Symmetry reductions in loop quantum gravity

based on classical gauge fixings

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based on work in collaboration with J. Lewandowski, J. Świeżewski, and A. Zipfel

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Talk in a nutshell

- **Aim:**
  - Symmetry reduce at quantum level
  - Extract dynamics from full theory
- **Results:**
  - Reduction to LQC
    - Bianchi I [NB '14]
    - $k = 0$ FRW [NB '15]
  - Reduction to spherical symmetry
    - SU(2) variables [NB, Lewandowski, Świeżewski '14-]
    - Commutators in SU(2) vars. [NB, Zipfel '15]
    - Abelian connections [NB '15]
- **What else**?
  - Simplified coarse graining & dynamics

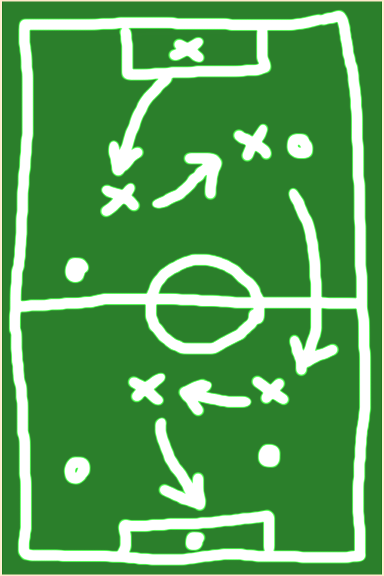
Plan of the talk

1. Strategy
2. Example of the formalism
3. $\bar{\mu}$ scheme in the full theory
4. Spherical symmetry and SU(2)
5. Conclusion

Outline

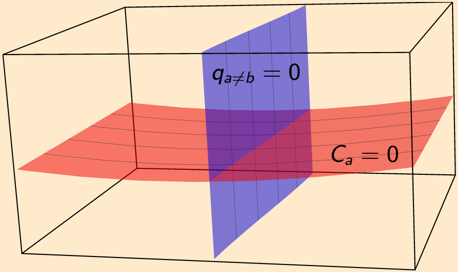
STRATEGY

1. **Suitable classical starting point**
   - Gauge fix
2. **Identify reduction constraints**
   - Symmetry $\Rightarrow$ $f_i(p,q) = 0$
3. **Quantise à la LQG**
4. **Impose reduction constraints**
   - $\hat{f}_i \left|\Psi\right\rangle_{\text{sym}} = 0, \qquad [\hat{\mathcal{O}}_{\text{sym}}, \hat{f}_i] = 0$
5. **Extract dynamics**



Outline

Phase space & gauge fixing



- ADM $\quad \{q_{ab}, P^{cd}\} = \delta^c_{(a}\delta^d_{b)}$

$$\downarrow$$

- **Diagonal metric** gauge

$$q_{ab} = \begin{pmatrix} q_{xx} & 0 & 0 \\ 0 & q_{yy} & 0 \\ 0 & 0 & q_{zz} \end{pmatrix}$$

Gauge fixes spatial diffeo constraint $C_a = 0$

$$C_a = 0 \quad \Rightarrow \quad P^{a\neq b}(q_{aa}, P^{bb})$$

$$\downarrow$$

- Gauge fixed phase space: $\{q_{aa}, P^{bb}\} = \delta^b_a$

Adapted variables

Full theory		Relation to LQC
$\alpha := \sqrt{q_{xx} q_{yy} q_{zz}}$	$P_\alpha := \frac{2}{3} \frac{P^{xx} q_{xx} + P^{yy} q_{yy} + P^{zz} q_{zz}}{\sqrt{q_{xx} q_{yy} q_{zz}}}$	$\int_\Sigma \alpha \propto v, \quad P_\alpha \propto b$
$\beta := P^{xx} q_{xx} - P^{yy} q_{yy}$	$P_\beta := \frac{1}{2} \log \frac{q_{yy}}{q_{xx}}$	$\beta = P_\beta = 0$
$\gamma := P^{xx} q_{xx} - P^{zz} q_{zz}$	$P_\gamma := \frac{1}{2} \log \frac{q_{zz}}{q_{xx}}$	$\gamma = P_\gamma = 0$
$\{\alpha, P_\alpha\} = \{\beta, P_\beta\} = \{\gamma, P_\gamma\} = \delta^{(3)}$		$\{b, v\} \propto 1$

