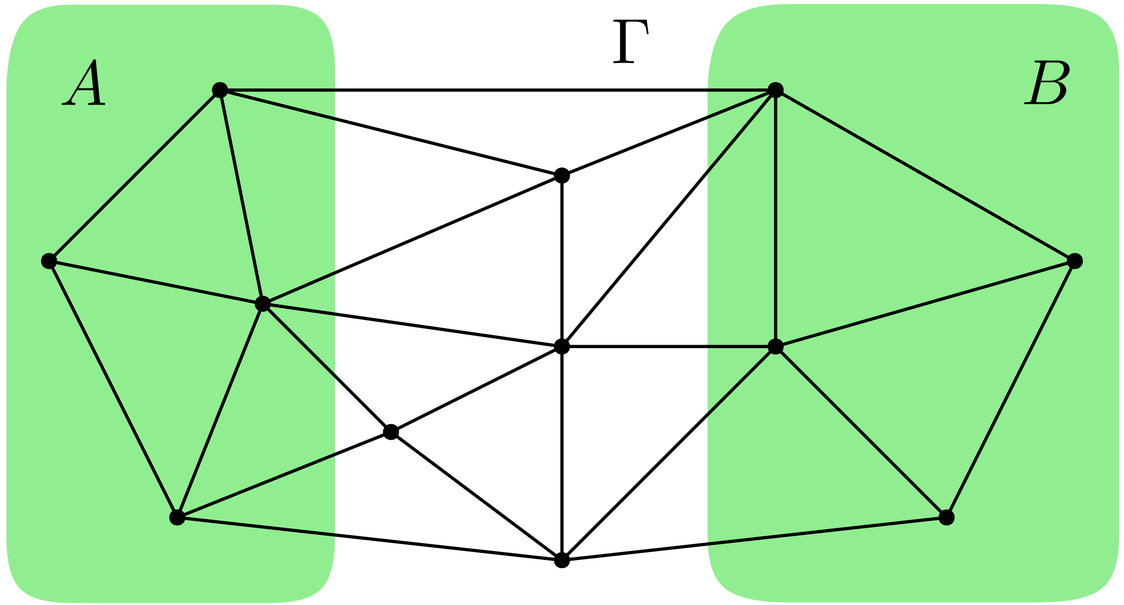
Correlation in loop quantum gravity

Correlations are exponentially suppressed in a random state

Two caveats:

1. Constraint on physical states

2. $\mathcal{H}_\Gamma$ is not a tensor product over nodes and not finite dimensional (because of the sum over spins)



Correlation in loop quantum gravity

Correlations are exponentially suppressed in a random state

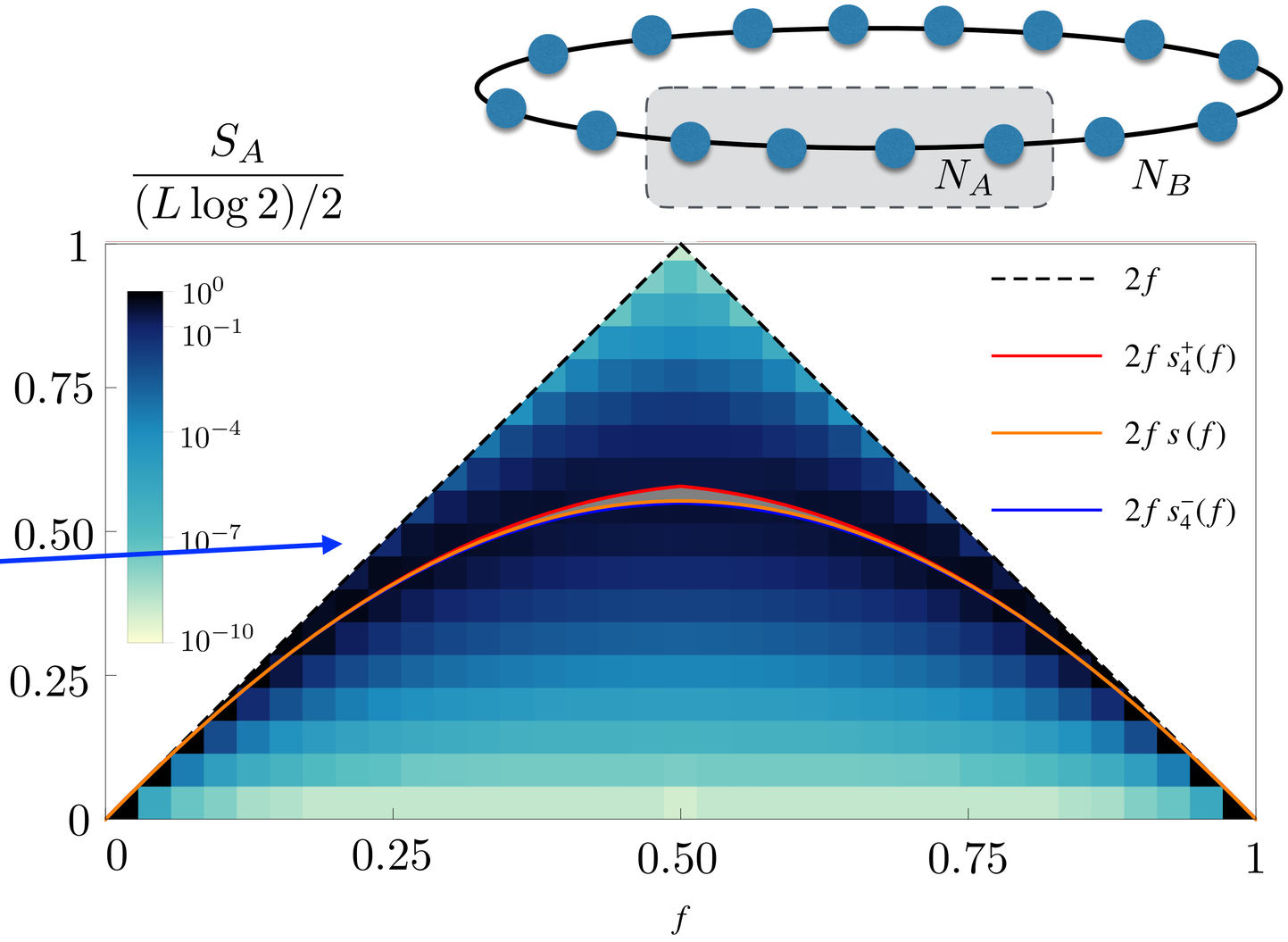
Two comments:

