Renormalization of Tensorial Group Field Theories

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Joint work with Daniele Oriti and Vincent Rivasseau: arXiv:1207.6734 [hep-th] and more.

Introduction and motivations

TGFTs are an approach to quantum gravity, which can be justified by two complementary logical paths:

- **The Tensor track** [Rivasseau '12]: matrix models, tensor models [Sasakura '91, Ambjorn et al. '91, Gross '92], $1/N$ expansion [Gurau, Rivasseau '10 '11], universality [Gurau '12], renormalization of tensor *field* theories... [Ben Geloun, Rivasseau '11 '12]

- **The Group Field Theory approach to Spin Foams** [Rovelli, Reisenberger '00, ...]
  - Quantization of simplicial geometry.
  - No triangulation independence $\Rightarrow$ lattice gauge theory limit [Dittrich et al.] or sum over foams.
  - GFT provides a prescription for performing the sum: simplicial gravity path integral = Feynman amplitude of a QFT.
  - Amplitudes are generically divergent $\Rightarrow$ renormalization?
  - Need for a continuum limit $\Rightarrow$ many degrees of freedom $\Rightarrow$ renormalization (phase transition along the renormalization group flow?)

Big question

Can we find a renormalizable TGFT exhibiting a phase transition from discrete geometries to the continuum, and recover GR in the classical limit?

Purpose of this talk

- State of the art: several renormalizable TGFTs with nice topological content:
  - $U(1)$ model in 4d: just renormalizable up to φ^6 interactions, asymptotically free [Ben Geloun, Rivasseau '11, Ben Geloun '12]
  - $U(1)$ model in 3d: just renormalizable up to φ^4 interactions, asymptotically free [Ben Geloun, Samary '12]
  - even more renormalizable models [Ben Geloun, Livine '12]
- Question: what happens if we start adding geometrical data (discrete connection)?

Main message of this talk

Introducing holonomy degrees of freedom is possible, and generically improves renormalizability. It implies a generalization of key QFT notions, including: connectedness, locality and contraction of (high) subgraphs.

Example I: $\mathrm{U}(1)$ super-renormalizable models in $4d$, for any order of interaction.

Example II: a just-renormalizable Boulatov-type model for $\mathrm{SU}(2)$ in $d = 3$!

Outline

1. A class of dynamical models with gauge symmetry
2. Multi-scale analysis
3. $U(1)$ 4d models
4. Just-renormalizable models

- Dynamical variable: rank-d complex field

$$\varphi : (g_1, \dots, g_d) \ni G^d \mapsto \mathbb{C}\,,$$

with G a (compact) Lie group.

- Partition function:

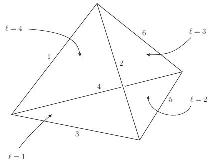
$$\mathcal{Z} = \int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \mathrm{e}^{-S(\varphi,\overline{\varphi})}\,.$$

  - $S(\varphi, \overline{\varphi})$ is the interaction part of the action, and should be a sum of **local** terms.
  - Dynamics + geometrical constraints contained in the **Gaussian measure** $\mathrm{d}\mu_C$ with covariance C (i.e. 2nd moment):

$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \varphi(g_\ell)\overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell)$$

- Natural assumption in d dimensional Spin Foams: elementary building block of space-time $= (d+1)$-simplex.
  In GFT, translates into a φ^{d+1} interaction, e.g. in 3d:

$$S(\varphi, \overline{\varphi}) \propto \int [\mathrm{dg}]^6 \varphi(g_1, g_2, g_3)\varphi(g_3, g_5, g_4)\varphi(g_5, g_2, g_6)\varphi(g_4, g_6, g_1) + \text{ c.c.}$$



Problems:

- Full topology of the simplicial complex not encoded in the 2-complex [Bonzom,Girelli, Oriti '; Bonzom, Smerlak '12];
- (Very) degenerate topologies.

