THE QUANTUM INFORMATION STRUCTURE OF SPACETIME

Entropy and Hilbert Spaces from Gravitational Path Integrals

Based on: arXiv:2310.02189 with Xi Dong, Donald Marolf and Zhencheng Wang

Eugenia Colafranceschi
Western University

International Loop Quantum Gravity Seminars

February 6, 2024

Motivations

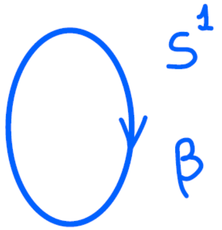
Entropy via the Euclidean path integral

Motivations

Entropy via the Euclidean path integral

➢ Standard quantum system:

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int_{S^1} \mathcal{D}q e^{-S_E} =$$

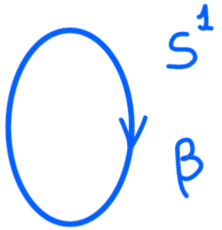


Motivations

Entropy via the Euclidean path integral

- Standard quantum system:

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int_{S^1} \mathcal{D}q e^{-S_E} =$$



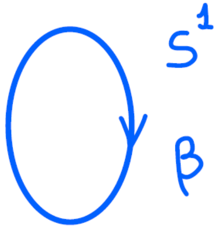
$$\longrightarrow \quad S = \beta \frac{dS_E}{d\beta} - S_E$$

Motivations

Entropy via the Euclidean path integral

➢ Standard quantum system:

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int_{S^1} \mathcal{D}q\, e^{-S_E} =$$



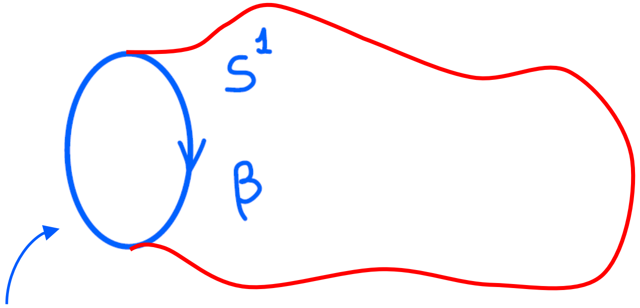
$$\rightarrow \quad S = \beta \frac{dS_E}{d\beta} - S_E$$

➢ Gravitational system?

guess:

(Hamiltonian in gravity is a boundary term)

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int_{S^1 \times \partial\Sigma} \mathcal{D}g\, e^{-S_E}$$

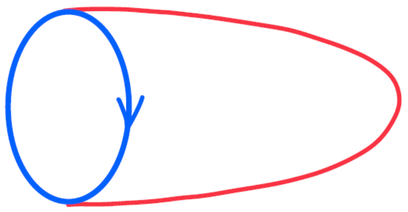


boundary conditions

Motivations

Indeed, for the Euclidean black hole:

$$Z = \mathrm{Tr}(e^{-\beta H}) \approx e^{-S_E}$$



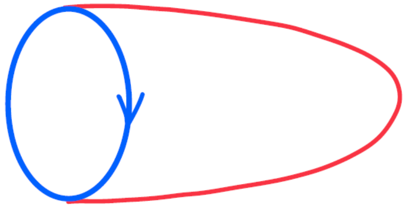
$$\Rightarrow \quad S = -S_E + \beta \frac{dS_E}{d\beta} = \frac{A_{\mathrm{horizon}}}{4G}$$

Bekenstein-Hawking
entropy

Motivations

Indeed, for the Euclidean black hole:

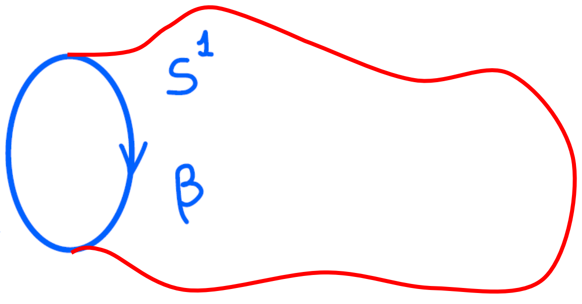
$$Z = \mathrm{Tr}(e^{-\beta H}) \approx e^{-S_E}$$



$$\Rightarrow \quad S = -S_E + \beta \frac{dS_E}{d\beta} = \frac{A_{\text{horizon}}}{4G}$$

Bekenstein-Hawking entropy

Note: this is a special case! $\rho = e^{-\beta H}$ equilibrium state



time-translation symmetry!

