The Quantum Nature of Spacetime Singularities

http://online.kitp.ucsb.edu/online/singular_m07

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Topics

Focus on *spacelike* (or possibly null) singularities to deal with dynamically relevant cosmological or collapse situations. (Also: cosmic censorship against timelike singularities.)

Singularity theorems not very specific about properties of singularities they predict. But classical structure of spacelike curvature singularities conjectured to be described well by *BKL picture,* reducing inhomogeneous situation to homogeneous (but anisotropic) one. Growing support numerically and in midi-superspace models.

Suggests simplifications for analysis of possible *quantum resolution,* but what does "resolution" mean?

Specific cases (mostly very special, but one has to start somewhere) presented with input from *string theory* and *loop quantum gravity.*

Kasner solution

Simplest anisotropic model: Bianchi I,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_1(t)^2\mathrm{d}x^2 + a_2(t)^2\mathrm{d}y^2 + a_3(t)^2\mathrm{d}z^2$$

Kasner solution: $a_I(t) = t^{p_I}$ with exponents satisfying $\sum_I p_I = 1 = \sum_I p_I^2$.

Parameterization by single variable:

$$p_1 = \frac{-u}{u^2+u+1} \quad , \quad p_2 = \frac{u+1}{u^2+u+1} \quad , \quad p_3 = \frac{u(u+1)}{u^2+u+1}$$

each value of $1 \leq u \leq \infty$ corresponding to a different solution.

During collapse ($t \to 0$, $V \propto t$), *one direction expands* (unless $u = \infty$).

Curvature singularity: $R^{abcd}R_{abcd} = \frac{16}{t^4}\frac{u^2(u+1)^2}{(u^2+u+1)^3}$.

BKL conjecture [Berger, Garfinkle]

In *approach to singularity,* some terms in Einstein's equation diverge and thus dominate over others.
Simpler asymptotic equations?
BKL conjecture: time derivatives most important, independent (but different) *minisuperspace model at each spatial point.*

Example: Gowdy models,
$\mathrm{d}s^2 = e^{(\lambda+t)/2}(-e^{-2t}\mathrm{d}t^2 + \mathrm{d}x^2) + e^{-t}(e^P(\mathrm{d}y + Q\mathrm{d}z)^2 + e^{-P}\mathrm{d}z^2)$
with three functions $\lambda(t,x)$, $P(t,x)$, $Q(t,x)$ of t and x only; singularity for $t\to\infty$. Einstein's equations:

$$P_{tt} - e^{2P}Q_t^2 - e^{-2t}P_{xx} + e^{2(P-t)}Q_x^2 = 0$$
$$Q_{tt} + 2P_tQ_t - e^{-2t}(Q_{xx} + 2P_xQ_x) = 0$$

with spatial derivatives suppressed exponentially (but $P-t$).

Supported numerically also for full equations, choosing a gauge where slices approach the singularity.

General situation

Method of consistent potentials, first ignore any spatial dependence, solve and then put (velocity term dominated) solution into full Einstein's equation.
Additional potential terms may arise which can be interpreted as "reflecting" one Kasner solution to another when they become large.

For instance, mixmaster (Bianchi IX): reflections described by chaotic BKL (Belinskii-Khalatnikov-Lifshitz) map $u_{n+1} = u_n - 1$ for $2 \leq u_n$, $u_{n+1} = (u_n - 1)^{-1}$ for $1 \leq u_n < 2$.

Suggests complicated structure of classical singularity (fractal?).
But: chaotic as a discrete system with infinitely many reflections, e.g. $\Delta \log V \approx 10^{60}$ required for 250 reflections to occur.

Typically, *only* $3 - 5$ *reflections between Planck and Hubble scale.*

Implications for quantum gravity?

Only one single *Kasner* motion may be enough (crossing Planck curvature). Then, any potentials or inhomogeneities in solutions can be ignored.

But: quantum gravity *changes dynamics,* scenario still realized? *Asymptotic statements* used in justifications. Usefulness depends on mechanism of resolution, e.g. asymptotic regime may be avoided in bounces.

How is a singularity to be resolved?
(i) *Non-singular solution(s)* for quantum degrees of freedom reproducing a semiclassical limit which would be singular.
(ii) *Well-behaved observables* at all times.
(iii) Theory of *initial conditions.*

Several special examples put forward.

String theory

(i) Main approach: *AdS/CFT correspondence,* relate bulk geometry of asymptotically AdS solutions to correlation functions of bounary field theory.

