Symmetry reductions in loop quantum gravity

based on classical gauge fixings

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based on work in collaboration with J. Lewandowski, J. Świeżewski, and A. Zipfel

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Talk in a nutshell

Aim:
- Symmetry reduce at quantum level
- Extract dynamics from full theory

Results:
- Reduction to LQC
  - Bianchi I [NB '14]
  - $k = 0$ FRW [NB '15]
- Reduction to spherical symmetry
  - SU(2) variables [NB, Lewandowski, Świeżewski '14-]
  - Commutators in SU(2) vars. [NB, Zipfel '15]
  - Abelian connections [NB '15]

What else?
- Simplified coarse graining & dynamics
Plan of the talk

1. Strategy
2. Example of the formalism
3. ¯μ scheme in the full theory
4. Spherical symmetry and SU(2)
5. Conclusion
**Strategy**

1. **Suitable classical starting point**
   - Gauge fix

2. **Identify reduction constraints**
   - Symmetry $\Rightarrow f_i(p,q) = 0$

3. **Quantise à la LQG**

4. **Impose reduction constraints**
   - $\hat{f}_i |\psi\rangle_{\text{sym}} = 0$, $[\hat{O}_{\text{sym}}, \hat{f}_i] = 0$

5. **Extract dynamics**
Outline

1. Strategy
2. Example of the formalism
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Phase space & gauge fixing

- **ADM** \[ \{ q_{ab}, P^{cd} \} = \delta_c^a \delta^d_b \]

\[ \downarrow \]

- **Diagonal metric** gauge

\[ q_{ab} = \begin{pmatrix} q_{xx} & 0 & 0 \\ 0 & q_{yy} & 0 \\ 0 & 0 & q_{zz} \end{pmatrix} \]

Gauge fixes spatial diffeo constraint \( C_a = 0 \)

\[ C_a = 0 \quad \Rightarrow \quad P^{a\neq b}(q_{aa}, P^{bb}) \]

\[ \downarrow \]

- **Gauge fixed phase space:** \( \{ q_{aa}, P^{bb} \} = \delta_b^b \)
### Adapted variables

<table>
<thead>
<tr>
<th>Full theory</th>
<th>Relation to LQC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha := \sqrt{q_{xx} q_{yy} q_{zz}})</td>
<td>(P_\alpha := \frac{2}{3} \frac{P^{xx} q_{xx} + P^{yy} q_{yy} + P^{zz} q_{zz}}{\sqrt{q_{xx} q_{yy} q_{zz}}}) (\integral \alpha \propto v, \quad P_\alpha \propto b)</td>
</tr>
<tr>
<td>(\beta := P^{xx} q_{xx} - P^{yy} q_{yy})</td>
<td>(P_\beta := \frac{1}{2} \log \frac{q_{yy}}{q_{xx}}) (\beta = P_\beta = 0)</td>
</tr>
<tr>
<td>(\gamma := P^{xx} q_{xx} - P^{zz} q_{zz})</td>
<td>(P_\gamma := \frac{1}{2} \log \frac{q_{zz}}{q_{xx}}) (\gamma = P_\gamma = 0)</td>
</tr>
</tbody>
</table>

\(\{\alpha, P_\alpha\} = \{\beta, P_\beta\} = \{\gamma, P_\gamma\} = \delta^{(3)}\) \(\integral \{b, v\} \propto 1\)
## Consequences of symmetry

### $\mathbb{T}^3$ FRW model

$$0 = \beta = P_\beta = \gamma = P_\gamma \quad \& \quad P^{a \neq b} = 0$$

### First class subset

- $\beta = \gamma = 0$
- Spatial diffeomorphisms: $\int_{\Sigma} d^3 \sigma \left( P_\alpha \mathcal{L}_{\vec{N}} \alpha + P_\phi \mathcal{L}_{\vec{N}} \phi \right) = 0$
Quantum kinematics and reduction

Scalar fields [Thiemann, QSD5]

- Point holonomies \( h^\rho_\sigma \) := \( e^{-i\rho_\alpha P_\alpha(\sigma)} \), \( \rho_\alpha \in \mathbb{R} \)

- \( \left\langle h^\rho_\sigma \mid h'^\rho_\sigma \right\rangle_{\text{kin}} = \delta_{\sigma,\sigma'} \delta_{\rho_\alpha,\rho'_\alpha} \)

- \( \hat{\alpha}(R) \mid h^\rho_\sigma \rangle := \int_R \alpha \mid h^\rho_\sigma \rangle = \rho_\alpha \mid h^\rho_\sigma \rangle \quad \forall \sigma \in R \)

Reduction: \( \downarrow \quad \hat{\beta}(R) = \hat{\gamma}(R) = 0 \quad + \quad \text{diff. invariance} \)