1. Constraint on physical states
2. $\mathcal{H}_\Gamma$ is not a tensor product over nodes and not finite dimensional (because of the sum over spins)

Entropy of a subalgebra of observables

[Casini, Huerta, Rosabal *PRD* ('14)]

[E.B., Satz *PRD* ('19)]



Eigenstate entanglement in the XY model

[E.B., Hackl, Rigol, Vidmar, *PRL* ('17), *PRL* ('18), *PRB* ('19)]

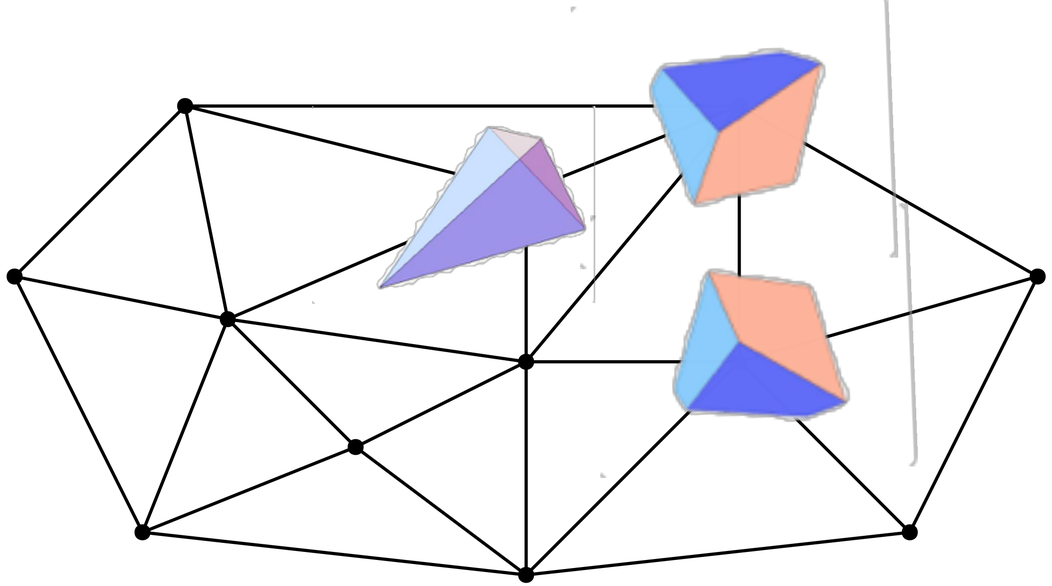
Plan:

- Typical entropy of a subsystem: Page curve and its variance

- Entropy and correlations in Bell-network states

Gluing polyhedra with entanglement in LQG

[E.B., Baytas, Yokomizo, *PRD* ('18)]



* Fluctuations of nearby quantum polyhedra are in general uncorrelated (twisted geometry): e.g.: in a spin-network basis state, and in a random state

Bell-network states:

use squeezed vacua techniques [E.B., Hackl, Guglielmon, Yokomizo *PRD* ('16)] to correlate back-to-back normals

- on links $|\mathcal{B},\lambda\rangle_\ell = \left(1-|\lambda|^2\right)\exp\left(\lambda\,\epsilon_{\alpha\beta}a_s^{\dagger\alpha}a_t^{\dagger\beta}\right)|0\rangle_s|0\rangle_t$ with $\lambda\in\mathbb{C}$ encoding average area and extr. curv.

- projections: $|\Gamma,\lambda_\ell,\mathcal{B}\rangle = P_\Gamma \otimes_{\ell\in\Gamma} |\mathcal{B},\lambda_\ell\rangle_\ell$

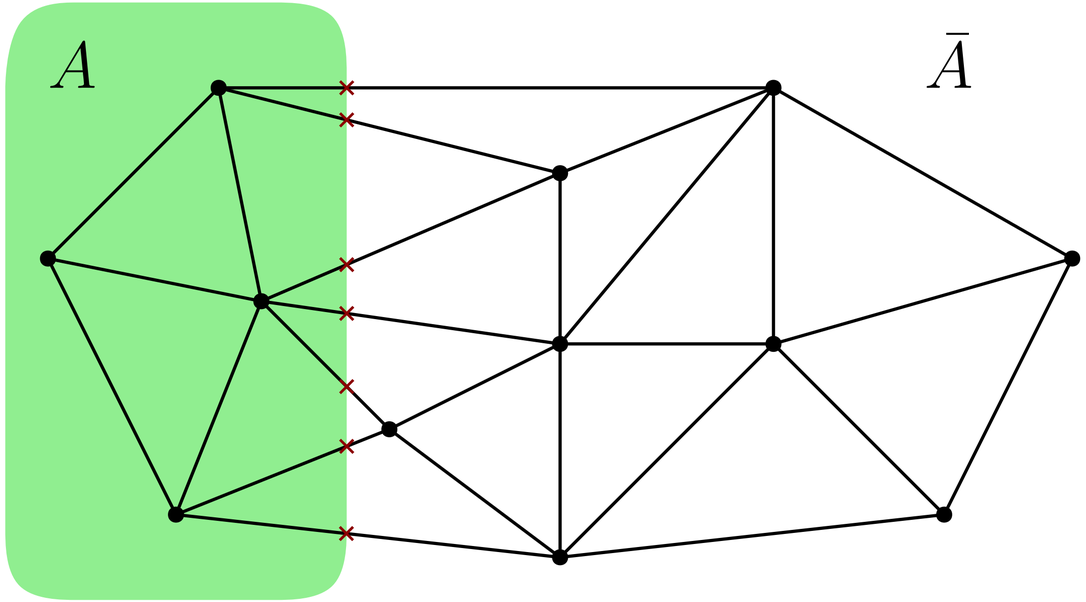
- result: $|\Gamma,\lambda_\ell,\mathcal{B}\rangle = \sum_{j_\ell}\left(\prod_\ell \left(1-|\lambda_\ell|^2\right)\lambda_\ell^{2j_\ell}\sqrt{2j_\ell+1}\right)|\Gamma,j_\ell,\mathcal{B}\rangle$