- A way out: add **colors** [Gurau '09]

$$S(\varphi, \overline{\varphi}) \propto \int [\mathrm{dg}]^6 \varphi_1(g_1, g_2, g_3)\varphi_2(g_3, g_5, g_4)\varphi_3(g_5, g_2, g_6)\varphi_4(g_4, g_6, g_1) + \text{ c.c.}$$

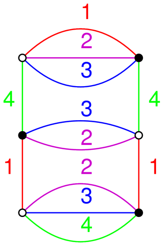
  ... then **uncolor** [Gurau '11; Bonzom, Gurau, Rivasseau '12] i.e. d auxiliary fields and 1 true dynamical field $\Rightarrow$ infinite set of **tensor invariant effective interactions**.

- Instead, **start** from **tensor invariant interactions**. They provide:
  - a good combinatorial control over topologies: full homology, pseudo-manifolds only etc.
  - analytical tools: $1/N$ expansion, universality theorems etc.

- S is a (finite) sum of **connected** tensor invariants, indexed by **d-colored graphs** (d-bubbles):

$$S(\varphi, \overline{\varphi}) = \sum_{b \in \mathcal{B}} t_b I_b(\varphi, \overline{\varphi}) \,.$$

- d-colored graphs are regular (valency d), bipartite, edge-colored graphs.



- Correspondence with tensor invariants:
  - white (resp. black) dot $\leftrightarrow$ field (resp. complex conjugate field);
  - edge of color ℓ $\leftrightarrow$ convolution of ℓ-th indices of φ and $\overline{\varphi}$.

$$\int [\mathrm{d}g_i]^{12} \varphi(g_1, g_2, g_3, g_4) \overline{\varphi}(g_1, g_2, g_3, g_5) \varphi(g_8, g_7, g_6, g_5)$$
$$\overline{\varphi}(g_8, g_9, g_{10}, g_{11}) \varphi(g_{12}, g_9, g_{10}, g_{11}) \overline{\varphi}(g_{12}, g_7, g_6, g_4)$$

- In general, the Gaussian measure has to implement the geometrical constraints:
  - gauge symmetry
$$\forall h \in G\,, \quad \varphi(hg_1, \ldots, hg_d) = \varphi(g_1, \ldots g_d)\,; \tag{1}$$
  - simplicity constraints.

$\Rightarrow$ C expected to be a projector, for instance

$$C(g_1, g_2, g_3; g_1', g_2', g_3') = \int \mathrm{d}h \prod_{\ell=1}^{3} \delta(g_\ell h g_\ell'^{-1}) \tag{2}$$

in 3d gravity (Ponzano-Regge amplitudes).

- But: not always possible in practice...
  - In 4d, with Barbero-Immirzi parameter: simplicity and gauge constraints don't commute $\rightarrow$ C not necessarily a projector.
  - Even when C is a projector, its cut-off version is not $\Rightarrow$ differential operators in radiative corrections e.g. Laplacian in the Boulatov-Ooguri model [Ben Geloun, Bonzom '11].

- Advantage: **built-in notion of scale** from C with non-trivial spectrum.

Gaussian measure II: non-trivial propagators

We would like to have a TGFT with:

- a built-in notion of scale i.e. a non-trivial propagator spectrum;
- a notion of discrete connection at the level of the amplitudes.

Particular realization that we consider:

- Gauge constraint:

$$\forall h \in G \,, \quad \varphi(hg_1, \ldots, hg_d) = \varphi(g_1, \ldots g_d) \,, \tag{3}$$

- supplemented by the non-trivial kernel (conservative choice, also justified by [Ben Geloun, Bonzom '11])

$$\left(m^2 - \sum_{\ell=1}^{d} \Delta_\ell \right)^{-1} . \tag{4}$$

This defines the measure $\mathrm{d}\mu_C$:

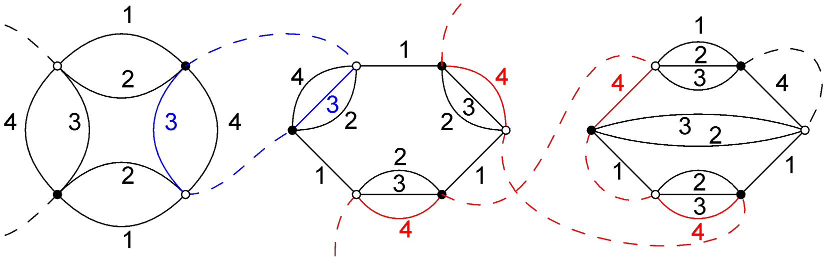
$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi}) \, \varphi(g_\ell)\overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell) = \int_0^{+\infty} \mathrm{d}\alpha \, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{d} K_\alpha(g_\ell h g'^{-1}_\ell) \,, \tag{5}$$

where K_α is the heat kernel on G at time α.