Single vertex states

\[
\left| h^\rho_\sigma, \rho_\phi \right\rangle = \sum_{\sigma \in \Sigma} \left| e^{-i\rho_\alpha P_\alpha(\sigma)} e^{-i\rho_\phi \phi(\sigma)} \right\rangle, \quad \left\langle h^\rho_\sigma, \rho_\phi \mid h'^\rho_\sigma, \rho'_\phi \right\rangle_{\text{diff}} = \delta_{\rho_\alpha,\rho'_\alpha} \delta_{\rho_\phi,\rho'_\phi}
\]
**Reduced operators**

**Diagonal operators**

\[
\alpha(\Sigma) | h_{\text{diff}}^{\rho_{\alpha}, \rho_{\phi}} \rangle = \rho_{\alpha} | h_{\text{diff}}^{\rho_{\alpha}, \rho_{\phi}} \rangle
\]

\[
P_{\phi}(\Sigma) | h_{\text{diff}}^{\rho_{\alpha}, \rho_{\phi}} \rangle = \rho_{\phi} | h_{\text{diff}}^{\rho_{\alpha}, \rho_{\phi}} \rangle
\]

**Polymerised “shift” operators**

\[
\frac{1}{\lambda} (\sin(\lambda P_{\alpha})\alpha)(\Sigma) | h_{\text{diff}}^{\rho_{\alpha}, \rho_{\phi}} \rangle = \frac{\rho_{\alpha}}{2i\lambda} \left( | h_{\text{diff}}^{\rho_{\alpha}-\lambda, \rho_{\phi}} \rangle - | h_{\text{diff}}^{\rho_{\alpha}+\lambda, \rho_{\phi}} \rangle \right)
\]

\[\lambda \leftrightarrow \text{cutoff for matter energy density } \propto P_{\alpha}^2\]
Quantum dynamics I

1. FRW part of Hamiltonian

2. Other terms vanish

\[
\left\langle \hat{P}_\phi(\Sigma)^2 \right| h^\rho_\alpha, \rho_\phi \right\rangle = \frac{3}{2\lambda^2} \left( \sin(\lambda \hat{P}_\alpha) |\alpha| (\Sigma) \right)^2 \left| h^\rho_\alpha, \rho_\phi \right\rangle
\]

\[\downarrow \quad \text{rescaling of variables}\]

LQC difference equation in \((v, b)\) variables [Ashtekar, Corichi, Singh '07]

\[
\partial^2_{\phi} |v, \phi \rangle = \frac{3\pi G}{4\lambda^2} |v| \left( |v + 2\lambda| |v + 4, \phi \rangle + |v - 2\lambda| |v - 4, \phi \rangle - (|v + 2\lambda| + |v - 2\lambda|) |v, \phi \rangle \right)
\]
Quantum dynamics II

1. FRW part of Hamiltonian
2. Other terms vanish

\[ \propto P^{ab}, \text{ but not } \propto P_{\alpha} \]

\[ \propto \beta, \gamma = 0 \]

Spatial derivatives

finite differences \(\Rightarrow\) vanish on single vertex states
Outline

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$\mathbb{T}^3$ Bianchi I: classical preparations

- ADM \( \{ q_{ab}, P^{cd} \} = \delta^c_{(a} \delta^d_{b)} \)

- Diagonal metric gauge

- Gauge fixed phase space: \( \{ q_{aa}, P^{bb} \} = \delta^b_a \)

- New variables: \( \{ K_a, E^b \} = \delta^b_a \)

\[
e_a e_a = q_{aa}, \quad E^a = \sqrt{\det q} e^a, \quad K_a = K_{ab} e^b,
\]
**$T^3$ Bianchi I: reduction constraints**

$T^3$ Bianchi I model

\[ \partial_a E^b = 0 = \partial_a K_b \quad \& \quad P^{a \neq b} = 0 \]

First class subset

- **Spatial diffeos:**
  \[ \int_\Sigma d^3\sigma \left( E^a \mathcal{L}_{\vec{N}} K_a + P_\phi \mathcal{L}_{\vec{N}} \phi \right) = 0 \]

- **Abelian Gauß law:**
  \[ \int_\Sigma d^3\sigma \omega \partial_a E^a = 0 \]
Quantum kinematics & reduction

Standard LQG quantisation for U(1): [Corichi, Krasnov '97]

1. Holonomies $h^\rho_\gamma = \exp \left( i \rho \int_\gamma K_a ds^a \right)$, fluxes $E(S) = \int_S E^a \epsilon_{abc} dx^b \wedge dx^c$

2. Reduction $\Rightarrow$ gauge / spatial diffeo invariance

Single vertex states

$|\rho_x, \rho_y, \rho_z\rangle \leftrightarrow |p_1, p_2, p_3\rangle_{LQC}$

[Ashtekar, Wilson-Ewing '09]