<u>superposition over spins of entangled interwiner state</u> $|\Gamma,j_\ell,\mathcal{B}\rangle = \frac{1}{\sqrt{Z}}\sum_{i_n}\overline{\mathcal{A}_\Gamma\left(j_\ell,i_n\right)}\bigotimes_n |i_n\rangle$

where $\mathcal{A}_\Gamma\left(j_\ell,i_n\right) = \sum_{\{m\}}\prod_n [i_n]^{m_1\cdots m_{F_n}}$ is the symbol of the graph

Entanglement entropy of Bell-network states

[E.B., Donà, Vilensky ('19)]



- Hilber space decomposition

$$\mathcal{H}_\Gamma = \bigoplus_{j_\ell} \left(\bigotimes_n \mathcal{H}_{i_n} \right)$$

- Subalgebra in A

- State $\quad |\Gamma, \lambda_\ell, \mathcal{B}\rangle = \sum_{j_\ell} q_{j_\ell} \, |\Gamma, j_\ell, \mathcal{B}\rangle$

- Entropy

$$S_A = -\sum_{j_\ell} p_{j_\ell} \log p_{j_\ell} + \sum_{j_\ell} p_{j_\ell} S_A\left(j_\ell\right)$$

easy:

Shannon entropy of

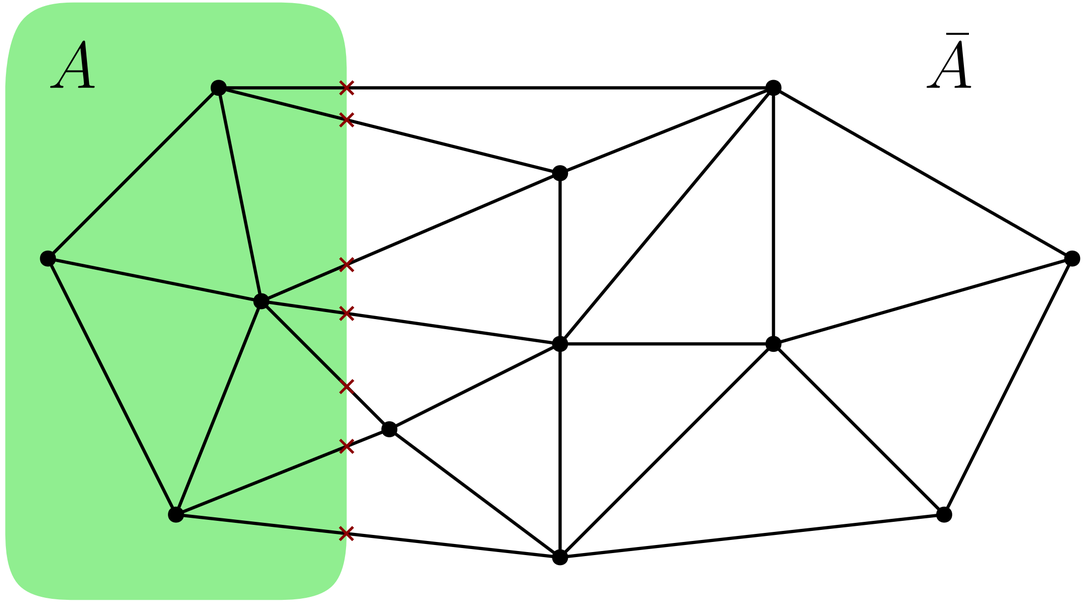
$$p_{j_\ell} = |q_{j_\ell}|^2 \Big/ \sum_{j_\ell} |q_{j_\ell}|^2$$

non-trivial:

<u>entropy of the entangled interwiner state</u>

$$|\Gamma, j_\ell, \mathcal{B}\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} \overline{\mathcal{A}_\Gamma\left(j_\ell, i_n\right)} \bigotimes_n |i_n\rangle$$

Analytical results: [E.B., Donà, Vilensky ('19)]



Large spin asymptotics of the entanglement entropy

- Entangled intertwiner state

$$|\Gamma, j_\ell, \mathcal{B}\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} \overline{\mathcal{A}_\Gamma\left(j_\ell, i_n\right)} \bigotimes_n |i_n\rangle$$

- Using the replica trick

$$\mathrm{Tr}\left(\rho_A^p\right) = \frac{\int d\vec{n}\, dg\; e^{f_p(j,\vec{n},g)}}{\left(\int d\vec{n}\, dg\; e^{f_1(j,\vec{n},g)}\right)^p} \qquad \text{with} \qquad \rho_A = \mathrm{Tr}_{\bar{A}} |\Gamma, j_\ell, \mathcal{B}\rangle\langle\Gamma, j_\ell, \mathcal{B}|$$

- Using spinfoam-asymptotics methods [Donà, Fanizza, Sarno, Speziale *CQG* ('18)]