Motivations

Without time-translation symmetry? Which boundary conditions?

Motivations

Without time-translation symmetry? Which boundary conditions?

Guess: the **boundary conditions** relevant to computing some gravitational quantity are given by whatever **path integral** would compute the **analogue quantity** in a **non-gravitational theory**

Motivations

Without time-translation symmetry? Which boundary conditions?

Guess: the **boundary conditions** relevant to computing some gravitational quantity are given by whatever **path integral** would compute the **analogue quantity** in a **non-gravitational theory**

Replica trick: $$S_{vN}(\rho_R) = -\underline{\mathrm{Tr}(\rho_R \ln \rho_R)} = \lim_{n\to 1} -\frac{1}{n-1} \ln \underline{\mathrm{Tr}(\rho_R^n)}$$

easier!

which boundary conditions for a gravitational system?

Motivations

Without time-translation symmetry? Which boundary conditions?

Guess: the **boundary conditions** relevant to computing some gravitational quantity are given by whatever **path integral** would compute the **analogue quantity** in a **non-gravitational theory**

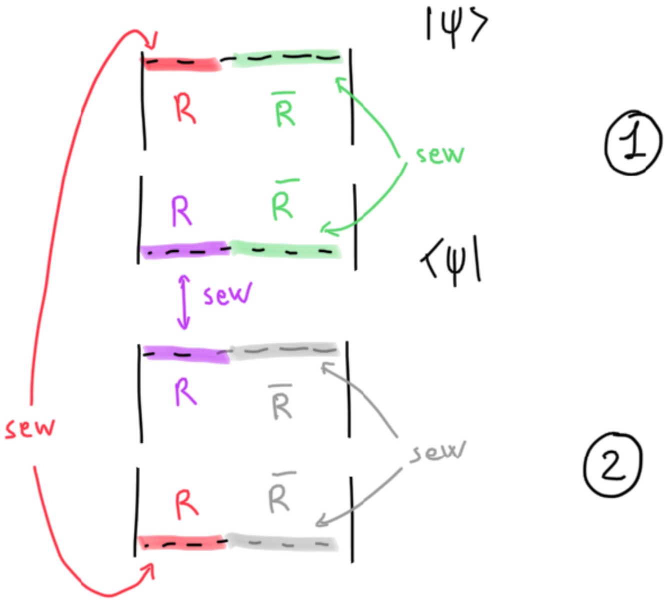
Replica trick: $S_{vN}(\rho_R) = \underline{-\mathrm{Tr}(\rho_R \ln \rho_R)} = \lim_{n\to 1} -\frac{1}{n-1} \ln \underline{\mathrm{Tr}(\rho_R^n)}$

easier!

which boundary conditions for a gravitational system?

QFT

$$e^{-S_2} = \mathrm{Tr}_R(\rho_R^2) =$$



closed manifold

Motivations

Without time-translation symmetry? Which boundary conditions?

