Black hole entropy from loop quantum gravity:

Generalized theories and higher dimensions

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based on work by NB, Thiemann, Thurn [arXiv:1304.2679]
NB, Neiman [arXiv:1304.3025]
NB [arXiv:1307.5029, to appear in PLB]

International Loop Quantum Gravity Seminar

October 1, 2013

Plan of the talk

1. Entropy calculation: Basic ingredients (in 3+1 dimensions)
2. Expectations for higher dimensions
3. Results in higher dimensions: Classical GR
4. Quantization
5. Generalized theories
6. Discussion / Remarks
7. Omitted points / further research
8. Conclusions

Outline

Entropy calculation: Basic ingredients

[Smolin '95; Krasnov '96; Rovelli '96; Ashtekar, Baez, Corichi, Krasnov '97-; ...; Engle, Noui, Perez '09-; ...]

Isolated horizon boundary of spacetime

Connection variables $\rightarrow$ boundary degrees of freedom

Idea: Count boundary degrees of freedom in agreement with total area

Important observation for BH entropy from LQG:

$$S_{\text{BH}} \propto \frac{1}{\gamma} A_H$$

γ = Barbero-Immirzi parameter, A_H = horizon area

Ingredients on horizon slice H

- Boundary symplectic structure $\int_H \delta_1 A^i \wedge \delta_2 A_i$
- Boundary condition $F^i(A) = \Sigma^i(E)$
- Area spectrum $8\pi G\gamma\sqrt{j(j+1)}$

Higher dimensions: Results in short

Known:

- Isolated horizon (IH) framework extendable to higher dimensions
  [Lewandowski, Pawlowski gr-qc/0410146; Korzynski, Lewandowski, Pawlowski gr-qc/0412108]
  [Ashtekar, Pawlowski, v. d. Broeck gr-qc/0611049; Liko, Booth 0705.1371]
- LQG extendable to higher dimensions
  [NB, Thiemann, Thurn 1106.1103]

New:

- Boundary symplectic structure on IH can be derived
- Boundary condition can be derived
- Quantum theory can be formulated
- State counting problem can be reduced almost to the $3 + 1$ dim. one
- Dimension independent log. correction in accordance with Carlip
- Extendable to Lanczos-Lovelock gravity and non-minimal coupling of scalars

Outline

Going to higher dimensions: Canonical variables

$3+1$ dimensions: Bulk variables

SU(2) connection $\boxed{{}^{(\gamma)}A_{ai}}$, densitized triad $\boxed{{}^{(\gamma)}E^{bj}}$, $\{A_{ai}, E^{bj}\} = \delta_a^b \delta_i^j$

${}^{(\gamma)}E^{ai}\,{}^{(\gamma)}E^{bi} = 1/\gamma^2 qq^{ab}$ $\gamma \in \mathbb{R}$ Barbero-Immirzi parameter, ${}^{(\gamma)}A_{ai} = \Gamma_{ai} + \gamma K_{ai}$, $a, b = 1, 2, 3$, $i, j = 1, 2, 3$

$\downarrow$ $\downarrow$

$D+1$ dimensions: Bulk variables

SO($D+1$) connection $\boxed{A_{aIJ}}$, densitized hybrid vielbein $\boxed{\pi^{bKL}}$, $\{A_{aIJ}, \pi^{bKL}\} = \delta_a^b \delta_{IJ}^{KL}$

$\pi^{aIJ}\pi^{b}{}_{IJ} = 1/\beta^2 2qq^{ab}$, $\beta \in \mathbb{R}$ free parameter, $A_{aIJ} = \Gamma_{aIJ} + \beta\, 2n_{[I}K_{a|J]} + \ldots$, n^I = internal normal

$a, b = 1, \ldots, D$, $I, J = 0, \ldots, D$

$\downarrow$ $\downarrow$

Holonomies from 1-forms

$$\int_c A_{aIJ}\,\tau^{IJ} dx^a$$

c = curve, τ^{IJ} = generators of SO($D+1$),

Fluxes from $D-1$ forms

$$\int_S \pi^{aIJ}\, n_{IJ} \epsilon_{ab_1 \ldots b_{D-1}} dx^{b_1} \wedge \ldots \wedge dx^{b_{D-1}}$$

$S = D-1$ surface, n_{IJ} smearing functions

$\rightarrow$ Holonomy-Flux algebra with $SO(D+1)$ structure

Going to higher dimensions: Constraints

List of constraints (all first class before using holonomies / fluxes)

- Hamiltonian constraint
- spatial diffeomorphism constraint
- Gauß constraint (solved by using spin networks)
- Simplicity constraint $\pi^{a[IJ}\pi^{b|KL]} = 0$ (no torsion constraints, gauge unfixing)

$\rightarrow$ Most interesting here: Simplicity constraint.