$$j_\ell \to \lambda j_\ell \qquad \text{with} \qquad \lambda \gg 1$$

- Renyi Entropy

$$R_A^{(p)} = -\frac{1}{p-1} \log \mathrm{Tr} \rho_A^p$$

- Entanglement Entropy $\quad S_A = \lim_{p\to 1} R_A^{(p)}$

Asymptotic bound on the entanglement entropy

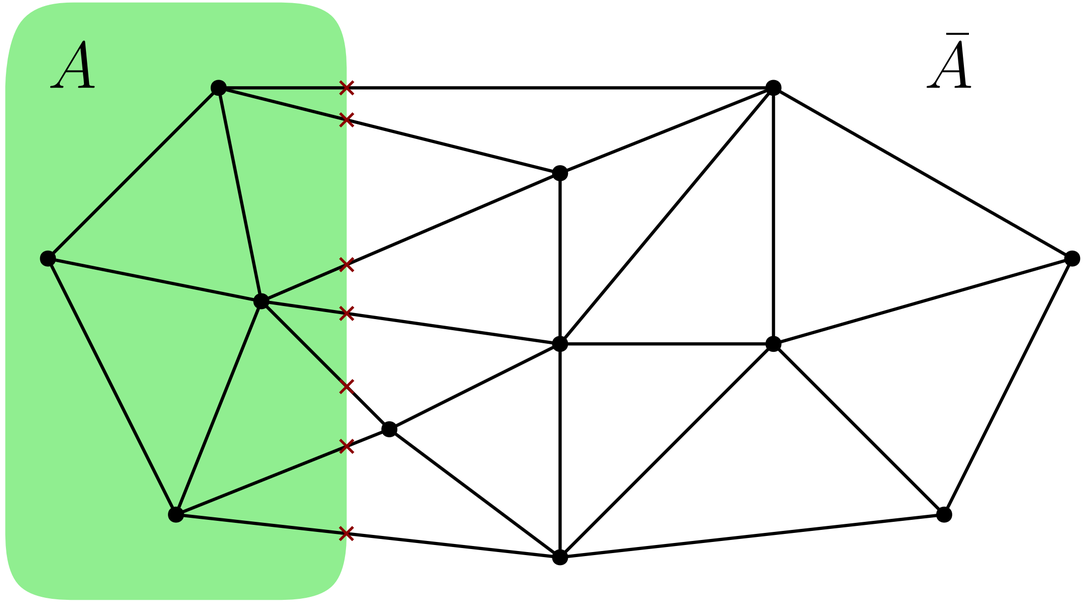
$$\left(|\partial A| - \frac{C_A^{(2)}}{2}\right) \log\lambda \;\leq\; S_A \;\leq\; \left(|\partial A| - 3\right) \log\lambda$$

Area-law from intertwiner entanglement

Numerical results: [E.B., Donà, Vilensky ('19)]

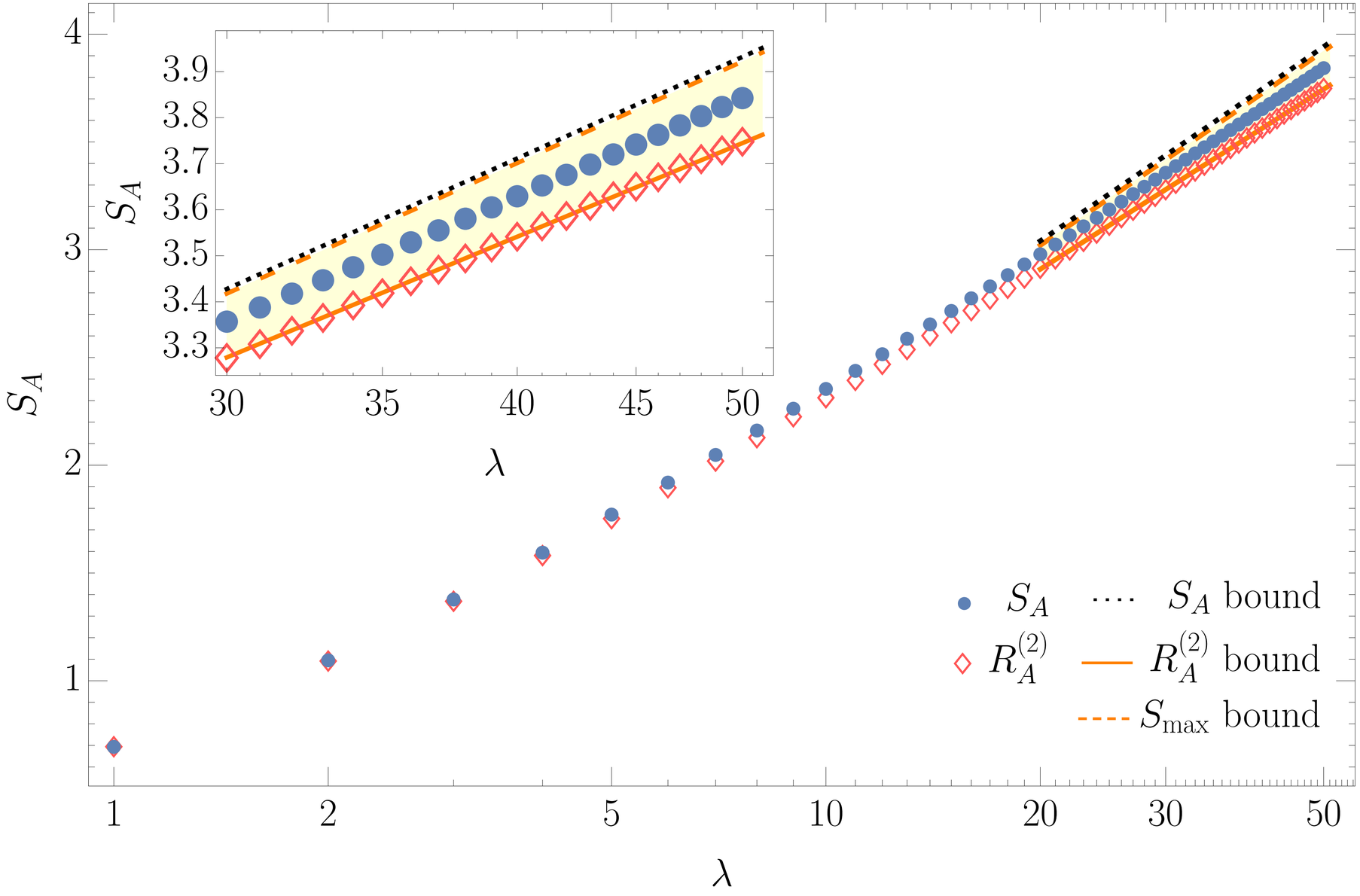
Algorithm 1 Numerical algorithm for the evaluation of the Bell-network state entropy