Boundary theory non-singular (unitary evolution), but can it reproduce all aspects of the bulk, especially those hidden behind horizons? And if so, how are they resolved specifically?

Other results:
(ii) *String perturbation theory,* brane gas, tachyon condensation.
(iii) *Matrix theory,* emergence of space-time.
(iv) *Classical supergravity,* σ-model on time axis, BKL.

AdS/CFT correspondence

Asymptotically AdS bulk geometry described through boundary properties of $\mathcal{N}=4$ SU(N) super Yang–Mills theory,
$g_s = g_{\rm YM}^2$ and $(L/\ell_s)^4 = g_{\rm YM}^2 N = \lambda$. (Bulk dimension $d+1$)

Background geometry	$\longleftrightarrow$	boundary state
scalar field ϕ of mass m	$\longleftrightarrow$	operator $\mathcal{O}$, conformal dim. $\Delta = d/2 + \sqrt{m^2 + d^2/4}$
bulk propagator $\mathcal{G}$ (Green's function of wave equation)	$\longleftrightarrow$	correlator G (Wightman function)

$$\lim_{r,r'\to\infty} (rr')^{\Delta} \mathcal{G}(x,r;x',r') = \frac{1}{\sqrt{m^2+d^2/4}} G(x,x')$$

Behavior of correlation functions determined for $N\to\infty$ and $\lambda\to\infty$ ($(\ell_{\rm P}/L)^4 = g_s(\ell_s/L)^4 = 1/N = 0$: no quantum gravity, $(\ell_s/L)^4 = 1/\lambda = 0$: no string effects).

$N\to\infty$ to probe classical structure, look for *changes at finite N.*

Schwarzschild-AdS black hole [Liu, Lowe]

Schwarzschild-AdS background of mass μ,

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2$$

with $f(r) = 1 + r^2 - \mu/r^2 = (r^2 - r_0^2)(r^2 + r_1^2)/r^2$, mapped to thermal state ρ_T of (Hawking) temperature $T = \beta^{-1} = (r_0^2 + r_1^2)/2\pi r_0$ (periodicity in Euclidean version).

Take large mass m of scalar, propagator describes *bulk geodesics* (geometric optics approximation) which can be extracted from boundary correlator $G(t) = \langle \mathcal{O}(t)\mathcal{O}(0)\rangle_T$. (No back-reaction if $N \to \infty$.)

Consider Fourier transform $G(\omega)$, compare correlator and geodesic equation.

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Emergence

Geodesics in Euclidean AdS reproduced for real ω: no trace of horizon or inside.
Analytic continuation to imaginary ω: *spacelike geodesics in Lorentzian AdS,* some stretching between asymptotic regions. *Cross horizon and approach singularity.*

Moreover: $G(\omega)$ has poles at
$\omega \approx 2\pi\sqrt{m^2 + d^2/4}/\mathcal{B} + \omega_0 + 4\pi n/\mathcal{B}$, $n \in \mathbb{N}$ along diagonals of complex plane ($\tilde{\beta} = 2\pi r_1/(r_0^2 + r_1^2)$, $\mathcal{B} = \tilde{\beta} + i\beta$): *quasinormal modes.* Classical results for $N \to \infty$, $\lambda \to \infty$ limit.

Analytic structure different at finite N: poles merge to branch cuts. No trace of quasinormal modes, horizon, singularity in quantum gravity. All *emerge* only in large N (classical) limit.

Geometrical picture? Do non-static backgrounds/back-reaction/evaporation require interactions in field theory, non-analytic Feynman propagator, choice of time?

Cosmology [Hertog]

Scalar field on asymptotically AdS: $\phi = \alpha/r + \beta/r^2 + O(r^{-3})$. Static background: $\beta = 0$; cosmological background: $\beta \neq 0$ (by studying Euclidean instantons).

Boundary theory for $\beta = 0$: ϕ dual to $\mathcal{O} \sim \phi^2$ ($\Delta = 1$), $\alpha \leftrightarrow \langle \mathcal{O} \rangle$.

$\beta \neq 0$: interacting CFT, $S = S_0 + k \int \mathcal{O}^3$. Cosmology requires *potential* $V = \phi^2 - k\phi^6$ *unbounded from below.* Runaway solution for each mode ϕ *reaching infinity in finite time.*

Quantum theory: Construct *self-adjoint extension* of Hamiltonian, reflecting boundary conditions at infinity. Taken as indication for bounce.

Only single mode considered in argument. Different extension parameter for each mode independently? Determined by string theory? What kind of bounce if different modes are reflected at different times?