Action of spin network edges:

- Selects "simple" $SO(D+1)$ representations [Freidel, Krasnov, Puzio hep-th/9901069]
- Labelled by single non-negative integer λ $(\lambda = 1 \Leftrightarrow j^+ = j^- = 1/2)$

Action of spin network vertices:

later... important for entropy calculation

Going to higher dimensions: Canonical analysis

Start with Palatini action: $\mathrm{SO}(1,D)$ connection A_{IJ} and $D+1$-bein e^I.

D+1 split of the Palatini action

$$\int *(e^I \wedge e^J) \wedge F_{IJ}(A) \Leftrightarrow \int_{\mathbb{R}} dt \int_\sigma d^D x \left(\frac{1}{2}\pi^{aIJ} \dot{A}_{aIJ} - N\mathcal{H} - N^a \mathcal{H}_a - \frac{1}{2}\lambda_{IJ} G^{IJ} - c^{\overline{M}}_{ab} S^{ab}_{\overline{M}} \right)$$

$\mathcal{H}$ = Hamiltonian constraint, $\mathcal{H}_a$=spatial diffeomorphism constraint, G^{IJ}=Gauss constraint, $S^{ab}_{\overline{M}}$=simplicity constraint

π^{aIJ}=momentum of A_{aIJ}, σ = spatial slice, N=lapse function, N^a = shift vector, $\overline{M} = M_1 \dots M_{D-3}$ = multiindex

$n^I e^a_I = 0$, $\sqrt{q} = \sqrt{\det q_{ab}}$, $q_{ab} = e^I_a e^J_b \eta_{IJ}$ = spatial metric, $a = 1, \dots, D$ spatial tensorial indices

$$S^{ab}_{\overline{M}} = \pi^{aIJ}\pi^{bKL}\epsilon_{IJKL\overline{M}} = 0 \quad \Rightarrow \quad \pi^{aIJ} = 2n^{[I}\sqrt{q}e^{aJ]}$$ [Freidel, Krasnov, Puzio hep-th/9901069]

Apply Dirac's stability algorithm [Peldan gr-qc/9305011]

Everything works, but one more constraint appears: $\{\mathcal{H}, S^{ab}_{\overline{M}}\} = D^{ab}_{\overline{M}}$.

Second class partner for the simplicity constraint: $\{S^{ab}_{\overline{M}}, D^{cd}_{\overline{N}}\}$ = invertible

Remove $D^{cd}_{\overline{N}}$ via gauge unfixing. $\rightarrow$ Modify Hamiltonian constraint for consistency.

Perform canonical transformation to $\mathrm{SO}(D+1)$ as gauge group.
Works because ADM phase space of Lorentzian and Euclidean gravity is the same.

Going to higher dimensions: Area operator

The usual construction of the area operator **generalizes directly**

Idea: area operator $\sim \sqrt{\text{flux}^2}$ integrated over $(D-1)$-surface

Area operator diagonal on edges labelled with λ:

Eigenvalues $8\pi G\beta\sqrt{\lambda(\lambda+D-1)}, \quad \lambda \in \mathbb{N}$

First expectation for entropy:

- Calculation goes through as in $3+1$ dimensions
- Horizon is modeled as a boundary of the spatial slice
- We count different possibilities to distribute the horizon area
- Area gap ensures finite entropy
- Arguments for statistics of the punctures are the same in higher dimensions
- (Actual calculation of course more detailed...)