Consequences of symmetry

$\mathbb{T}^3$ FRW model

$$0 = \beta = P_\beta = \gamma = P_\gamma \qquad \& \qquad P^{a \neq b} = 0$$

$$\Downarrow$$

First class subset

- $\beta = \gamma = 0$
- Spatial diffeomorphisms: $\int_\Sigma d^3\sigma \, \left(P_\alpha \mathcal{L}_{\vec{N}} \alpha + P_\phi \mathcal{L}_{\vec{N}} \phi \right) = 0$

Quantum kinematics and reduction

Scalar fields [Thiemann, QSD5]

- Point holonomies $h_\sigma^{\rho_\alpha} := e^{-i\rho_\alpha P_\alpha(\sigma)}$, $\rho_\alpha \in \mathbb{R}$
- $\left\langle h_\sigma^{\rho_\alpha} \,\middle|\, h_{\sigma'}^{\rho'_\alpha} \right\rangle_{\text{kin}} = \delta_{\sigma,\sigma'}\delta_{\rho_\alpha,\rho'_\alpha}$
- $\widehat{\alpha(R)}\,|h_\sigma^{\rho_\alpha}\rangle := \widehat{\int_R \alpha}\,|h_\sigma^{\rho_\alpha}\rangle = \rho_\alpha\,|h_\sigma^{\rho_\alpha}\rangle \qquad \forall\ \sigma \in R$

Reduction: $\downarrow$ $\quad \widehat{\beta(R)} = \widehat{\gamma(R)} = 0 \quad + \quad$ diff. invariance

Single vertex states

$$\left|h_{\text{diff}}^{\rho_\alpha,\rho_\phi}\right\rangle = \sum_{\sigma\in\Sigma}\left|e^{-i\rho_\alpha P_\alpha(\sigma)} e^{-i\rho_\phi \phi(\sigma)}\right\rangle, \qquad \left\langle h_{\text{diff}}^{\rho_\alpha,\rho_\phi} \,\middle|\, h_{\text{diff}}^{\rho'_\alpha,\rho'_\phi} \right\rangle_{\text{diff}} = \delta_{\rho_\alpha,\rho'_\alpha}\delta_{\rho_\phi,\rho'_\phi}$$

Reduced operators

Diagonal operators

$$\widehat{\alpha(\Sigma)}\left|h_{\text{diff}}^{\rho_\alpha,\rho_\phi}\right\rangle = \rho_\alpha \left|h_{\text{diff}}^{\rho_\alpha,\rho_\phi}\right\rangle$$

$$\widehat{P_\phi(\Sigma)}\left|h_{\text{diff}}^{\rho_\alpha,\rho_\phi}\right\rangle = \rho_\phi \left|h_{\text{diff}}^{\rho_\alpha,\rho_\phi}\right\rangle$$

Polymerised "shift" operators

$$\frac{1}{\lambda}\,\widehat{(\sin(\lambda P_\alpha)\alpha)}\,(\Sigma)\left|h_{\text{diff}}^{\rho_\alpha,\rho_\phi}\right\rangle = \frac{\rho_\alpha}{2i\lambda}\left(\left|h_{\text{diff}}^{\rho_\alpha-\lambda,\rho_\phi}\right\rangle - \left|h_{\text{diff}}^{\rho_\alpha+\lambda,\rho_\phi}\right\rangle\right)$$

$$\boxed{\lambda \quad \leftrightarrow \quad \text{cutoff for matter energy density} \propto P_\alpha^2}$$

Quantum dynamics I

1. $\boxed{\text{FRW part of Hamiltonian}}$
2. Other terms vanish

Dynamics

$$\widehat{P_\phi(\Sigma)}^2 \left| h_{\text{diff}}^{\rho_\alpha,\rho_\phi} \right\rangle = \frac{3}{2\lambda^2} \left(\left(\sin(\lambda \widehat{P_\alpha) |\alpha} | \right) (\Sigma) \right)^2 \left| h_{\text{diff}}^{\rho_\alpha,\rho_\phi} \right\rangle$$

$\Downarrow$ rescaling of variables

LQC difference equation in (v, b) variables [Ashtekar, Corichi, Singh '07]

$$\partial_\phi^2 \left| v, \phi \right\rangle = \frac{3\pi G}{4\lambda^2} |v| \Big(|v + 2\lambda| \left| v + 4, \phi \right\rangle + |v - 2\lambda| \left| v - 4, \phi \right\rangle - \left(|v + 2\lambda| + |v - 2\lambda| \right) \left| v, \phi \right\rangle \Big)$$