Feynman graphs

- The amplitudes are indexed by $(d+1)$-colored graphs, obtained by connecting d-bubble vertices through propagators (dotted, color-0 lines).

  Example: 4-point graph with 3 vertices and 6 (internal) lines.



- Nomenclature:
  - $L(\mathcal{G})$ = set of (dotted) lines of a graph $\mathcal{G}$.
  - **Face of color ℓ** = connected set of (alternating) color-0 and color-ℓ lines.
  - $F(\mathcal{G})$ (resp. $F_{ext}(\mathcal{G})$) = set of internal (resp. external) i.e. closed (resp. open) faces of $\mathcal{G}$.

Amplitudes and gauge symmetry

- The amplitude of $\mathcal{G}$ depends on oriented products of group elements along its faces:

$$\mathcal{A}_{\mathcal{G}} = \left[\prod_{e \in L(\mathcal{G})} \int \mathrm{d}\alpha_e \, e^{-m^2 \alpha_e} \int \mathrm{d}h_e \right] \left(\prod_{f \in F(\mathcal{G})} K_{\alpha(f)} \left(\overrightarrow{\prod_{e \in \partial f}} {h_e}^{\epsilon_{ef}} \right) \right)$$
$$\left(\prod_{f \in F_{ext}(\mathcal{G})} K_{\alpha(f)} \left(g_{s(f)} \left[\overrightarrow{\prod_{e \in \partial f}} {h_e}^{\epsilon_{ef}} \right] g_{t(f)}^{-1} \right) \right) ,$$
$$= \left[\prod_{e \in L(\mathcal{G})} \int \mathrm{d}\alpha_e \, e^{-m^2 \alpha_e} \right] \{ \textit{ Regularized Boulatov-like amplitudes } \}$$

where $\alpha(f) = \sum_{e \in \partial f} \alpha_e$, $g_{s(f)}$ and $g_{t(f)}$ are boundary variables, and $\epsilon_{ef} = \pm 1$ **when** $e \in \partial f$ **is the incidence matrix between oriented lines and faces**.

- A **gauge symmetry** associated to vertices ($h_e \mapsto g_{t(e)} h_e g_{s(e)}^{-1}$) allows to impose $h_e = \mathbf{1}$ along a maximal tree of (dotted) lines.

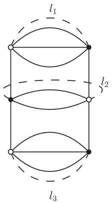
Spin Foam wisdom: lines $\to$ faces; faces $\to$ bubbles.

Amplitudes depend on holonomies along faces, built from group elements associated to lines $\Rightarrow$ new notion of connectedness: incidence relations between lines and faces instead of incidence relations between vertices and lines.

Definition

- A **subgraph** $\mathcal{H} \subset \mathcal{G}$ is a subset of (dotted) lines of $\mathcal{G}$.
- **Connected components** of $\mathcal{H}$ are the subsets of lines of the maximal factorized rectangular blocks of its ϵ_{ef} incidence matrix.

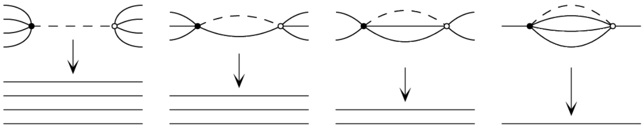
Equivalently, two lines of $\mathcal{H}$ are elementarily connected if they have a common internal face in $\mathcal{H}$, and we require transitivity.



- $\mathcal{H}_1 = \{l_1\}$, $\mathcal{H}_{12} = \{l_1, l_2\}$ are connected;
- $\mathcal{H}_{13} = \{l_1, l_3\}$ has two connected components (despite the fact that there is a single vertex!).