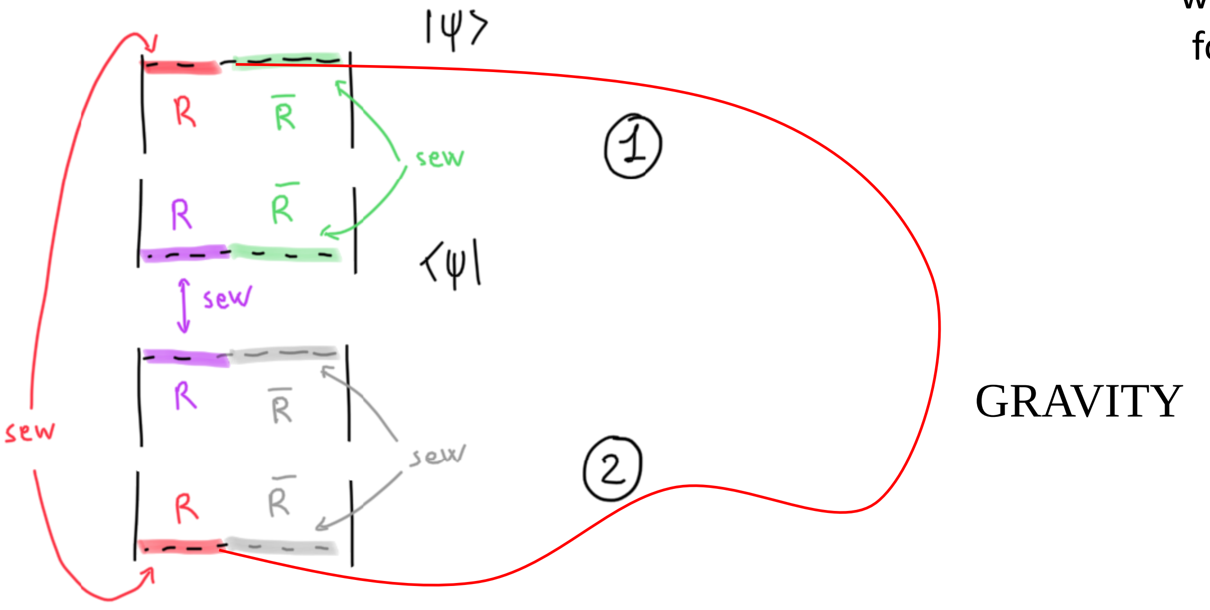
Guess: the **boundary conditions** relevant to computing some gravitational quantity are given by whatever **path integral** would compute the **analogue quantity** in a **non-gravitational theory**

Replica trick: $$S_{vN}(\rho_R) = \underline{-\mathrm{Tr}(\rho_R \ln \rho_R)} = \lim_{n\to 1} -\frac{1}{n-1} \ln \underline{\mathrm{Tr}(\rho_R^n)}$$

easier!

which boundary conditions for a gravitational system?

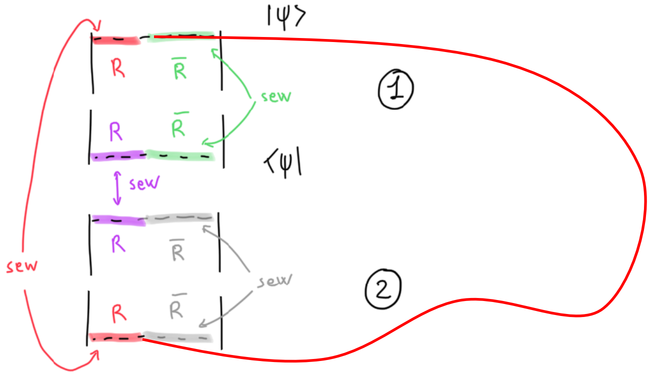
$$e^{-S_2} = \mathrm{Tr}_R(\rho_R^2) =$$



closed manifold
= BCs for the gravitational path integral

Motivations

Lewkowycz, Maldacena (2013)

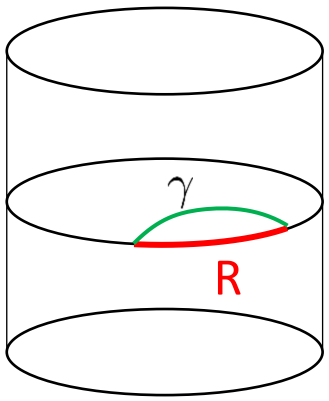


$\rightarrow$ $S_R = \dfrac{A_\gamma}{4G}$

semiclassical
approx.

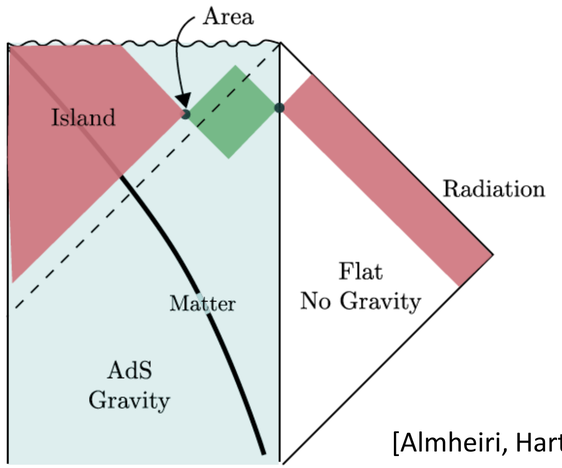
Ryu-Takayanagi formula

(first derived in AdS/CFT)



- ➢ Holography does not enter the derivation, but
- ➢ It is required for the interpretation as standard entropy, i.e. $S_R = -\mathrm{Tr}(\rho_R \ln \rho_R)$

Motivations



[Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini 2019]

Implication for the case of **Hawking radiation** from AdS to a bath:

In appropriate semiclassical limits, the von Neumann entropy of the bath is given by the **island formula**, a **special case of the quantum-corrected RT formula**, and follows the **Page curve**.