Quantum dynamics II

1. FRW part of Hamiltonian
2. Other terms vanish

$\propto P^{ab}$, but not $\propto P_\alpha$

$$\propto \beta, \gamma = 0$$

Spatial derivatives

finite differences $\Rightarrow$ vanish on single vertex states

Outline

$\mathbb{T}^3$ Bianchi I: classical preparations

- ADM $\quad \{q_{ab}, P^{cd}\} = \delta^c_{(a}\delta^d_{b)}$

$$\downarrow$$

- Diagonal metric gauge

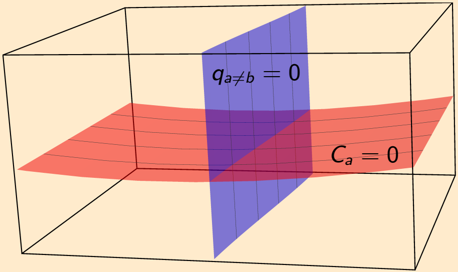
$$\downarrow$$

- Gauge fixed phase space: $\{q_{aa}, P^{bb}\} = \delta^b_a$

$$\downarrow$$

- New variables: $\boxed{\{K_a, E^b\} = \delta^b_a}$

$$e_a e_a = q_{aa}, \quad E^a = \sqrt{\det q}\, e^a, \quad K_a = K_{ab} e^b,$$



$\mathbb{T}^3$ Bianchi I: reduction constraints

$\mathbb{T}^3$ Bianchi I model

$$\partial_a E^b = 0 = \partial_a K_b \qquad \& \qquad P^{a\neq b} = 0$$

$$\Downarrow$$

First class subset

- Spatial diffeos: $\int_\Sigma d^3\sigma \, \left(E^a \mathcal{L}_{\vec{N}} K_a + P_\phi \mathcal{L}_{\vec{N}} \phi\right) = 0$
- Abelian Gauß law: $\int_\Sigma d^3\sigma \, \omega \, \partial_a E^a \quad = 0$

Quantum kinematics & reduction

Standard LQG quantisation for U(1): [Corichi, Krasnov '97]

1. Holonomies $h_\gamma^\rho = \exp\left(i\rho\int_\gamma K_a ds^a\right)$, fluxes $E(S) = \int_S E^a \epsilon_{abc} dx^b \wedge dx^c$
2. Reduction $\Rightarrow$ gauge / spatial diffeo invariance

$\downarrow$

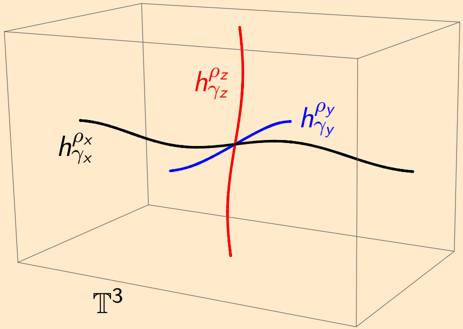
Single vertex states

$|\rho_x, \rho_y, \rho_z\rangle \quad \leftrightarrow \quad |p_1, p_2, p_3\rangle_{\mathrm{LQC}}$

[Ashtekar, Wilson-Ewing '09]

Reduced operators

- Areas $A(\mathbb{T}_x^2), A(\mathbb{T}_y^2), A(\mathbb{T}_z^2)$
- Reduced Wilson loops



Quantum dynamics

Polymerisation $\int K_a ds^a \approx \sin(\lambda \int K_a ds^a)/\lambda$

- U(1) $\rightarrow$ $\lambda = 1$ $\Rightarrow$ "old" LQC dynamics

  [Ashtekar, Bojowald, Lewandowski '03, has been formulated using $\mathbb{R}_{\text{Bohr}}$]

- $\mathbb{R}_{\text{Bohr}}$ $\rightarrow$ $\lambda \in \mathbb{R}$ $\Leftarrow$ "new" LQC dynamics

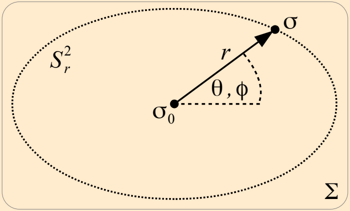
  $1/\lambda_x =$ size of universe in x-direction [Ashtekar, Pawlowski, Singh '06; Ashtekar, Wilson-Ewing '09]

Full theory lessons

- LQG on fixed graph [Giesel, Thiemann '06] $\leftrightarrow$ U(1)
  - Problems for coarse states (?)
  - FRW: $\int K_a ds^a \propto \sqrt{\rho_\phi} \times$ distance see also [Charles, Livine '15]
- $\bar{\mu}$ dynamics from coarse graining? [Gielen, Oriti, Sindoni '13; Alesci, Cianfrani '14]