"Dimensional" analysis

Boundary condition in $3+1$ dim.: $F^i(A) = \Sigma^i(\mathrm{E})$

- Boundary: Curvature **2-form** $F(A)^i$
- Bulk: **2-form** Σ^i build from densitized triad as $\Sigma^i_{ab} = E^{ci}\epsilon_{abc}$

Higher dimensions

- Boundary: Curvature **2-form** $F(A)^{IJ}$
- Bulk: $(D-1)$-**form** $\pi^{cIJ}\epsilon_{a_1\ldots a_{D-1}c}$ $\quad\rightarrow$ mismatch in tensor structure

Two possibilities:

- Modify form of boundary condition

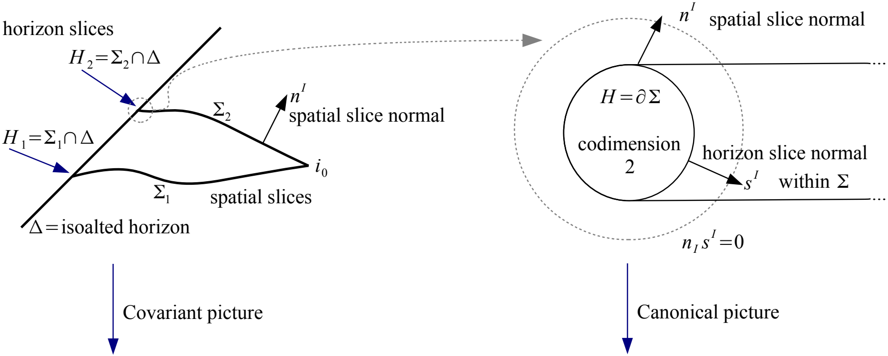
  $\epsilon\pi^{IJ} \sim (F(A) \wedge \ldots \wedge F(A))^{IJ}$ (with appropriate internal index contraction)

- Use different variables on the horizon (i.e. not a connection)

  $\epsilon\pi^{IJ} = L^{IJ}\underset{\Leftarrow}{\epsilon}$ ($L^{IJ}\underset{\Leftarrow}{\epsilon}$ = internal horizon bi-normal as $(D-1)$-form)

Outline

Geometry of the problem: Two pictures



Show that action principle is well defined using the isolated horizon boundary condition on Δ.

Perform canonical transformations to e.g. Chern-Simons type variables.
Switch gauge group to SO(D+1).

Action principle and covariant canonical framework

Start with the $D+1$-dimensional Lorentzian Palatini action

$$S = S[A,e] = \int \Sigma^{IJ}(e) \wedge F(A)_{IJ}, \qquad \Sigma^{IJ} \sim *(e^I \wedge e^J)$$

No additional boundary contribution on Δ.

Use that Δ is a non-rotating isolated horizon

$$\Rightarrow \quad \int_\Delta \underset{\leftarrow}{\Sigma}_{IJ} \wedge \delta \underset{\leftarrow}{A}^{IJ} = 0, \qquad \Rightarrow \text{ usual equations of motion are enforced}$$

Second variation on Δ reduces to a boundary contribution on H_2, H_1:

$$\int_\Delta \delta_{[1} \underset{\leftarrow}{\Sigma}^{IJ} \wedge \delta_{2]} \underset{\leftarrow}{A}_{IJ} = \left(\int_{H_2} - \int_{H_1} \right) \text{Boundary symplectic structure}$$

$\rightarrow$ Leads to a boundary contribution in the symplectic structure.

Boundary symplectic structure and boundary condition

Bulk variables:

SO$(D+1)$ connection $\boxed{A_{aIJ}}$, densitized hybrid vielbein $\boxed{\pi^{bKL}}$, $\{A_{aIJ}, \pi^{bKL}\} = \delta_a^b \delta_{IJ}^{KL}$

n^I = internal normal, $\pi^{aIJ} \approx 2/\beta n^{[I} E^{aJ]}$, $s^I = s^a e_a^I$ = horizon slice normal, $\tilde{s}^I = \sqrt{h}\, s^I$, $\sqrt{h}$ = area density on H