[Penington 2019; Almheiri, Engelhardt, Marolf, Maxfield 2019; Penington, Shenker, Stanford, Yang 2019; Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini 2019]

✓ Possible solution to the bulk interpretation problem: the gravitational replica trick computes the entropy of the emitted Hawking radiation in a superselection sector. [Marolf, Maxfield, 2020]

Although inspired from AdS/CFT, the argument relies only on properties of the gravitational path integral!

Can this story be generalized?

Motivations

- Consider a gravitational system with closed asymptotic boundaries B_L and B_R

- The Hilbert space $\mathcal{H}_{LR}$ for this two-boundary gravitational system *a priori does not factorize*

  - If the gravitational system has a holographic dual, $\mathcal{H}_{LR} = \mathcal{H}_L \otimes \mathcal{H}_R$ and we can then associate a "state-counting" entropy to B_L and B_R. But we are not going to assume holography.

- Goal: construct, from purely-bulk arguments, a Hilbert space $\mathcal{H}_L$ associated with B_L such that the associated Ryu-Takayanagi entropy can be understood in terms of a standard trace on $\mathcal{H}_L$:

$$S_{\mathrm{vN}}(\rho_L) := -\mathrm{Tr}_L(\rho_L \ln \rho_L)$$

$$\rho_L = \mathrm{Tr}_R(\rho)$$

Motivations

- Recent works [Chandrasekaran, Longo, Penington, Witten, Jensen, Sorce, Speranza, Kudler-Flam ,Leutheusser, Satishchandran, …] have shown that, in various contexts, the Ryu-Takayanagi entropy can be derived (up to an infinite constant) as the entropy of a type II von Neumann algebra. This provides a "statistical interpretation" for the RT entropy (thanks to the type II trace).

- For a standard quantum mechanical system, we have an entropy in terms of a Hilbert space trace, which provides a "state-counting interpretation". A Hilbert space trace corresponds to a **type I trace.** $\mathrm{Tr}(\cdot) = \sum_i \langle i| \cdot |i\rangle$

$\Rightarrow$ Can we understand the Ryu-Takayanagi entropy in terms of a Hilbert space trace, i.e. as a state-counting entropy?

THIS TALK:

In a UV-complete, asymptotically locally AdS theory of quantum gravity in which the **Euclidean path integral** satisfies a **simple set of axioms**, it is possible to associate a **von Neumann entropy** to B_L which, in the semiclassical limit, is given by the Ryu-Takayanagi formula. No need to invoke holography!

Outline

1. Axioms for the Euclidean Path Integral

2. Hilbert Space from the Path Integral

3. Operator Algebras from the Path Integral

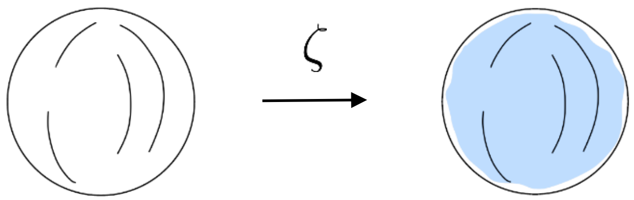
4. Type I von Neumann Factors

5. Hilbert Space Interpretation of the Ryu-Takayanagi Entropy

1. Axioms for the Euclidean Path Integral

The Euclidean Gravitational Path Integral

We consider a UV-complete finite-coupling asymptotically-locally-AdS theory of gravity with an ‘Euclidean path integral’, an object that, to every closed codimension-1 boundary M (with appropriate boundary conditions), assigns a complex number; e.g.