Outline

Spherical symmetry: classical preparations



$$q_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ & q_{AB} & \\ 0 & & \end{pmatrix}$$

1. Radial gauge $q_{ra} = \delta_{ra}$
   [Duch, Kamiński, Lewandowski, Świeżewski '14]
   [NB, Lewandowski, Świeżewski '14, '15]

2. $\boxed{\textbf{SU}(2) \text{ connection variables } A^i_A,\ E^B_j}$

3. $C_a = 0 \quad \Rightarrow \quad P^{ra}(A^i_A, E^B_j)$

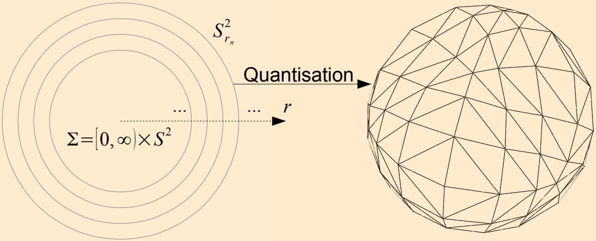
Reduction constraints

$$P^{rA} = 0 \quad \Leftrightarrow \quad \text{spatial diffeomorphisms preserving } S^2_r$$

QUANTUM KINEMATICS & REDUCTION [NB, Lewandowski, Świeżewski '14]

Standard LQG quantisation

1. Kinematics $\Rightarrow$ spin networks $\subset S^2_{r_1} \cup \ldots \cup S^2_{r_n}$
2. Reduction $\Rightarrow$ diff invariance on S^2_r



Symmetric operators

1. Areas of the S^2_r $\rightarrow$ $R(r)^2 := \frac{1}{4\pi} \int_{S^2_r} d^2\theta \sqrt{\det q_{AB}}$
2. Averaged trace of momenta $\rightarrow$ $P_R(r) := \frac{2}{R(r)} \int_{S^2_r} d^2\theta \, P^{AB} q_{AB}$

Regularising $[\hat{R}, \hat{P}_r]$

[NB, Zipfel '15]

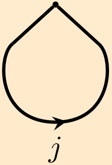
Poisson bracket tricks [Thiemann: QSD1, QSD4]

$$R(r)^2 \propto \int_{S_r^2} \sqrt{V^k V_k} \qquad P_R(r) \propto \frac{1}{R(r)} \int_{S_r^2} d^2\theta \, \{H, V\}$$

$$V^k \propto \epsilon^{ijk} E_i^A E_j^B \epsilon_{AB} \qquad H := F_{AB}^i \, n^i \epsilon^{AB} \qquad n^i = \frac{\epsilon^{ijk} E_j^A E_k^B \epsilon_{AB}}{\|\epsilon^{ijk} E_j^A E_k^B \epsilon_{AB}\|}$$

Simplest non-trivial spin network:

- Operators non-trivial at kink
- Graph-preserving regularisation
- Graphical calculus [Alesci, Liegener, Zipfel '13]



Kink state

RESULTS OF $[\hat{R}, \hat{P}_r]$

[NB, Zipfel '15]

$$P_R \approx \frac{e^{i\lambda P_R} - e^{-i\lambda P_R}}{2i\lambda} \qquad \leftrightarrow \qquad F^i_{AB} \sim h_{\alpha_{AB}} - h^{-1}_{\alpha_{AB}}$$

Classical reduction $\quad \left\langle \rho \pm \lambda \,\middle|\, \left[\hat{R}, \hat{P}_R\right] \,\middle|\, \rho \right\rangle = 0.5\, i$

Quantum reduction $\quad \left\langle j \pm \frac{1}{2} \,\middle|\, \left[\hat{R}, \hat{P}_R\right] \,\middle|\, j \right\rangle \approx 0.1\, i + \mathcal{O}(j^{-1})$

Several problems

- Strong regularisation dependence
- Kink state degenerate
- Problems absent for trivalent vertex $\quad \to \quad$ future work

Outline

Conclusion

- **Strategy**
  - Gauge fixing
  - $\hat{f}_i \, |\Psi\rangle_{\mathrm{sym}} = 0, \quad [\hat{\mathcal{O}}_{\mathrm{sym}}, \hat{f}_i] = 0$
- $\rightarrow$ **Loop quantum cosmology**
  - $\bar{\mu}$ scheme in full theory
  - Single-vertex truncation
- $\rightarrow$ **Spherical symmetry**
  - Partial results in SU(2) variables
- **Lessons / open questions**
  - $\bar{\mu}$-scheme for coarse states
  - Coarse graining

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Thank you for your attention!