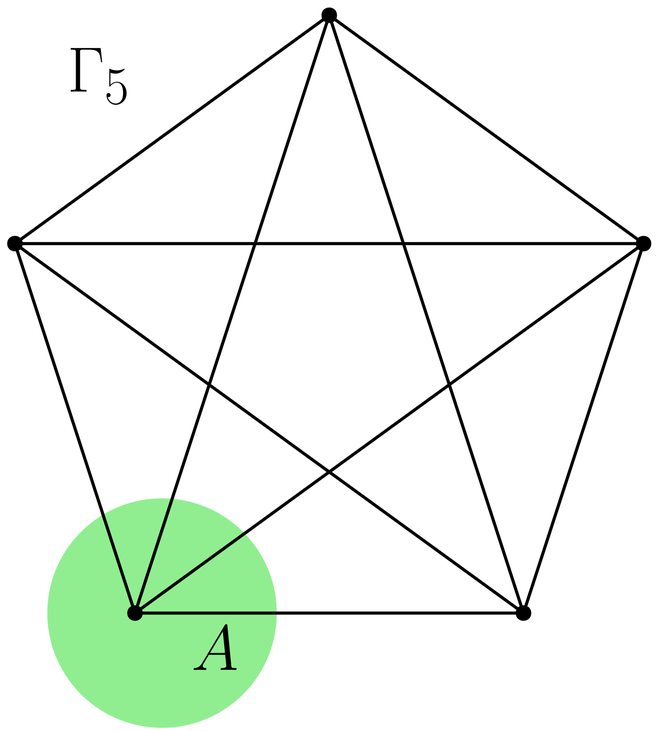
Precompute the $\{6j\}$ symbols with `wigxjpf` if needed
Precompute the $\{9j\}$ symbols with `wigxjpf` if needed
Load the symbol tables in the memory
for each j_ℓ **do**
  for each i_n **do**
    Assemble the matrix $(\mathcal{A}_\Gamma)_{ik} = \mathcal{A}_\Gamma(j_\ell, i_n, k_n)$ with the intertwiners i_n, k_n in A, $\bar{A}$
  Compute the matrix $M_{ii'} = \left(\mathcal{A}_\Gamma^T \cdot \mathcal{A}_\Gamma\right)_{ii'}$
  Normalize it to obtain the density matrix $(\rho_A)_{ii'} = M_{ii'}/Tr(M)$
  Find its eigenvalues $\rho_A \to \nu_i$
  Compute the entanglement entropy $S_A = -\sum_i \nu_i \log \nu_i$
  Compute the Rényi entropy $R_A^{(p)} = -\log \sum_i \nu_i^p$



Code available at: https://bitbucket.org/pietrodona/bellnetworkentropy

Numerical results: [E.B., Donà, Vilensky ('19)]





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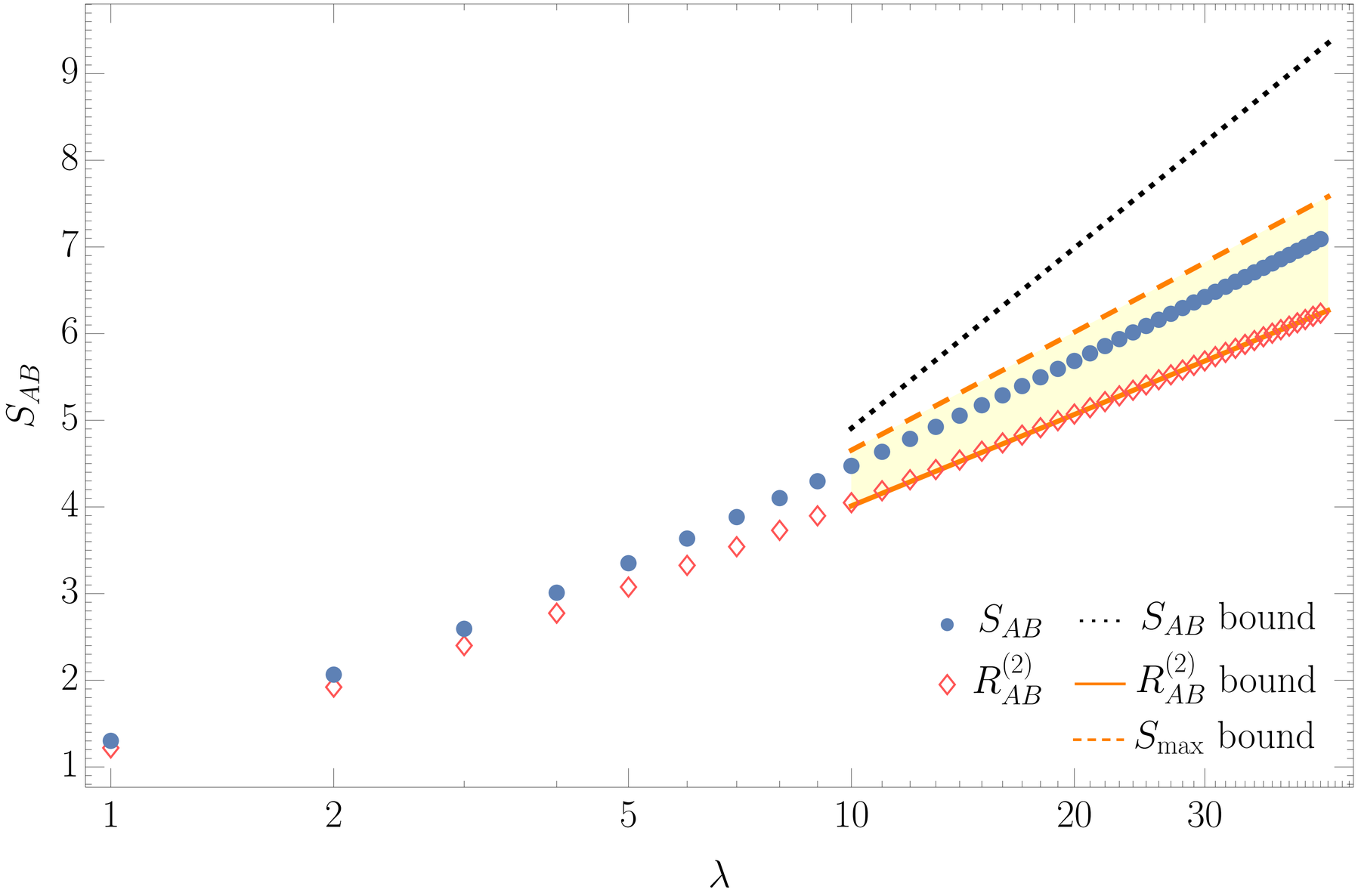