Boundary symplectic structure

3+1: $\int_H \delta_{[1} e^i \wedge \delta_{2]} e_i \;\propto\; \int_H \delta_{[1} A^i \wedge \delta_{2]} A_i$

$$\int_H \delta_{[1} \tilde{s}^I \delta_{2]} n_I \quad \propto \quad \frac{A_H}{\chi\ \beta} \int_H \epsilon^{IJKLM_2N_2 \ldots M_nN_n} \left(\delta_{[1} {\Gamma^0}_{IJ}\right) \wedge \left(\delta_{2]} {\Gamma^0}_{KL}\right) \wedge {R^0}_{M_2N_2} \wedge \ldots \wedge {R^0}_{M_nN_n}$$

$Dn^I = Ds^I = D \underset{\Leftarrow}{e}{}^I = 0$, $D = \partial + \Gamma^0$, $R^0_{IJ} = R(\Gamma^0)_{IJ}$, A_H = horizon area, χ = Euler characteristic of H, $n = (D-1)/2$

Boundary condition

3+1: $F^i(A) = \Sigma^i(\mathrm{E})$

$$\hat{s}_a \pi^{aIJ} \underset{\Leftarrow}{\epsilon} \quad \propto \quad \frac{2}{\beta} n^{[I} \tilde{s}^{J]} \underset{\Leftarrow}{\epsilon} \quad \propto \quad \frac{A_H}{\chi\ \beta} \epsilon^{IJK_1L_1 \ldots K_nL_n} {R^0}_{K_1L_1} \wedge \ldots \wedge {R^0}_{K_nL_n}$$

Two choices of boundary variables

- Connection (Chern-Simons-theory): hard, local DOF for $D > 3$, $D+1$ even
- Metric variables n^I, $\tilde{s}^I$: easier to handle, dimension independent treatment

Outline

Higher dimensions: Problems with Chern-Simons theory

Symplectic structure

$$\frac{A_H}{\chi\ \beta}\int_H \epsilon^{IJKLM_2N_2\dots M_nN_n}\left(\delta_{[1}\Gamma^0{}_{IJ}\right)\wedge\left(\delta_{2]}\Gamma^0{}_{KL}\right)\wedge R^0{}_{M_2N_2}\wedge\dots\wedge R^0{}_{M_nN_n}$$

$\rightarrow$ In general very complicated Poisson brackets

Boundary condition: area-related degrees of freedom

$$\hat{s}_a\pi^{aIJ}\overset{BC}{=}L^{IJ}:=2/\beta n^{[I}\tilde{s}^{J]}\propto\epsilon^{IJI_2J_2\dots I_nJ_n}R^0_{I_2J_2}\wedge\dots\wedge R^0_{I_nJ_n}$$

$$\left\{L^{IJ}(x),L^{KL}(y)\right\}=4\,\delta^{(D-1)}(x-y)\,\delta^{L][J}L^{I][K}(x)$$

Same result from bi-normal symplectic structure and Chern-Simons symplectic structure

Other phase space functions in the Chern-Simons theory seem physically irrelevant for the entropy computation. Remove them with stronger boundary condition?

Problem is avoided from the beginning when sticking to bi-normals as variables.

Higher dimensions: Quantization

Smeared binormals: Algebra of fluxes

Use $\boxed{L^{IJ}_S = \int_S 2/\beta n^{[I}\tilde{s}^{J]}}$ $\quad \{L^{IJ}_S, L^{KL}_S\} = 4\,\delta^{L][J} L^{I][K}_S$ $\quad \rightarrow$ so$(D+1)$ Lie algebra

SU(2) case: [Engle, Noui, Perez, Pranzetti 1006.0634]

Boundary Hilbert space

Product of SO$(D+1)$ representation spaces

$\rightarrow$ Non-trivial at points where bulk spin network punctures boundary

$\rightarrow$ Related to bulk rep. by boundary condition $\boxed{\hat{s}_a \pi^{aIJ} = L^{IJ}}$

Off-diagonal horizon simplicity constraints

Non-rotating isolated horizon $\rightarrow$ Off-diagonal simplicity constraints $\boxed{L^{[IJ}_{S_1} L^{KL]}_{S_2} \approx 0}$

$\underset{\Leftarrow}{D} k^I = 0$ and $L^{IJ} \propto l^{[I} k^{J]}$, $\quad k^I \propto (n^I + s^I)$, demand off-diagonal simplicity on two contractible charts

$\rightarrow$ Breaks local gauge invariance to global invariance on H.