Contraction of a subgraph

- The **contraction** of a line is implemented by so-called **dipole moves**, which in $d = 4$ are:



Definition: k-dipole = line appearing in exactly k closed faces of length 1.

- The contraction of a subgraph $\mathcal{H} \subset \mathcal{G}$ is obtained by successive contractions of its lines.

Net result

The contraction of a subgraph $\mathcal{H} \in \mathcal{G}$ amounts to delete all the internal faces of $\mathcal{H}$ and reconnect its external legs according to the pattern of its external faces.

$\Rightarrow$ well-suited for coarse-graining / renormalization steps!

Remark Would be interesting to analyse these moves in a coarse-graining context [Dittrich et al.].

Strategy

1) Decompose amplitudes according to slices of "momenta" (Schwinger parameter);
2) Replace high divergent subgraphs by effective local vertices;
3) Iterate.

$\Rightarrow$ Effective multi-series (1 effective coupling per interaction at each scale).

Can be reshuffled into a renormalized series (1 renormalized coupling per interaction).

Advantages of the effective series:

- Physically transparent, in particular for overlapping divergencies;
- No "renormalons": $|\mathcal{A}_{\mathcal{G}}| \leq K^n$.

- The Schwinger parameter α determines a momentum scale, which can be sliced in a geometric way. One fixes $M > 1$ and decomposes the propagators as

$$C = \sum_i C_i \,, \tag{6}$$

$$C_0(g_\ell; g'_\ell) = \int_1^{+\infty} \mathrm{d}\alpha \, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{d} K_\alpha(g_\ell h g'^{-1}_\ell) \tag{7}$$

$$C_i(g_\ell; g'_\ell) = \int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha \, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{d} K_\alpha(g_\ell h g'^{-1}_\ell) \,. \tag{8}$$

- A natural regularization is provided by a cut-off on i: $i \leq \rho$. To be removed by renormalization.

- The amplitude of a connected graph $\mathcal{G}$ is decomposed over scale attributions $\mu = \{i_e\}$ where i_e runs over all integers (smaller than ρ) for every line e:

$$\mathcal{A}_{\mathcal{G}} = \sum_\mu \mathcal{A}_{\mathcal{G},\mu} \,.$$

Strategy

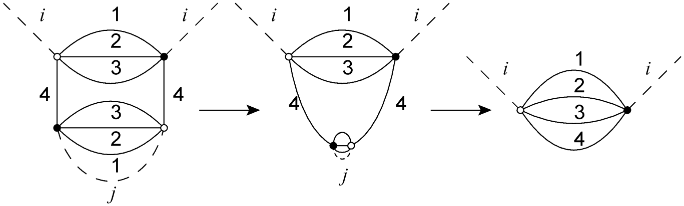
Find optimal bounds on each $\mathcal{A}_{\mathcal{G},\mu}$, in terms of the scales μ.

High subgraphs

To a couple $(\mathcal{G}, \mu)$ is associated a set of **high subgraphs** $\mathcal{G}_i^{(k)}$: for each i, one defines $\mathcal{G}_i$ as the subgraph made of all lines with scale higher or equal to i, and $\{\mathcal{G}_i^{(k)}\}$ its connected components.

Necessary condition: **divergent high subgraphs must be quasi-local, i.e. look like (connected) tensor invariants.**

Example: $i < j$



Contractiblity and traciality

2 sources of loss of locality:

- When $i \to +\infty$, $H_f(\{h_e\}) \to \mathbf{1}$ in $\mathcal{G}_i^{(k)}$, but not necessarily $h_e \to \mathbf{1}$;
- Combinatorial loss of connectedness when contracting a $\mathcal{G}_i^{(k)}$.