$$\zeta(M) = \int_{\phi \sim M} \mathcal{D}\phi e^{-S[\phi]}$$

$$\phi = g, \phi^{\mathrm{matter}}$$

$$M \supset g_M, \phi_M^{\mathrm{matter}}$$

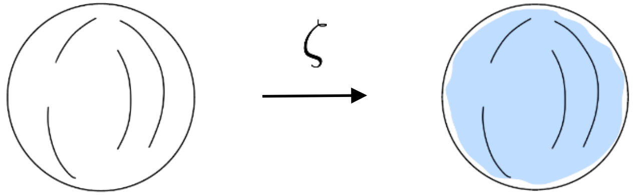
“source manifold”

Should be: finite, continuous, and $[\zeta(M)]^* = \zeta(M^*)$

complex conjugated sources

The Euclidean Gravitational Path Integral

We consider a UV-complete finite-coupling asymptotically-locally-AdS theory of gravity with an ‘Euclidean path integral’, an object that, to every closed codimension-1 boundary M (with appropriate boundary conditions), assigns a complex number; e.g.



$$\zeta(M) = \int_{\phi\sim M} \mathcal{D}\phi e^{-S[\phi]}$$

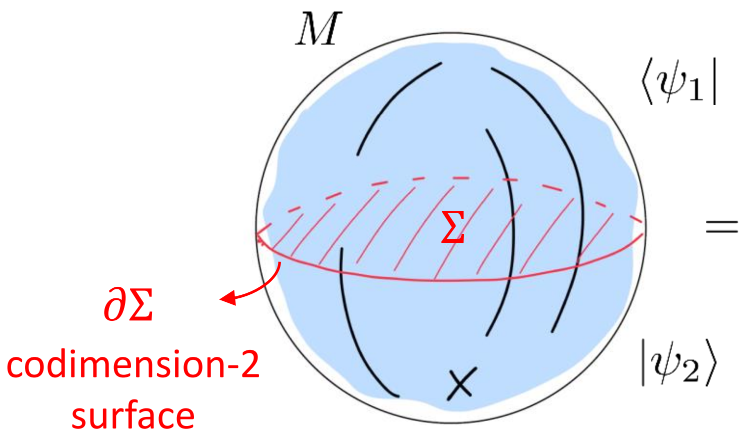
$$\phi = g, \phi^{\text{matter}}$$

$$M \supset g_M, \phi_M^{\text{matter}}$$

“source manifold”

Should be: finite, continuous, and $[\zeta(M)]^* = \zeta(M^*)$

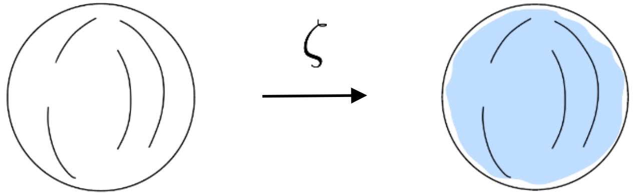
complex conjugated sources



$$= \langle\psi_1|\psi_2\rangle \text{ on } \mathcal{H}_{\partial\Sigma}$$

The Euclidean Gravitational Path Integral

We consider a UV-complete finite-coupling asymptotically-locally-AdS theory of gravity with an 'Euclidean path integral', an object that, to every closed codimension-1 boundary M (with appropriate boundary conditions), assigns a complex number; e.g.



$$\zeta(M) = \int_{\phi \sim M} \mathcal{D}\phi e^{-S[\phi]}$$

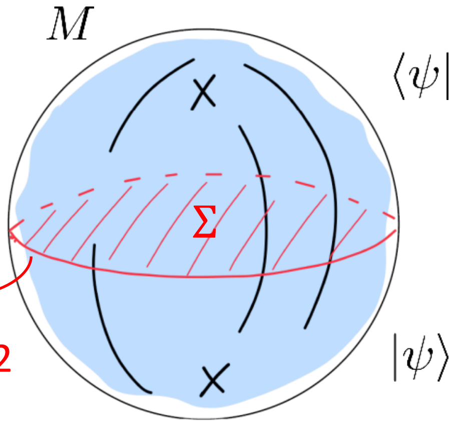
$$\phi = g, \phi^{\text{matter}}$$

$$M \supset g_M, \phi_M^{\text{matter}}$$

"source manifold"