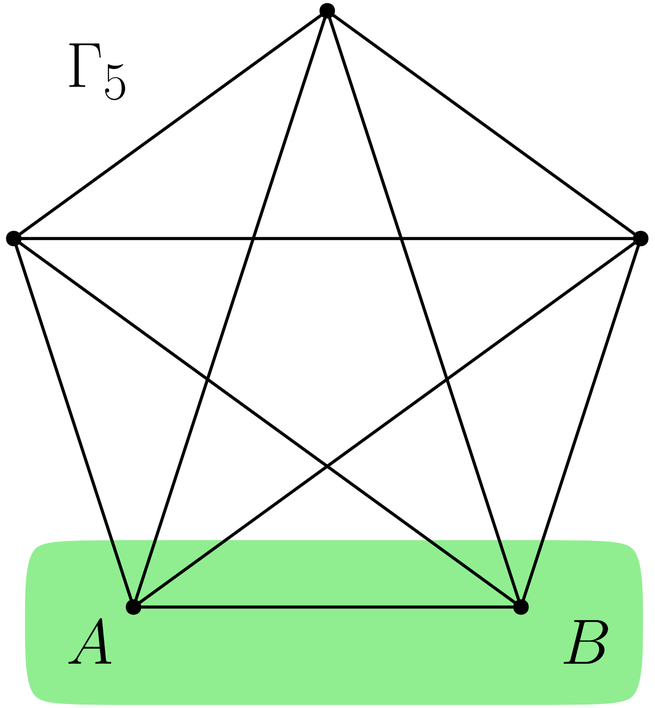

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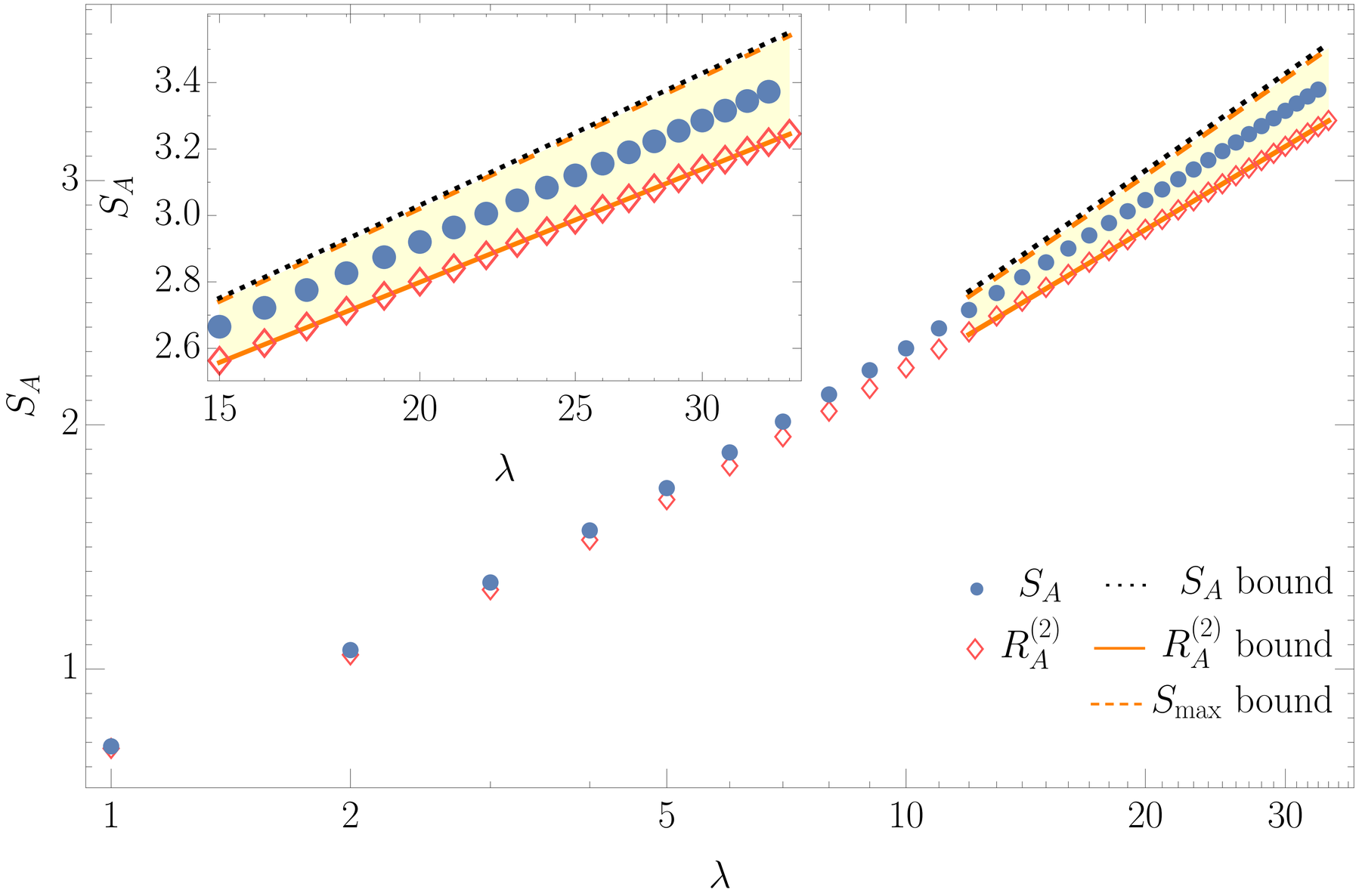