We therefore define

Definition

- A connected subgraph $\mathcal{H} \subset \mathcal{G}$ is called **contractible** if there exists a maximal tree of lines $\mathcal{T} \subset L(\mathcal{H})$ such that

$$\left(\forall f \in F_{int}(\mathcal{H}) \, , \, \overrightarrow{\prod_{e \in \partial f}} {h_e}^{\epsilon_{ef}} = \mathbf{1} \right) \Rightarrow \left(\forall e \in L(\mathcal{H}) \, , \; h_e = \mathbf{1} \right)$$

  for any assignment of group elements $(h_e)_{e \in L(\mathcal{H})}$ that verifies $h_e = \mathbf{1}$ for any $e \in \mathcal{T}$. (approximate invariance)
- A connected subgraph $\mathcal{H} \subset \mathcal{G}$ is called **tracial** if it is contractible and its contraction in $\mathcal{G}$ conserves its connectedness. (approximate connected invariance)

Theorem

(i) If G has dimension D, there exists a constant K such that the following bound holds:

$$|\mathcal{A}_{\mathcal{G},\mu}| \leq K^{L(\mathcal{G})} \prod_{(i,k)} M^{\omega[\mathcal{G}_i^{(k)}]} , \tag{9}$$

where the **degree of divergence** ω is given by

$$\omega(\mathcal{H}) = -2L(\mathcal{H}) + D(F_{int}(\mathcal{H}) - r(\mathcal{H})) \tag{10}$$

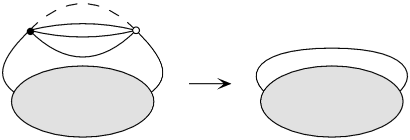
and $r(\mathcal{H})$ is the rank of the ϵ_{ef} incidence matrix of $\mathcal{H}$.

(ii) These bounds are optimal when G is Abelian, or when $\mathcal{H}$ is contractible.

- Subgraphs with $\omega < 0$ are **convergent** i.e. have finite contributions when $\rho \to \infty$.
- Subgraphs with $\omega \geq 0$ are **divergent** and need to be renormalized. Traciality (or at the very least contractiblity) of divergent subgraphs is therefore needed for renormalizability to hold.

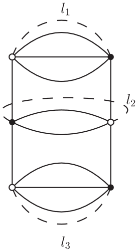
$U(1)$ 4d models

The renormalization of such models is triggered by so-called **melopoles**. They are the tadpole connected subgraphs that can be reduced to a single line by successive 4-dipole contractions.



Example:

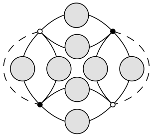
- $\mathcal{H} = \{l_1\}$, $\mathcal{H} = \{l_1, l_2\}$ or $\mathcal{H} = \{l_1, l_2, l_3\}$ are melopoles;
- $\mathcal{H} = \{l_2\}$ and $\mathcal{H} = \{l_1, l_3\}$ are not (the last one because it is not connected).



Classification of subgraphs

Theorem

- If $\omega(\mathcal{H}) = 1$, then $\mathcal{H}$ is a **vacuum melopole**.
- If $\omega(\mathcal{H}) = 0$, then $\mathcal{H}$ is either a **non-vacuum melopole**, or a **submelonic vacuum graph**.
- Otherwise, $\omega(\mathcal{H}) \leq -1$ and $\omega(\mathcal{H}) \leq -\frac{N(\mathcal{H})}{4}$, $N(\mathcal{H})$ being the number of external legs of $\mathcal{H}$.



Submelonic vacuum graph: grey blobs represent melopole insertions.

Corollary

For a given finite set of non-zero couplings, the theory has a finite set of divergent subgraphs.

Melordering

Lemma

Melopoles are tracial.

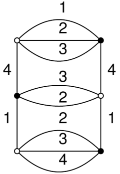
Renormalization is therefore possible in the realm of connected tensor invariants.

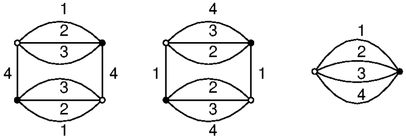
- One can use a **Wick ordering** procedure to remove divergencies. It is given by a linear map:

$$\Omega_\rho : \{invariants\} \to \{invariants\}$$

  depending on the cut-off ρ.

- Precise expression of $\Omega_\rho(I_b)$ given as a sum over all possible contractions of melopoles in b.





