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An Invitation to the New Variables with Possible Applications

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(work by NB, T. Thiemann, AT [arXiv:1106.1103])

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Plan of the talk

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Why Higher Dimensional Loop Quantum (Super-)Gravity?

Quantum Gravity:

- Perturbative: Superstring theory / M-theory (ST / MT), require
  - Additional particles
  - Supersymmetry
  - Higher dimensions
- Non-perturbative: Loop Quantum Gravity
  - Various matter couplings & SUSY possible
  - 3+1 dimensions (Ashtekar Barbero variables) [however, Melosch, Nicolai '97; Nieto '04, '05]
  - What if LHC finds evidence for higher dimensions?

→ Make contact between them? [Thiemann '04; Fairbairn, Noui, Sardelli '09, '10]

- Compare results in 3+1 dimensions:
  Landscape problem: Dimensional reduction of ST / MT highly ambiguous
- Compare results in higher dimensions: Starting points:
  - Higher dimensional Supergravities
    - ★ are considered as the low-energy limits of ST / MT
    - ★ have action of the type S_{GR} + more
  - Symmetry reduced models (higher dim. & SUSY black holes or cosmology)

→ Extend loop quantisation programme to higher dimensions and Supergravities

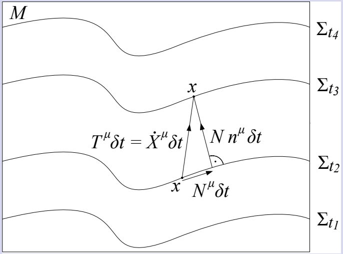
[Jacobson '88; Fülöp '93; Armand-Ugon, Gambini, Obrégon, Pullin '95; Ling, Smolin '99-; Sawaguchi '01; Smolin '05,...]

Plan of the talk

ADM Formulation [Arnowitt, Deser, Misner '62]

D+1 split

- **Foliation of $\mathcal{M}$:**
  $\mathcal{M}$ top. $\mathbb{R} \times \sigma$, $\quad \Sigma_t = X_t(\sigma)$, $\quad X_t : \sigma \to \mathcal{M}$
- Important fields on σ:
  Lapse, Shift: N, N^a
  Spatial metric $q_{ab} = (X^* g)_{ab}$,
  Extrinsic curvature $K_{ab} = (X^* \mathcal{L}_n q)_{ab}$
  $= \frac{1}{N}(\dot{q}_{ab} - (\mathcal{L}_{\vec{N}} q)_{ab})$



$$\Rightarrow \quad S_{EH} = \int dt \int_\sigma d^D x \; N \sqrt{\det q} \; (R^{(D)} \pm [K_{ab} K^{ab} - (K_a{}^a)^2]) \qquad [a, b = 1, \ldots, D]$$

ADM phase space Γ

- **Canonical variables:** q_{ab}, P^{ab} ($\sim$ extrinsic curvature K_{ab})
- **Poisson brackets:** $\quad \{q_{ab}(x), P^{cd}(y)\}_{ADM} = \delta^c_{(a} \delta^d_{b)} \delta^{(D)}(x, y)$
- 1^{st} **class constraints:**

  Totally constrained Hamiltonian: $H = \int_\sigma d^D x (N\mathcal{H} + N^a \mathcal{H}_a)$
  Spatial diffeomorphism constraint $\mathcal{H}_a(q, P)$
  Hamiltonian constraint $\mathcal{H}(q, P) = \pm\sqrt{\det q} \; R^{(D)} + \frac{1}{\sqrt{\det q}}[P_{ab} P^{ab} - \frac{1}{D-1}(P_a{}^a)^2]$

Extended ADM I

Extension of ADM phase space I

- Introduce SO(D)-valued vielbein:

$$q_{ab} = e_a{}^i e_b{}^j \; \delta_{ij} \qquad K_{ab} = K_{ai} e_b^i \qquad E^{ai} = \sqrt{\det q} \; e^{ai} \quad i, j, \ldots \in \{1, \ldots, D\} \tag{1}$$

- Poisson bracket relations: $\{E^{ai}, K_{bj}\} = \delta_b^a \delta_j^i$
- Increased number of degrees of freedoms $\Rightarrow$ new constraint needed:

$$K_{[ab]} = 0 \quad \Leftrightarrow \quad K_{[a}{}^i e_{b]i} = 0 \quad \Leftrightarrow \quad G_{ij} := K_{a[i} E^a{}_{j]} = 0 \tag{2}$$

Valid extension?