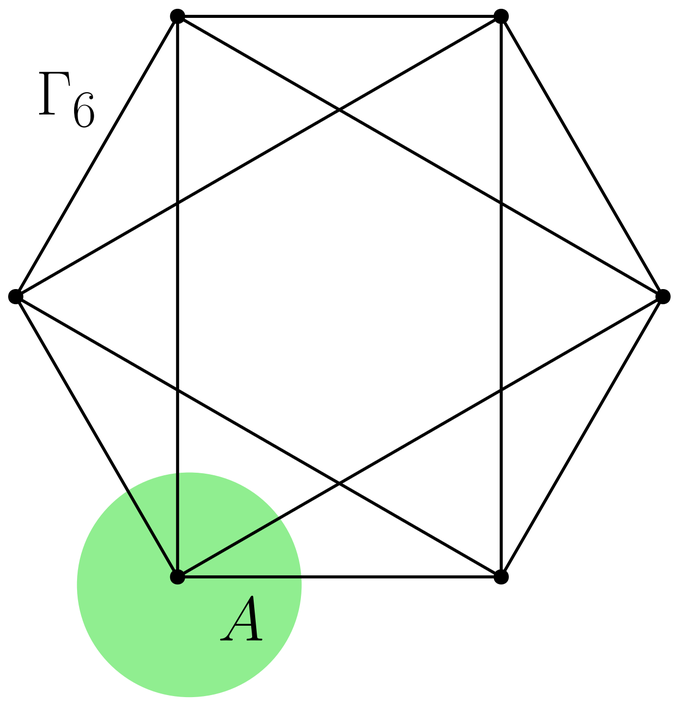

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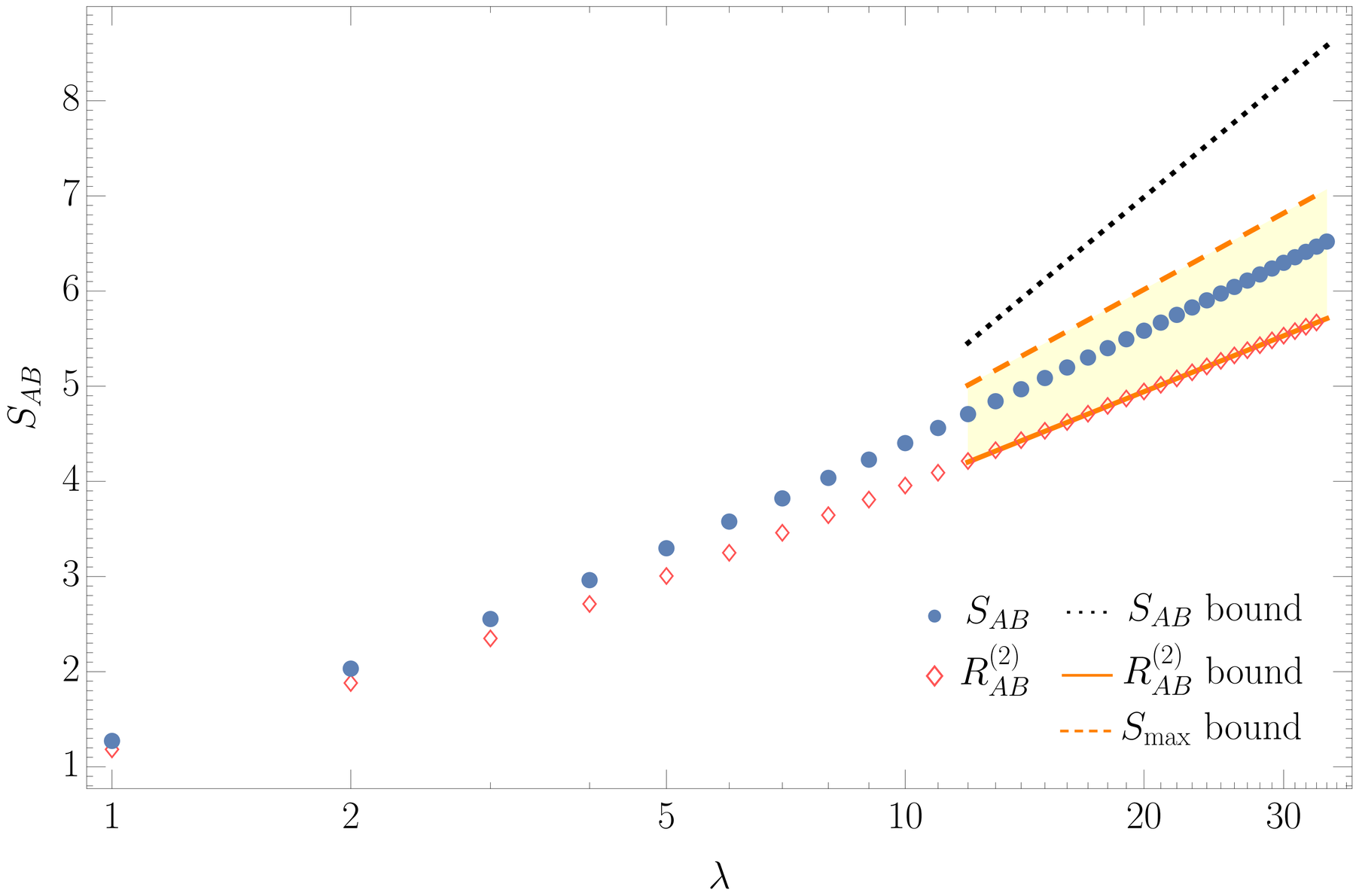

