Page theorem and the entropy of Bell-network states

Eugenio Bianchi

Penn State - Institute for Gravitation & the Cosmos

http://relativity.phys.lsu.edu/ilqgs/

International Loop Quantum Gravity Seminar — Spring 2019
Correlation in loop quantum gravity

Setup:
- Hilbert space $\mathcal{H}_\Gamma$ (fixed graph)
- State $|\psi\rangle \in \mathcal{H}_\Gamma$
- Region $A \subset \Gamma$
- Subalgebra of observables $O_A \in \mathcal{A}_A$
- Correlations

$$\mathcal{C} = \langle \psi | O_A O_B | \psi \rangle - \langle \psi | O_A | \psi \rangle \langle \psi | O_B | \psi \rangle$$

This talk: Information-theoretic bounds on correlations

$$\frac{1}{2} \left( \frac{\mathcal{C}(O_A, O_B)}{\|O_A\| \|O_B\|} \right)^2 \leq S_A(|\psi\rangle) + S_B(|\psi\rangle) - S_{AB}(|\psi\rangle)$$
Plan:

- Typical entropy of a subsystem: Page curve and its variance
- Entropy and correlations in Bell-network states
Typical entropy of a subsystem

Example: spin chain
- Hilbert space \( \mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \)
- Random state \( |\psi\rangle \in \mathcal{H} \)
- Subsystem \( A \), \( d_A = \dim \mathcal{H}_A = 2^{N_A} \)
- Restricted state \( \rho_A = \text{Tr}_B(|\psi\rangle\langle \psi|) \)
- Entanglement entropy
  \[ S_A(|\psi\rangle) = -\text{Tr}_A(\rho_A \log \rho_A) \]
- Average over all states
  \[ \langle S_A \rangle = \int d\mu(|\psi\rangle) \ S_A(|\psi\rangle) \]

[Page PRL('95)] *The average entropy is close to maximal*

\[ d_B \gg 1 \quad , \quad \langle S_A \rangle \approx \log d_A - \frac{1}{d_A d_B} \frac{d_A^2 - 1}{2} \]
Typical entropy of a subsystem

Example: spin chain

- Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$
- Random state $|\psi\rangle \in \mathcal{H}$
- Subsystem $A$, $d_A = \text{dim} \mathcal{H}_A = 2^{N_A}$
- Restricted state $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$
- Entanglement entropy $S_A(|\psi\rangle) = -\text{Tr}_A(\rho_A \log \rho_A)$
- Average over all states $\langle S_A \rangle = \int d\mu(|\psi\rangle) S_A(|\psi\rangle)$

[Page PRL('95)] The average entropy is close to maximal $d_B \gg 1$, $\langle S_A \rangle \approx \log d_A - \frac{1}{d_A d_B} \frac{d_A^2 - 1}{2}$

[E.B. & P.Donà ('19)] The variance is small $\Delta S_A \approx \frac{1}{d_A d_B} \sqrt{\frac{d_A^2 - 1}{2}}$
Average entropy of a subsystem [Page PRL’95]

System: \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \)

Step 1. \( \langle \text{Tr} \rho_A^r \rangle = \int d\mu(|\psi\rangle) \text{Tr} \rho_A^r = \int \left( \sum_{a=1}^{d_A} \lambda_a^r \right) \mu(\lambda_1, \ldots, \lambda_{d_A}) \prod_{k=1}^{d_A} d\lambda_k \).