- ADM Possion bracket relations reproduced on extended phase space

$$\{q_{ab}(E), P^{cd}(E,K)\}|_{G=0} = \{q_{ab}, P^{cd}\}_{ADM} = \delta^c_{(a}\delta^d_{b)}\delta^{(D)}(x,y) \tag{3}$$

- New constraints close amongst themselves: $\{G, G\} \sim G$
- $q_{ab}(E), P_{cd}(E,K)$ (and in particular $\mathcal{H}, \mathcal{H}_a$) are Dirac observables w.r.t. new constraint G_{ij}

$\Rightarrow$ $\mathcal{H}, \mathcal{H}_a$ and G_{ij} constitute 1^{st} class constraint algebra by construction

Ashtekar-Barbero Formulation

Canonical transformation to Ashtekar-Barbero variables [Sen; Ashtekar; Immirzi; Barbero]

- Introduce spin connection $\Gamma^{SPIN}_{aij}[e]$ s.t. $\partial_a e_{bi} - \Gamma^c_{ab} e_{ci} + \Gamma^{SPIN}_{aij}[e] \; e_b{}^j = 0$
- Crucial: Defining and adjoint representation of SU(2) equivalent!
- *Only* in $D = 3$: *Canonical* transformation

$$\{E^{ai}, K_{bj}\} \longrightarrow \{\frac{1}{\gamma} E^{ai}, A_{bj} := 1/2 \; \epsilon_j{}^{kl} \; \Gamma^{SPIN}_{bkl}[e] + \gamma K_{bj}\} \qquad \gamma \in \mathbb{R}/\{0\}\text{: Immirzi Parameter} \tag{4}$$

$\Rightarrow$ Simple Poisson algebra $\{A, E\} \sim 1$ and 1^{st} class constraint algebra

- Canonicity of the above transformation *non-trivial*
- New constraint $G_{ij} = K_{a[i} E^a{}_{j]} \quad \Rightarrow \quad$ SU(2) Gauß law constraint:

$$\begin{aligned} G_{ij} &= \gamma K_{a[i} \, \frac{1}{\gamma} E^a{}_{j]} + \frac{1}{2\gamma} \, \epsilon_{ij}{}^k (\partial_a E^a{}_k + \Gamma^{SPIN}_{akl}[e] E^{al}) \\ &= \frac{1}{2\gamma} \, \epsilon_{ij}{}^k (\partial_a E^a{}_k + \epsilon_k{}^{lm} A_{al} E^a{}_m) \end{aligned} \tag{5}$$

Higher dimensions?

No obvious way of combining K_{ai} and $\Gamma^{SPIN}_{aij}[e]$ to a connection conjugate to E^{bj} in a mathematically sensible way!

Extended ADM II

Extension of ADM phase space II

- Introduce $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ *"hybrid"* vielbein:

$$q_{ab} = e_a{}^I e_b{}^J \eta_{IJ} \qquad K_{ab} = K_{aJ} e_b^J \qquad E^{aJ} = \sqrt{\det q}\; e^{aJ} \quad I, J, \ldots \in \{0, 1, \ldots, D\} \qquad (6)$$

- Motivation: 2^{nd} order Palatini formulation of General Relativity
- Poisson bracket relations: $\{E^{aI}, K_{bJ}\} = \delta^a_b \delta^I_J$
- New constraints: $K_{[ab]} = 0 \quad \Leftrightarrow \quad K_{[a}{}^I e_{b]I} = 0$ insufficient! Use

$$\Leftarrow \quad G^{IJ} := K_a{}^{[I} E^{aJ]} \qquad (7)$$

- Proof of validity of extension II analogous to extension I case

Connection formulation?

- *"Hybrid"* spin connection [Peldan '94] $\Gamma^{HYB}_{aIJ}[e]$ s.t. $\partial_a e_{bI} - \Gamma^c_{ab} e_{cI} + \Gamma^{HYB}_{aIJ}[e]\; e_b{}^J = 0$
- BUT: No obvious way of combining K_{aJ} and $\Gamma^{HYB}_{aIJ}[e]$ to a connection conjugate to E^{bJ} in a mathematically sensible way (if $D \neq 2$)!