Locally covariant quantization in the context of Chern-Simons theory?

Higher dimensions: Quantization

Restrictions on horizon Hilbert space

Gauge invariance / tracing: SO$(D+1)$ intertwiner — See also [Rovelli, Krasnov 0905.4916]

Bulk simplicity: Simple representations of SO$(D+1)$ on H, label $\lambda \in \mathbb{N}$
[Freidel, Krasnov, Puzio hep-th/9901069]

Off-diagonal simplicity: Simple intertwiner (Intertwining repr. simple)

1 to 1 mapping of simple SO$(D+1)$ and SU(2) intertwiners

→ Using dimension formulas from SU(2) counting:
$$S = \mathrm{const(D)}\frac{A_H}{\beta} - \frac{3}{2}\mathrm{Log}\,A_H$$

→ (Up to β) Same result as Carlip and Solodukhin using CFT methods
[Carlip hep-th/9812013, gr-qc/0005017; Solodukhin hep-th/9812056; log correct.: Kaul and Majumdar gr-qc/0002040]

- Compatible with generalized theories — $S \propto S_{\text{Wald}}$
- Compatible with analytic continuation of β — $S = A_H/(4G) + \text{corrections}$
  [Frodden, Geiller, Noui, Perez 1212.4060]

Outline

Generalized gravity theories: Wald entropy

$$S_{\text{Generalized}} = \int \sqrt{-g}\,\mathcal{L}$$

$$\mathcal{L} = \mathcal{L}\left(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_{\xi_1} R_{\mu\nu\rho\sigma}, \ldots, \nabla_{(\xi_1} \ldots \nabla_{\xi_n)} R_{\mu\nu\rho\sigma}, \psi, \nabla_{\xi_1}\psi, \ldots, \nabla_{(\xi_l} \ldots \nabla_{\xi_l)}\psi\right)$$

Entropy from classical first law [Wald gr-qc/9307038]

$$S_{\text{Wald}} = \frac{1}{4G}\int_H \sqrt{h}\,\frac{-\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} \neq \frac{A_H}{4G}$$

$\mathcal{L}$ = Lagrangian, $\sqrt{h}$ = area density on H

$\epsilon_{\mu\nu} = 2n_{[\mu}s_{\nu]}$ = horizon slice bi-normal

Here:

Restrict to GR phase space plus standard matter (no higher time derivatives):

- Lanczos-Lovelock gravity plus non-minimally coupled scalars
- Presentation in $3+1$, works also in higher dimensions

Generalized gravity theories

Pure GR

The connection and the momentum both have standard geometric interpretation!

$$A_{ai} = \Gamma_{ai} + \gamma K_{ai}, \qquad 1/\gamma^2 q q^{ab} = E^{ai} E_i^b, \qquad \{A_{ai}, E^{bj}\} = \delta_a^b \delta_i^j$$

Γ_{ai} = spin connection, K_{ai} = extrinsic curvature, γ = Barbero-Immirzi parameter, q^{ab} = spatial metric

Generalized theory

The momentum P^{ai} conjugate to $A_{ai} = \Gamma_{ai} + \gamma K_{ai}$ is not the densitized triad E^{ai}!

$$P^{ai} \propto \frac{\partial \mathcal{L}}{\partial \dot{A}_{ai}} \qquad \Rightarrow \quad \{A_{ai}, P^{bj}\} = \delta_a^b \delta_i^j$$

e.g. non-minimally coupled scalar: $\mathcal{L} = a(\Phi) R + \ldots \quad \Rightarrow \quad P^{ai} = a(\Phi) E^{ai}$

(More discussion on this in the Loops13 talk, available at pirsa.org)

Area $\to$ Wald entropy

$\sqrt{(\hat{s}_a P^a)^2}$ = Wald entropy density on horizon slice H

$$\hat{s}_a\, P^{ai} \propto \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \times \hat{s}_a\, E^{ai}$$

$$\propto \frac{\text{Wald entropy density}}{\text{area density}} \times \underbrace{\hat{s}_a\, E^{ai}}_{\text{vector-valued area density}}$$