Eigenvalues of the density matrix and induced integration measure

Step 2. \( \langle S_A \rangle = -\langle \text{Tr} \rho_A \log \rho_A \rangle = - \lim_{r \to 1} \partial_r \langle \text{Tr} \rho_A^r \rangle \)

Exact result: \( \langle S_A \rangle = \psi(d_A d_B + 1) - \psi(d_B + 1) - \frac{d_A - 1}{2d_B} \)

diGamma function

Asymptotics: \( \langle S_A \rangle = \log d_A - \frac{1}{d_A d_B} \frac{d_A^2 - 1}{2} + O(1/d_B^2) \)
Variance of the entropy of a subsystem [E.B. & P. Donà (‘19)]

System: \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \)

Step 1. \( \langle \text{Tr}(\rho_A^{r_1}) \text{Tr}(\rho_A^{r_2}) \rangle = \int \left( \sum_{a=1}^{d_A} \lambda_a^{r_1} \right) \left( \sum_{b=1}^{d_A} \lambda_b^{r_2} \right) \mu(\lambda_1, \ldots, \lambda_{d_A}) \prod_{k=1}^{d_A} d\lambda_k. \)

Eigenvalues of the density matrix and induced integration measure \([\text{Lloyd} \& \text{Pagels, Ann. Phys. ('88)}]\)

Step 2. \( \langle S_A^2 \rangle = \langle \text{Tr} \rho_A \log \rho_A \rangle^2 = \lim_{r_1 \to 1} \lim_{r_2 \to 1} \partial_{r_1} \partial_{r_2} \langle \text{Tr} \rho_A^{r_1} \text{Tr} \rho_A^{r_2} \rangle \)

Exact result: \( (\Delta S_A)^2 = \langle S_A^2 \rangle - \langle S_A \rangle^2 = -\frac{(d_A-1)(d_A+2d_B-1)}{4d_B^2(d_A^2d_B+1)} + \frac{d_A+d_B}{d_A^2d_B+1} \psi'(d_B + 1) - \psi'(d_Ad_B + 1). \)

\( \psi(x) = \Gamma'(x)/\Gamma(x) \)

Asymptotics: \( (\Delta S_A)^2 = \frac{d_A^2 - 1}{2d_A^2d_B^2} + O(1/d_B^3) \)
Numerical data vs Normal distribution with given mean and variance [E.B. & P.Donà ('19)]

\[d_A = 2^4\]
\[d_B = 2^6\]

sample = 10^5
bins = 200

\[P_{n.d.}(S_A) \, dS_A = \frac{1}{\sqrt{2\pi}\sigma} \, e^{-\frac{(S_A - \mu)^2}{2\sigma^2}} \, dS_A\]

Narrow distribution for \(1 \ll N_A \leq N_B\)

\[\frac{\Delta S_A}{S_{\text{max}} - \langle S_A \rangle} \approx \sqrt{2} \, e^{-N_A \log 2}\]
Comparison to previous bounds on the support of the distribution [E.B. & P. Donà ('19)]

\[ d_A = 2^4 \]
\[ d_B = 2^6 \]

Concentration of measure bound

\[ \mathcal{P}(S_A \leq \alpha) \leq e^{-\frac{(\alpha - \mu_b)^2}{2\sigma_b^2}} \]

\[ \mu_b = \log d_A - \frac{d_A}{2d_B}, \quad \sigma_b = \frac{2\pi \log d_A}{\sqrt{d_A d_B} - 1} \]

Normal distribution
Information theoretic bound on correlations
[Wolf, Verstraete, Hastings and Cirac, PRL ('08)]

\[ C = \langle \psi | O \_A \ O \_B | \psi \rangle - \langle \psi | O \_A | \psi \rangle \langle \psi | O \_B | \psi \rangle \]

\[ \frac{1}{2} \left( \frac{C(O_A, O_B)}{\|O_A\| \|O_B\|} \right)^2 \leq S(A(\psi)) + S(B(\psi)) - S(A\_B(\psi)) \]

For a random state:

\[ \left\langle \frac{1}{2} \left( \frac{C(O_A, O_B)}{\|O_A\| \|O_B\|} \right)^2 \right\rangle \leq \left( \log d_A - \frac{1}{d_A d_B d_C} \left( \frac{d_A^2 - 1}{2} \right) \right) + \left( \log d_B - \frac{1}{d_A d_B d_C} \left( \frac{d_B^2 - 1}{2} \right) \right) - \left( \log d_A d_B - \frac{1}{d_A d_B d_C} \left( \frac{d_A^2 d_B^2 - 1}{2} \right) \right) \]

\[ = O(2^{-N_C}) \]