Plan of the talk

The New Variables - Hamiltonian Viewpoint

Extension of ADM phase space III

- Introduce "generalised" vielbein, transforming in the adjoint representation of $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$:

$$q_{ab} = e_{aIJ} e_b{}^{IJ} \qquad K_{ab} = K_{aIJ} e_b{}^{IJ} \qquad \pi^{aIJ} = \sqrt{\det q}\; e^{aIJ} \qquad I, J, \ldots \in \{0, 1, \ldots, D\} \tag{8}$$

- Motivation: 1^{st} order Palatini formulation of General Relativity (cf. next to next slide)
- Poisson bracket relations: $\{\pi^{aIJ}, K_{bKL}\} = \delta^a_b \, \delta^I_{[K} \delta^J_{L]}$
- New constraints: Gauß and simplicity constraint

$$G^{IJ} := K_a{}^{[I|K} \pi^a{}_K{}^{J]} \qquad \text{and} \qquad S^{aIJ\;bKL} := \pi^{a[IJ|} \pi^{b|KL]} \tag{9}$$

- Proof of validity of extension analogous to extension I and II case

Canonical transformation to new connection formulation

- $\Gamma^{HYB}_{aIJ}[\pi]$: Extension of $\Gamma^{HYB}_{aIJ}[e]$ off the simplicity constraint surface

$$S = 0 \;\Leftrightarrow\; \pi^{aIJ} = n^{[I} E^{a|J]} \tag{10}$$

[Freidel, Krasnov, Puzio '99]

- *Canonical* transformation (non-trivial):

$$\{\pi^{aIJ}, K_{bKL}\} \longrightarrow \{\frac{1}{\beta}\pi^{aIJ}, A_{bKL} := \Gamma^{HYB}_{bKL}[\pi] + \beta K_{bKL}\} \qquad \beta \in \mathbb{R}/\{0\}, \neq \gamma! \tag{11}$$

- G^{IJ} becomes $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ Gauß law constraint:

$$G^{IJ} = \partial_a \pi^{aIJ} + A_a{}^{[I|}{}_K \pi^{aK|J]} \tag{12}$$

- Formulation works with $\mathrm{SO}(D+1)$ and $\mathrm{SO}(1,D)$ independent of spacetime signature!

Comparison with Ashtekar-Barbero Formulation

Ashtekar-Barbero formulation

- Canonical variables A^{LQG}_{aj}, E^{bk} are real
- Simple Poisson algebra $\{A^{LQG}, E\} \sim 1$
- Compact gauge group SU(2)
- First class constraints $\mathcal{H}, \mathcal{H}_a$ and G^i
- Physical information:

$$A^{LQG}_{aij} - \Gamma^{SPIN}_{aij}[e] = \gamma \, \epsilon_{ij}{}^{k} \, K_{ak} \tag{13}$$

- Relation to other formulations: AB $\xrightarrow{G=0}$ ADM

New formulation, $D = 3$

- Canonical variables A^{NEW}_{aIJ}, π^{bKL} are real
- Simple Poisson algebra $\{A^{NEW}, \pi\} \sim 1$
- Compact gauge group SO(4)
- First class constraints $\mathcal{H}, \mathcal{H}_a, G^{IJ}$ and $S^{aIJ\ bKL}$
- Physical information:

$$A^{NEW}_{aij} - \Gamma^{HYB}_{aij}[\pi] \approx S - gauge, \qquad A^{NEW}_{a0j} - \Gamma^{HYB}_{a0j}[\pi] \approx \beta K_{aj} \tag{14}$$

- NEW $\xrightarrow{S=0}$ Ex. ADM II $\xrightarrow{\text{time gauge}}$ Ex. ADM I $\xrightarrow{G=0}$ ADM

The New Variables - Lagrangian Viewpoint

Canonical analysis of the 1^{st} order Palatini action [Peldan '94]

$$S_P = \int \left(\pi^{aIJ} \dot{A}_{aIJ} - N\mathcal{H} - N^a\mathcal{H}_a - \Lambda \cdot G - c \cdot S \right) \tag{15}$$

- Gauß and simplicity constraint: Exactly like before
- Dirac constraint analysis: Additional constraint D, second class partner to S
- A_{aIJ} not self-commuting w.r.t. corresponding Dirac bracket [Alexandrov '00]

⇒ Loop quantisation not (directly) applicable! [see, however: Alexandrov & Roche '10; Geiller, Lachieze-Rey, Noui, Sardelli '11]