Finiteness

One defines the renormalized theory through melordering:

$$\mathcal{Z}_{\Omega_\rho} = \int \mathrm{d}\mu_{C_\rho}(\varphi, \overline{\varphi})\, \mathrm{e}^{-S_{\Omega_\rho}(\varphi, \overline{\varphi})},$$

$$S_{\Omega_\rho}(\varphi, \overline{\varphi}) = \sum_{b \in \mathcal{B}} t_b^R\, \Omega_\rho(I_b)(\varphi, \overline{\varphi}).$$

Theorem

For any finite set of non-zero renormalized couplings $\{t_b^R\}$, the amplitudes are convergent when $\rho \to +\infty$.

Conclusion: U(1) 4d models with gauge symmetry are super-renormalizable at any order of perturbation theory.

Hypotheses:

- rank-d tensors;
- G of dimension D;
- v_{max} = maximal order of interactions.

Question: necessary conditions on d, D and v_{max} in order to construct just-renormalizable models (i.e. with infinite sets of divergent graphs) ?

Notations:

- $n_{2k}(\mathcal{H})$ = number of vertices with valency $2k$ in $\mathcal{H}$;
- $N(\mathcal{H})$ = number of external legs attached to vertices of $\mathcal{H}$;
- $\mathcal{H}/\mathcal{T}$ = contraction of $\mathcal{H}$ along a tree of lines (gauge-fixing).

Proposition

Let $\mathcal{H}$ be a non-vacuum subgraph. Then:

$$\begin{aligned}\omega(\mathcal{H}) &= D\,(d-2) - \frac{D(d-2)-2}{2}N &(11)\\ &- \sum_{k=1}^{v_{max}/2-1} [D\,(d-2) - (D(d-2)-2)\,k]n_{2k} &(12)\\ &+ D\rho(\mathcal{H}/\mathcal{T})\,, &(13)\end{aligned}$$

with

$$\rho(\mathcal{G}) \leq 0 \qquad \text{and} \qquad \rho(\mathcal{G}) = 0 \Leftrightarrow \mathcal{G} \text{ is a melopole}\,. \qquad (14)$$

Type	d	D	v_{max}	ω
A	3	3	6	$3 - N/2 - 2n_2 - n_4 + 3\rho$
B	3	4	4	$4 - N - 2n_2 + 4\rho$
C	4	2	4	$4 - N - 2n_2 + 2\rho$
D	5	1	6	$3 - N/2 - 2n_2 - n_4 + \rho$
E	6	1	4	$4 - N - 2n_2 + \rho$

Table: Classification of potentially just-renormalizable models.

φ^6 model on $\mathrm{SU}(2)$, in $d = 3$

$$\omega(\mathcal{H}) = 3 - \frac{N}{2} - 2n_2 - n_4 + 3\rho(\mathcal{H}/\mathcal{T}) \tag{15}$$

N	n_2	n_4	ρ	ω
6	0	0	0	0
4	0	0	0	1
4	0	1	0	0
2	0	0	0	2
2	0	1	0	1
2	0	2	0	0
2	1	0	0	0

Table: Classification of non-vacuum divergent graphs for $d = D = 3$. All of them are melonic.

Theorem

The φ^6 $\mathrm{SU}(2)$ model in $3d$ is renormalizable. Divergencies generate coupling constants, mass and wave-function counter-terms.

Conclusions and outlook

Summary:

- Introducing connection degrees of freedom is possible in renormalizable TGFTs.
- Generically improves renormalizability.
- $U(1)$ 4d models with any finite number of interactions are super-renormalizable.
- 5 types of just-renormalizable models, including a $SU(2)$ model in $d = 3$.

What's next?

- Flow of the $SU(2)$ model in $3d$ [wip]: asymptotic freedom? relation to Ponzano-Regge?
- Constructibility (of $\mathrm{U}(1)$ models first) [Gurau wip].
- Generalization to 4d gravity models [wip]: EPRL, FK, BO, etc.
  - geometry: interplay between simplicity constraints and tensor invariance?
  - with or without Laplacian (or other differential operator)?

Thank you for your attention