Correlations are exponentially suppressed in a random state
Correlation in loop quantum gravity

Correlations are 
*exponentially suppressed* 
in a random state

Two caveats:

1. Constraint on physical states

2. $\mathcal{H}_\Gamma$ is not a tensor product over nodes 
and not finite dimensional 
(because of the sum over spins)
Correlation in loop quantum gravity

Correlations are exponentially suppressed in a random state

Two comments:

1. Constraint on physical states
2. $\mathcal{H}_\Gamma$ is not a tensor product over nodes and not finite dimensional (because of the sum over spins)

Entropy of a subalgebra of observables

[Casini, Huerta, Rosabal *PRD* (’14)]
[E.B., Satz *PRD* (’19)]

Eigenstate entanglement in the XY model

[E.B., Hackl, Rigol, Vidmar, *PRL* (’17), *PRL* (’18), *PRB* (’19)]
Plan:

- Typical entropy of a subsystem: Page curve and its variance
- Entropy and correlations in Bell-network states
* Fluctuations of nearby quantum polyhedra are in general uncorrelated (twisted geometry):
e.g.: in a spin-network basis state, and in a random state

**Bell-network states:**

use squeezed vacua techniques [E.B., Hackl, Guglielmon, Yokomizo PRD ('16)]
to correlate back-to-back normals

- on links  $|\mathcal{B}, \lambda \rangle_\ell = \left(1 - |\lambda|^2\right) \exp\left(\lambda \epsilon_{\alpha\beta} a^\dagger_\alpha a_\beta\right) |0\rangle_s |0\rangle_\ell$  
  with  $\lambda \in \mathbb{C}$  encoding average area and extr. curv.

- projections:  $|\Gamma, \lambda_\ell, \mathcal{B}\rangle = P_\Gamma \otimes_{\ell \in \Gamma} |\mathcal{B}, \lambda_\ell\rangle_\ell$

- result:  $|\Gamma, \lambda_\ell, \mathcal{B}\rangle = \sum_{j_\ell} \left(\prod_{\ell} \left(1 - |\lambda_\ell|^2\right) \lambda^2_{2j_\ell} \sqrt{2j_\ell + 1}\right) |\Gamma, j_\ell, \mathcal{B}\rangle$

superposition over spins of entangled interwiner state  $|\Gamma, j_\ell, \mathcal{B}\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} A_\Gamma (j_\ell, i_n) \otimes_n i_n$  
where  $A_\Gamma (j_\ell, i_n) = \sum m_n [i_n]^{m_1 \cdots m_p}$  is the symbol of the graph
Entanglement entropy of Bell-network states
[E.B., Donà, Vilensky (’19)]

- Hilbert space decomposition

$$\mathcal{H}_\Gamma = \bigoplus_{j_\ell} \bigotimes_n \mathcal{H}_{i_n}$$

- Subalgebra in $A$

- State $|\Gamma, \lambda_, B\rangle = \sum_{j_\ell} q_{j_\ell} |\Gamma, j_\ell, B\rangle$

- Entropy

$$S_A = - \sum_{j_\ell} p_{j_\ell} \log p_{j_\ell} + \sum_{j_\ell} p_{j_\ell} S_A(j_\ell)$$

easy:

Shannon entropy of $p_{j_\ell} = |q_{j_\ell}|^2 / \sum_{j_\ell} |q_{j_\ell}|^2$

non-trivial:

entropy of the entangled interwiner state

$$|\Gamma, j_\ell, B\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} ^{\mathcal{A}_\Gamma (j_\ell, i_n)} \bigotimes_n |i_n\rangle$$
Large spin asymptotics of the entanglement entropy

- Entangled intertwiner state

$$|\Gamma, j_\ell, \mathcal{B}\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} A_\Gamma (j_\ell, i_n) \otimes |i_n\rangle$$