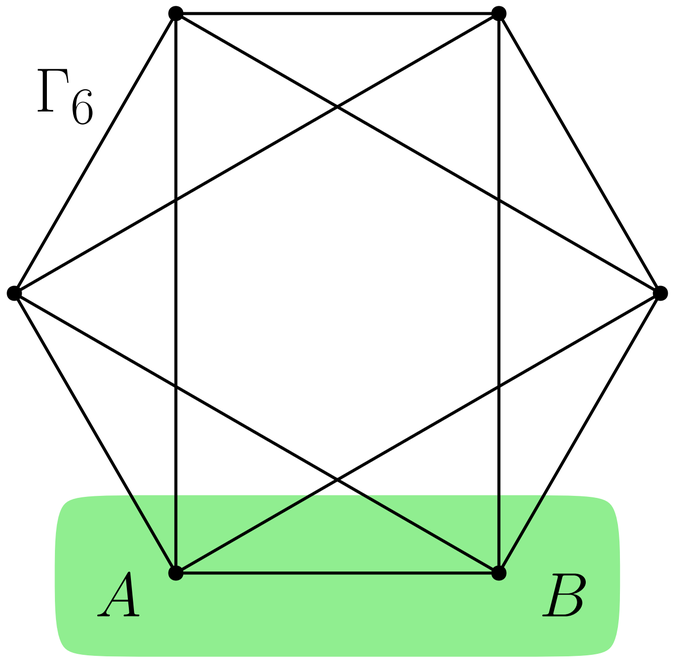

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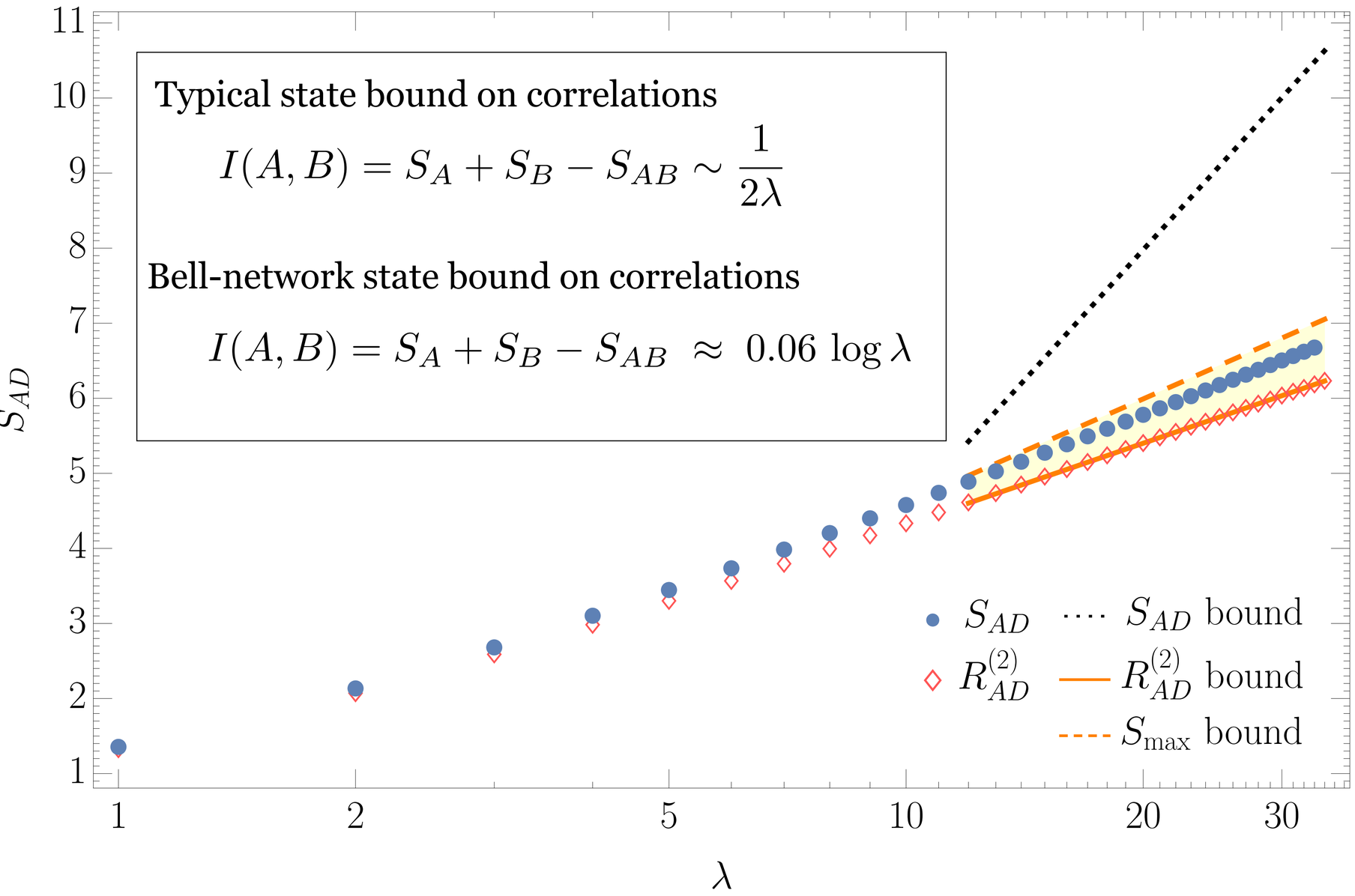

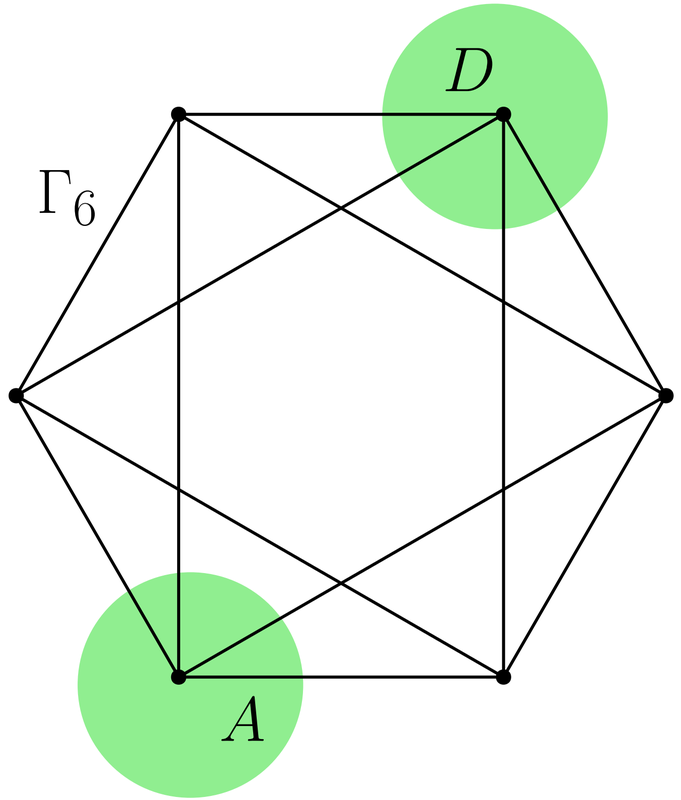

Page theorem and the entropy of Bell-network states

Summary:

- Page curve and its variance:
  - Random states have typical entropy
  - Unlikely to have maximum entropy
  - Concentration of measure
  - Vanishing correlations

- Entanglement entropy in a Bell-network state
  - Analytic asymptotics
  - Area-law from intertwiner entanglement
  - Numerical code
  - Non-vanishing intertwiner correlations