Reduced operators

- Areas $A(\mathbb{T}^2_x), A(\mathbb{T}^2_y), A(\mathbb{T}^2_z)$
- Reduced Wilson loops
Quantum dynamics

Polymerisation $\int K_a ds^a \approx \sin(\lambda \int K_a ds^a)/\lambda$

- $U(1) \rightarrow \lambda = 1 \Rightarrow \text{“old” LQC dynamics}$
  
  [Ashtekar, Bojowald, Lewandowski '03, has been formulated using $R_{\text{Bohr}}$]

- $R_{\text{Bohr}} \rightarrow \lambda \in \mathbb{R} \Leftarrow \text{“new” LQC dynamics}$

$1/\lambda_x = \text{size of universe in } x\text{-direction}$ [Ashtekar, Pawlowski, Singh '06; Ashtekar, Wilson-Ewing '09]

Full theory lessons

- LQG on fixed graph [Giesel, Thiemann '06] $\leftrightarrow U(1)$
  
  - Problems for coarse states (?)
  - FRW: $\int K_a ds^a \propto \sqrt{\rho \phi} \times \text{distance}$ see also [Charles, Livine '15]

- $\bar{\mu}$ dynamics from coarse graining? [Gielen, Oriti, Sindoni '13; Alesci, Cianfrani '14]
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**Spherical symmetry: classical preparations**

1. Radial gauge $q_{ra} = \delta_{ra}$
   
   [Duch, Kamiński, Lewandowski, Świeżewski '14]
   
   [NB, Lewandowski, Świeżewski '14, '15]

2. **SU(2) connection variables** $A^i_A, E^B_j$

3. $C_a = 0 \quad \Rightarrow \quad P^{ra}(A^i_A, E^B_j)$

**Reduction constraints**

$P^{rA} = 0 \iff \text{spatial diffeomorphisms preserving } S^2_r$
Quantum kinematics & reduction [NB, Lewandowski, Świeżewski '14]

Standard LQG quantisation

1. Kinematics $\Rightarrow$ spin networks $\subset S_{r_1}^2 \cup \ldots \cup S_{r_n}^2$

2. Reduction $\Rightarrow$ diff invariance on $S_r^2$

Symmetric operators

1. Areas of the $S_r^2$ $\rightarrow$ $R(r)^2 := \frac{1}{4\pi} \int_{S_r^2} d^2\theta \sqrt{\det q_{AB}}$

2. Averaged trace of momenta $\rightarrow$ $P_R(r) := \frac{2}{R(r)} \int_{S_r^2} d^2\theta P^{AB} q_{AB}$
**Regularising** $[\hat{R}, \hat{P}_r]$

[Thiemann: QSD1, QSD4]

<table>
<thead>
<tr>
<th>Poisson bracket tricks</th>
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<tbody>
<tr>
<td>$R(r)^2 \propto \int_{S^2_r} \sqrt{V^k V_k}$</td>
</tr>
<tr>
<td>$V^k \propto \epsilon^{ijk} E^A_i E^B_j \epsilon_{AB}$</td>
</tr>
<tr>
<td>$n^i = \frac{\epsilon^{ijk} E^A_j E^B_k \epsilon_{AB}}{</td>
</tr>
</tbody>
</table>

**Simplest non-trivial spin network:**

- Operators non-trivial at kink
- Graph-preserving regularisation
- Graphical calculus [Alesci, Liegener, Zipfel '13]
Results of $[\hat{R}, \hat{P}_r]$

$P_R \approx \frac{e^{i\lambda P_R} - e^{-i\lambda P_R}}{2i\lambda} \iff F_{AB}^i \sim h_{\alpha AB} - h_{\alpha AB}^{-1}$

Classical reduction

$\left\langle \rho \pm \lambda \left| \left[ \hat{R}, \hat{P}_R \right] \right| \rho \right\rangle = 0.5 i$

Quantum reduction

$\left\langle j \pm \frac{1}{2} \left| \left[ \hat{R}, \hat{P}_R \right] \right| j \right\rangle \approx 0.1 i + \mathcal{O}(j^{-1})$

Several problems

- Strong regularisation dependence
- Kink state degenerate
- Problems absent for trivalent vertex → future work
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Conclusion

Strategy

- Gauge fixing
- \( \hat{f}_i |\psi\rangle_{\text{sym}} = 0, \quad [\hat{O}_{\text{sym}}, \hat{f}_i] = 0 \)

→ Loop quantum cosmology

- \( \bar{\mu} \) scheme in full theory
- Single-vertex truncation

→ Spherical symmetry

- Partial results in SU(2) variables

Lessons / open questions

- \( \bar{\mu} \)-scheme for coarse states
- Coarse graining

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Thank you for your attention!