Other [Silverstein, Brandenberger, Verlinde, Nicolai]

Tachyon condensation: String winding modes with antiperiodic Fermion boundary conditions give negative contribution to mass. *Tachyon* leads to instability, condensate provides matter contribution effectively avoiding singularity (null, not curvature). Picture: Not a bounce but topological closure.

Matrix models: Coordinates as diagonal components of interacting matrices. *Emergence* of classical space-time when diagonal, no geometry if not. Suggests replacing singularity by non-geometrical quantum state. Sspecific examples for null (not curvature) singularity, sometimes bouncing sometimes not.

Eleven dimensional supergravity: combined with BKL ideas suggests minisuperspace model but with enhanced E_{11} symmetry. Near singularity, *infinite-dimensional* σ*-model* on time axis. Space-time to *emerge* at large volume.

LQG [Ashtekar, MB, Pullin, Thiemann]

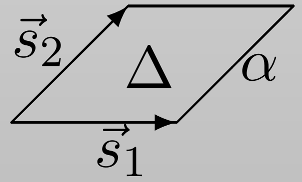
Underlying features: use of bounded *holonomies* (Ashtekar connection A_a^i, extrinsic curvature),
discrete spectra for fluxes (densitized triad E_i^a, spatial metric).

Allows *well-defined quantizations of Hamiltonian constraint.*
Main problem: express curvature components through holonomies, quantize inverse powers of triad components as required in $\int \mathrm{d}^3x \epsilon_{ijk} F_{ab}^i E_i^a E_j^b / \sqrt{|\det E_l^c|}$.

Use identities such as

$$\left\{ A_a^i, \int \sqrt{|\det E|} \mathrm{d}^3x \right\} = 2\pi\gamma G \epsilon^{ijk} \epsilon_{abc} \frac{E_j^b E_k^c}{\sqrt{|\det E|}}$$

and $s_1^a s_2^b F_{ab}^i \tau_i = \Delta^{-1}(h_\alpha - 1) + O(\Delta)$.



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Implications

Quantizations of inverse volume are *finite in isotropic models.* Expectation values in general remain *bounded* if computed in coherent states whose peak follows classical isotropic trajectory toward singularity.

Properties of full operator corresponding to inverse volume? (Difficult to relate to singularities.) Peak trajectory with classical singularity structure put in, quantum corrections to/dissolution of coherent states?

Dynamics of symmetric models often described by *difference equation* for wave function in triad representation. Can be analized more easily than full constraint.

Isotropic model with free scalar allows determination of *physical Hilbert space, observables:* not only complete construction of quantum model but also detailed (but model specific?) picture.

Difference equation

Hamiltonian constraint:

$$\hat{C}|\mu\rangle = \frac{3}{16\pi G\delta^3\gamma^3\ell_{\rm P}^2}(V_{\mu+\delta} - V_{\mu-\delta})(|\mu+4\delta\rangle - 2|\mu\rangle + |\mu-4\delta\rangle)$$

Operator equation $(\hat{C} + \hat{H}_{\rm matter})|\psi\rangle = 0$ to be solved for states $|\psi\rangle = \sum_\mu \psi_\mu(\phi)|\mu\rangle$ where $\psi_\mu(\phi)$ represents the state in the *triad representation.*

Results in *difference equation* for $\psi_\mu(\phi)$ (one version):

$$(V_{\mu+5\delta} - V_{\mu+3\delta})\psi_{\mu+4\delta}(\phi) - 2(V_{\mu+\delta} - V_{\mu-\delta})\psi_\mu(\phi)$$
$$+(V_{\mu-3\delta} - V_{\mu-5\delta})\psi_{\mu-4\delta}(\phi) = -\tfrac{4}{3}\pi G\delta^3\gamma^3\ell_{\rm P}^2\hat{H}_{\rm matter}(\mu)\psi_\mu(\phi)$$

with volume eigenvalues $V_\mu = (\gamma\ell_{\rm P}^2|\mu|/6)^{3/2}$.

$\hat{H}_{\rm matter}$ fully quantized, including metric coefficients, e.g.

$\hat{H}_\phi = -\frac{1}{2}\hbar^2\widehat{|p|^{-3/2}}\partial^2/\partial\phi^2 + |\hat{p}|^{3/2}W(\phi)$.