Gauge Unfixing [Mitra & Rajaraman '89 '90; Henneaux & Teitelboim '92; Anishetty & Vytheeswaran '93]

- Well defined procedure: 2^{nd} class $\Rightarrow$ 1^{st} class constrained system
- Applied to GR: Drop D at the cost of a more complicated $\mathcal{H}$
- Resulting theory coincides with result of Hamiltonian derivation iff
  - Internal and external signatures match
  - Free parameter $\beta = 1$

Quantisation, Generalisations

Quantisation [Rovelli, Smolin, Ashtekar, Isham, Lewandoski, Marolf, Mourao, Thiemann...]

- Most results of loop quantisation formulated independently of
  - Dimension of spacetime
  - Choice of compact gauge group
- Sole new ingredient for canonical theory: Implementation of simplicity constraint (but well-known from covariant approach, cf. below)

Generalisations

- Extension to diverse matter fields and supergravity:
  - Dirac, Weyl, Majorana fermions
  - Gauge fields with compact gauge groups
  - Scalar fields
  - Rarita-Schwinger fields (gravitinos)
  - Abelian higher *p*-form fields
- Not treatable so far:
  - Non-abelian higher *p*-form fields (higher gauge theory?)
  - Non-compact gauge groups

$\Rightarrow$ Includes, inter alia, supergravity theories in 4, 10 and 11 dimensions

Plan of the talk

Solutions to the Simplicity Constraint

There exist multiple, plausible suggestions for solving the simplicity constraint, e.g.

- Weak implementation [Engle, Pereirra, Rovelli '07; Livine, Speziale '07]
- Coherent states [Freidel, Krasnov '07]
- Holomorphic simplicity constraints [Dupuis, Freidel, Livine, Speziale '11]
- Maximally commuting subsets [NB, Thiemann, AT '11]
- ...

It is however in general unclear, if they lead to the same dynamics.

Application of the new variables

- Test different implementations of the simplicity constraint within the new canonical framework for dynamical equivalence
- New requirement: Anomaly-freedom of the constraint algebra including the Hamiltonian constraint, i.e. implement

$$\{S[...], H[N]\} = S[...] \quad \rightarrow \quad \left[\hat{S}, \hat{H}\right] = \hat{S} \tag{16}$$

Canonical = Covariant Formulation?

[Reisenberger, Rovelli '97]

Basic idea: The spinfoam provides a rigging map for the Hamiltonian constraint.

$$\langle \phi \mid \psi \rangle_{\text{phys}} = \sum_{\kappa : \psi \to \phi} Z[\kappa] \tag{17}$$

New question

For which canonical quantisation should we test the above equation?

$\updownarrow$

Are the quantum theories based on the Ashtekar-Barbero and the newly proposed variables equivalent?

Ashtekar-Barbero	**New variables**
simplicity solved classically $\Rightarrow$ Hilbert spaces have to be related	simplicity can be quantised $\Rightarrow$ Hilbert spaces are the same
usual Hamiltonian constraint $\Rightarrow$ calculations "easier"	Hamiltonian constraint more complicated $\Rightarrow$ calculations "harder"

Supersymmetry Constraint

Important open problem:
Understand the solution space of the Hamiltonian constraint, including matter.

Hint from Supergravity: Super Dirac algebra [Teitelboim '77]

$$\{\mathcal{S}, \mathcal{S}\} = H + H_a + \mathcal{S}, \qquad \mathcal{S} : \text{supersymmetry constraint} \tag{18}$$

Assuming an anomaly-free implementation of the super Dirac algebra:

Solution to the supersymmetry constraint operator
$\Downarrow$
Solution to the Hamiltonian constraint operator

$\rightarrow$ Supergravity as a simplified version of General Relativity coupled to matter

Important progress with implementing the supersymmetry constraint has been made in the GSU(2) framework. [Armand-Ugon, Gambini, Obrégon, Pullin '95]

Comparing LQG to other Approaches to Quantum Gravity

General considerations

- Supergravity has been extensively studied as a low energy limit of String- / M-theory
- A great deal of "technology" has been developed in order to deal with String- / M-theory and Supergravity

Comparing LQG to String- / M-theory

- Dimensional reduction to 4 dimensions is not unique
  $\rightarrow$ Work in the natural dimensions of String- / M-theory
- Generic calculations are hard both in LQG and String- / M-theory
  $\rightarrow$ Work in symmetry reduced situations

Black Hole Entropy

Calculation of black hole entropy

- Thermodynamic analogy [Bekenstein '73]; QFTCS [Hawking '74]
- String theory [Strominger, Vafa; ... '96]
- Loop quantum gravity [Krasnov '96; Rovelli 96'; Ashtekar, Baez, Corichi, Krasnov '97-, ...]