- Using the replica trick

$$\text{Tr} (\rho_A^p) = \frac{\int d\tilde{n} \, d\tilde{g} \, e^{|f_p(j, \tilde{n}, \tilde{g})|}}{\left( \int d\tilde{n} \, d\tilde{g} \, e^{|f_1(j, \tilde{n}, \tilde{g})|} \right)^p} \quad \text{with} \quad \rho_A = \text{Tr}_A |\Gamma, j_\ell, \mathcal{B}\rangle \langle \Gamma, j_\ell, \mathcal{B}|$$

- Using spinfoam-asymptotics methods [Donà, Fanizza, Sarno, Speziale CQG ('18)]

$$j_\ell \rightarrow \lambda j_\ell \quad \text{with} \quad \lambda \gg 1$$

- Renyi Entropy

$$R_A^{(p)} = -\frac{1}{p-1} \log \text{Tr} \rho_A^p$$

- Entanglement Entropy

$$S_A = \lim_{p \to 1} R_A^{(p)}$$

Asymptotic bound on the entanglement entropy

$$\left( |\partial A| - \frac{C_A^{(2)}}{2} \right) \log \lambda \leq S_A \leq (|\partial A| - 3) \log \lambda$$

Area-law from intertwiner entanglement
Algorithm 1 Numerical algorithm for the evaluation of the Bell-network state entropy

1: Precompute the \( \{6j\} \) symbols with \texttt{wigxjpf} if needed
2: Precompute the \( \{9j\} \) symbols with \texttt{wigxjpf} if needed
3: Load the symbol tables in the memory
4: for each \( j \) do
5: for each \( i \) in \( \bar{A} \) do
6: Assemble the matrix \( \left( A_{ij} \right)_{ik} = A_{ij} \left( j, i, k \right) \) with the intertwiners \( i, k \) in \( A, \bar{A} \)
7: Compute the matrix \( M_{i'i'} = \left( A_{ij}^T \cdot A_{ij} \right)_{i'i'} \)
8: Normalize it to obtain the density matrix \( \left( \rho_A \right)_{i'i'} = M_{i'i'}/Tr\left( M \right) \)
9: Find its eigenvalues \( \rho_A \rightarrow \nu_i \)
10: Compute the entanglement entropy \( S_A = -\sum_i \nu_i \log \nu_i \)
11: Compute the Rényi entropy \( R_A^{(p)} = -\log \sum_i \nu_i^p \)

Code available at:  
https://bitbucket.org/pietrodona/bellnetworkentropy
Numerical results: [E.B., Donà, Vilensky ('19)]
Numerical results: [E.B., Donà, Vilensky ('19)]
Numerical results: [E.B., Donà, Vilensky ('19)]
Numerical results: [E.B., Donà, Vilensky ('19)]
Numerical results: [E.B., Donà, Vilensky ('19)]

Typical state bound on correlations

\[ I(A, B) = S_A + S_B - S_{AB} \approx \frac{1}{2\lambda} \]

Bell-network state bound on correlations

\[ I(A, B) = S_A + S_B - S_{AB} \approx 0.06 \log \lambda \]
Page theorem and the entropy of Bell-network states

Summary:

- Page curve and its variance:
  - Random states have typical entropy
  - Unlikely to have maximum entropy
  - Concentration of measure
  - Vanishing correlations

- Entanglement entropy in a Bell-network state
  - Analytic asymptotics
  - Area-law from intertwiner entanglement
  - Numerical code
  - Non-vanishing intertwiner correlations

![Graph showing the typical state bound on correlations and the Bell-network state bound on correlations.](image)

Typical state bound on correlations

\[ I(A, B) = S_A + S_B - S_{AB} \sim \frac{1}{2\lambda} \]

Bell-network state bound on correlations

\[ I(A, B) = S_A + S_B - S_{AB} \approx 0.06 \log \lambda \]