Properties

$\longrightarrow$ $\psi_\mu(\phi)$ replaces $\psi(a,\phi)$ in Wheeler–DeWitt quantization; Wheeler–DeWitt equation good approximation on large scales. *Extension of Hilbert space* provided. Wave function *uniquely extended:* non-singular in internal time μ.
Every solution extended: *circumvent physical inner product.*

$\longrightarrow$ Extended Hilbert space provided by basic variables (densitized triad); extendability of wave functions consequence of dynamics. Restricts ordering choices.
Generalized to more complicated models: (i) For *Kasner* one metric component diverges at the classical singularity, but all densitized triad components approach zero. (ii) Coupled difference equations in *midisuperspace models* (BKL?).

Physical states? Detailed semiclassical behavior of solutions away from singularities? What about full situation lacking triad representation (non-Abelian)?

Free scalar model

Further specialization of matter content to free, massless scalar allows determination of physical Hilbert space. Matter Hamiltonian $\hat{H}_\phi = -\frac{1}{2}\hbar^2 \widehat{|p|^{-3/2}} \frac{\partial^2}{\partial\phi^2}$.

"Klein–Gordon" type equation

$$-\frac{\partial^2}{\partial\phi^2}\psi(p,\phi) \propto (\widehat{\sin(c)p})^2 \psi(p,\phi) =: \hat{H}^2\psi(p,\phi)$$

with difference operator $\hat{H}$.
Physically normalizable solutions given by $i\hbar\dot{\psi} = \hat{H}\psi = |(\widehat{\sin(c)p})|\psi$ where ϕ is viewed as internal time.

Collapsing solutions with *Gaussian initial state* at large p always *bounce* back to large volume, according to $p^2\sin^2 c = p_\phi^2$.

Relation to solvable model

Free scalar model closely related to *solvable* one, obtained with Hamiltonian $\hat{H} = \frac{1}{2i}(\hat{J} - \hat{J}^\dagger)$ with $\hat{J} := \widehat{p\exp(ic)}$.

Hamiltonian linear except for norm, linear algebra ($\mathrm{sl}(2,\mathbb{R})$) with basic operators:

$$[\hat{p}, \hat{J}] = \hbar\hat{J} \quad , \quad [\hat{p}, \hat{J}^\dagger] = -\hbar\hat{J}^\dagger \quad , \quad [\hat{J}, \hat{J}^\dagger] = -2\hbar\hat{p} - \hbar^2$$

Look at states for which $\Delta H \ll \langle\hat{H}\rangle$ (conserved) as semiclassicality condition and to justify dropping norm.

Allows complete solutions for expectation values as well as fluctuations. Justifies taking $p^2\sin^2 c$ as effective constraint since $\langle\hat{H}\rangle = \frac{1}{2i}(\langle\hat{J}\rangle - \overline{\langle\hat{J}\rangle})$ while $\langle\widehat{\sin c}\,\hat{p}\rangle \neq \langle\widehat{\sin c}\rangle\langle\hat{p}\rangle$.

Also shows limitations: Matter potential? Anisotropies? Inhomogeneity? Quantum back-reaction in general important.

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Perceived strengths and weaknesses

String theory positive: Trusted framework in AdS/CFT correspondence, much information in analytic boundary correlators, independent perturbative string theory scenarios.

String theory negative: Lack of non-perturbative formulation applicable to cosmology, perturbation theory in time-dependent backgrounds not well-formulated (Lorentzian metrics on Riemann surfaces?), space-time geometry obtained only indirectly.

Loop quantum gravity positive: Rather general basis for statements about non-singular evolution, indications for bounded curvature, some bounce models with detailed geometry.

Loop quantum gravity negative: Definition of full constraint/algebra, local Lorentz invariance? Uniqueness? (Non-)Separability of Hilbert space?

Common problems

How *state dependent* is singularity resolution?

Is singularity resolution gauge dependent?
(Mini-/midi-superspaces, backgrounds)

Problem of *observables:* Required to extract specific information from wave function, or local bulk properties from boundary field theory.

Problem of *time:* observables, interpretation of wave function, boundary correlators for non-static bulks.

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What is being said about singularities?

$\longrightarrow$ Mostly *special examples.* (Compare with situation before singularity theorems: best-known behavior of solutions may not play large role in general statements.)

$\longrightarrow$ Often, many *difficulties* to be faced at the same time in generalizations. Rarely clear how any one of them *may change picture.*

$\longrightarrow$ Specific examples refer to systems where *classical singularity structure is precisely known;* not good for general statements. (Here, BKL might help.)

$\longrightarrow$ All refer to *diverging curvature;* possibly most pressing, but general?

Useful as long as seen as *consistency tests* of quantum gravity, but not yet suitable for presice and robust statements.