$\Rightarrow$ Calculation possible in different theories!

Application of the new variables

- Calculate entropy of a supersymmetric extremal black hole in higher dimensions
- Compare to results coming from string theory

Cosmology

Cosmology from different points of view

- Wheeler-DeWitt quantum cosmology [Wheeler '64-; DeWitt '67; Misner '69]
- String cosmology [Veneziano; ... '91]
- Loop quantum cosmology [Bojowald '01-, Ashtekar, Kaminski, Lewandowski, Pawlowski, Singh, ... '02-]

$\Rightarrow$ Calculation possible in different theories!

Application of the new variables

- Investigate SLQC in higher dimensions
- Compare to results coming from string cosmology and possibly from experiments

$\rightarrow$ hints of higher dimensions and supersymmetry in cosmological observables?

AdS / CFT Correspondence

[Maldacena '97; Gubser, Klebanov, Polyakov '98; Witten '98]

Conjectured exact equivalence

Type IIB String Theory on $AdS^5 x S^5$ $\longleftrightarrow$ $\mathcal{N} = 4$ Super Yang-Mills Theory in $4d$

$4\pi g_s = g_{YM}^2$

$T = \frac{1}{2\pi}\sqrt{g_{YM}^2 N}$

String coupling g_s, String tension T — YM coupling g_{YM}, number of coulors N

- weak string coupling
- strong string tension (only massless states)

$\downarrow$

- weak YM-coupling
- strong 't-Hooft coupling (only planar diagrams)

Well tested low energy equivalence

Type IIB Supergravity in $AdS^5 x S^5$ $\longleftrightarrow$ $\mathcal{N} = 4$ Super Yang-Mills Theory in $4d$ at strong 't Hooft coupling

$4\pi g_s = g_{YM}^2$

$T = \frac{1}{2\pi}\sqrt{g_{YM}^2 N}$

$g_s \to 0, \quad T \to \infty$

$g_{YM} \to 0, \quad g_{YM}^2 N \to \infty$

AdS / CFT Correspondence

[Maldacena '97; Gubser, Klebanov, Polyakov '98; Witten '98]

Conjectured exact equivalence

Type IIB String Theory on $AdS^5 x S^5$ $\longleftrightarrow$ $\mathcal{N} = 4$ Super Yang-Mills Theory in $4d$

$$4\pi g_s = g_{YM}^2$$
$$T = \frac{1}{2\pi}\sqrt{g_{YM}^2 N}$$

String coupling g_s, String tension T

YM coupling g_{YM}, number of coulors N

New non-perturbative limit?

Loop quantized Type IIB Supergravity (in $AdS^5 x S^5$?) $\longleftrightarrow$ $\mathcal{N} = 4$ Super Yang-Mills Theory in $4d$

?

$g_s = ?, \quad T = ?$

$g_{YM} = ?, \quad g_{YM}^2 N = ?$

Well tested low energy equivalence

Type IIB Supergravity in $AdS^5 x S^5$ $\longleftrightarrow$ $\mathcal{N} = 4$ Super Yang-Mills Theory in $4d$ at strong 't Hooft coupling

$$4\pi g_s = g_{YM}^2$$
$$T = \frac{1}{2\pi}\sqrt{g_{YM}^2 N}$$

$g_s \to 0, \quad T \to \infty$

$g_{YM} \to 0, \quad g_{YM}^2 N \to \infty$

Plan of the talk

Conclusion

- $D+1$ dim. GR formulated on an $SO(D+1)$ Yang-Mills phase space
- LQG methods apply $\rightarrow$ rigorous quantisation exists
- Extensions to interesting Supergravities exist
- Possible applications include
  - Better understanding the simplicity constraint
  - Supergravity as "simplified" matter coupled GR
  - Higher dimensional (supersymmetric) black hole entropy
  - Higher dimensional (supersymmetric) quantum cosmology
  - New tests / applications of the AdS/CFT correspondence?

Thank you for your attention!