$\epsilon^{\mu\nu}$ = binormal on horizon slice, $\mathcal{L}$ undensitized Lagrangian, $\hat{s}_a$ = horizon slice co-normal

Generalized area operator

Idea: $\widetilde{\text{Area}} \propto \int \sqrt{|P|^2}$

$$\Rightarrow \text{Spec}(\widehat{\widetilde{\text{Area}}}) = \gamma \sqrt{j(j+1)}, \quad j \in \mathbb{N}_0/2$$

Generalized area density $\propto$ Wald entropy density

Isolated horizon framework

Calculations as before, just with Wald entropy instead of area

(Horizon connection build from **some** $(D+1)$-bein with area density $\sim$ Wald entropy density)

Outline

Discussion / Remarks

- Area operator $\rightarrow$ "Wald entropy operator" (on isolated horizon only)
  - The only operator from which we know that is has an "easy" spectrum measures Wald entropy. Interpretation of spin networks?
  - Twisted geometry interpretation for generalized theories?

    Faces labelled by entropy, not area

- Quantization of Wald entropy expected from general arguments

  [Bekenstein gr-qc/9710076; Kothawala, Padmanabhan, Sarkar 0807.1481]

- Most important ingredient in the horizon theory: **area density**

  Lots of freedom for canonical transformations, different connections, different free parameter on horizon, ...

Outline

Omitted points / further research

Omitted points:

- Ambiguity in the choice of horizon connection variables

  See Loops13 talk available at pirsa.org and [NB, Stottmeister, Thurn 1203.6525, NB, Neiman 1304.3025]

- Polyhedral interpretation

  Generalizing [Bianchi, Dona', Speziale 1009.3402] to higher dimensions

Further research:

- Issue of obtaining prefactor $1/4$ appears also in higher dimensions

  See [Gosh, Frodden, Perez 11-; Frodden, Geiller, Noui, Perez 1212.4060; NB, Stottmeister, Thurn 1203.6525, NB, Neiman 1303.4752, Pranzetti 1305.6714]

- Quantization of higher-dim. Chern-Simons theory on boundary

  More rigorous quantization? Gauge invariance and simplicity constraint?

- Extension to generic isolated horizons

  (Non-rotating condition used only on the covariant side and for Wald entorpy)

- Topology corrections

  $3+1$ dim. [Kloster, Brannlund, DeBenedictis: gr-qc/0702036]

- Supergravity

  Isolated horizon: [Liko, Booth 0712.3308]

Outline

Summary

1. **Classical** results:
   - Start: Non-rotating isolated horizon
   - Action principle well defined
   - Boundary symplectic structure
   - Boundary condition
2. **Generalized theory** of gravity:
   - $\sqrt{\text{flux}^2} \propto$ Wald entropy density

     $\Rightarrow$ Entropy $S \propto S_{\text{Wald}}$
3. **Quantization**
   - Chern-Simons theory description possible, but hard to quantize
   - Largely dimension-independent result from using densitized bi-normals
   - Useful for studying how to impose simplicity constraints

Thank you for your attention!

Non-rotating isolated horizon

Definition

A sub-manifold Δ of (M, g) is said to be a non-expanding horizon (NEH) if

1. Δ is topologically $\mathbb{R} \times H$ and null.
2. Any null normal l of Δ has vanishing expansion $\theta_l := h^{\mu\nu} \nabla_\mu l_\nu$
3. All field equations hold at Δ and $-T^\mu_\nu l^\nu$ is a future-causal vector for any future directed null normal l.

Definition

A pair $(\Delta, [l])$, where Δ is a NEH and $[l]$ an equivalence class of null normals, is said to be a weakly isolated horizon (WIH) if for any $l \in [l]$

4. $\mathcal{L}_l \omega \hat{=} 0$. $\quad (\nabla_{\underleftarrow{\mu}} l^\nu \hat{=} \omega^l_\mu l^\nu)$

Definition

1 A non-rotating isolated horizon (NRIH) is a WIH where to each $l \in [l]$ there is a k with the property "good foliation" (see paper for details), such that

5. k is shear-free with nowhere vanishing spherically symmetric expansion and vanishing Newman - Penrose coefficients $\pi_J \hat{=} l^\mu m^\nu_J \nabla_\mu k_\nu$ on Δ.