Should be: finite, continuous, and $[\zeta(M)]^* = \zeta(M^*)$

complex conjugated sources



$$= \langle\psi|\psi\rangle \geq 0$$

$\partial\Sigma$

codimension-2 surface

Axioms

1. Finiteness: The path integral gives a well-defined map ζ from boundary conditions defined by smooth manifolds to the complex numbers $\mathbb{C}$

Axioms

1. Finiteness: The path integral gives a well-defined map ζ from boundary conditions defined by smooth manifolds to the complex numbers $\mathbb{C}$

2. Reality: ζ is a real function of (possibly complex) boundary conditions, i.e. $[\zeta(M)]^* = \zeta(M^*)$

Axioms

1. Finiteness: The path integral gives a well-defined map ζ from boundary conditions defined by smooth manifolds to the complex numbers $\mathbb{C}$

2. Reality: ζ is a real function of (possibly complex) boundary conditions, i.e. $[\zeta(M)]^* = \zeta(M^*)$

3. Reflection Positivity: ζ is reflection-positive

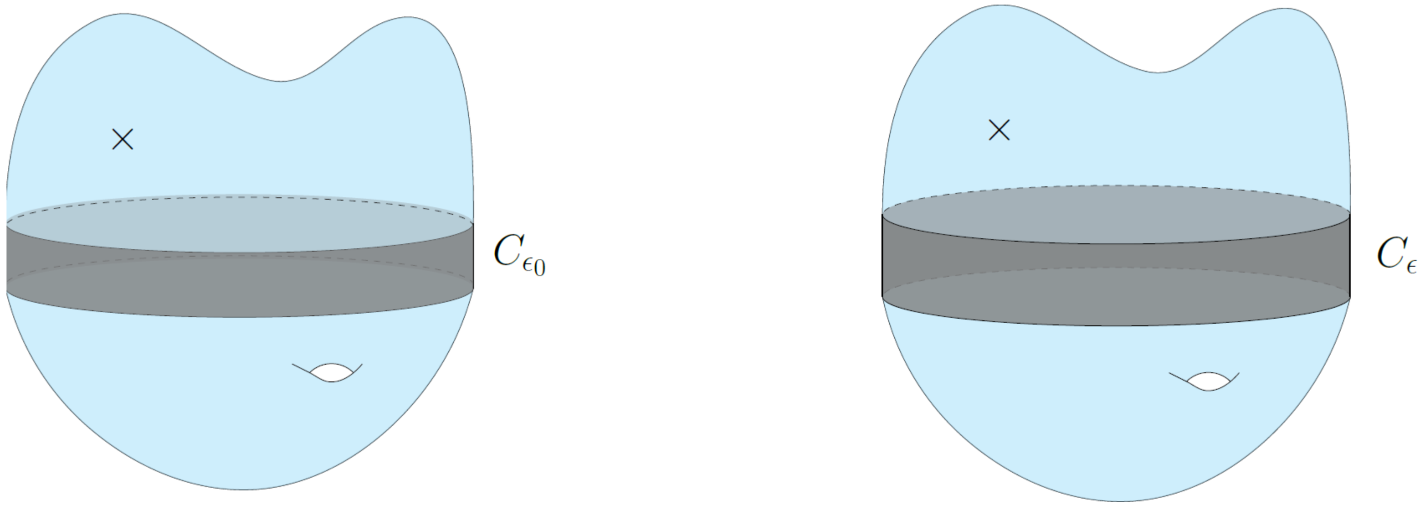
Axioms

1. Finiteness: The path integral gives a well-defined map ζ from boundary conditions defined by smooth manifolds to the complex numbers $\mathbb{C}$

2. Reality: ζ is a real function of (possibly complex) boundary conditions, i.e. $[\zeta(M)]^* = \zeta(M^*)$

3. Reflection Positivity: ζ is reflection-positive

4. Continuity: if the boundary manifold contains a cylinder of size ε, ζ is continuous under changes of ε



Axioms

1. Finiteness: The path integral gives a well-defined map ζ from boundary conditions defined by smooth manifolds to the complex numbers $\mathbb{C}$

2. Reality: ζ is a real function of (possibly complex) boundary conditions, i.e. $[\zeta(M)]^* = \zeta(M^*)$

3. Reflection Positivity: ζ is reflection-positive

4. Continuity: if the boundary manifold contains a cylinder of size ε, ζ is continuous under changes of ε

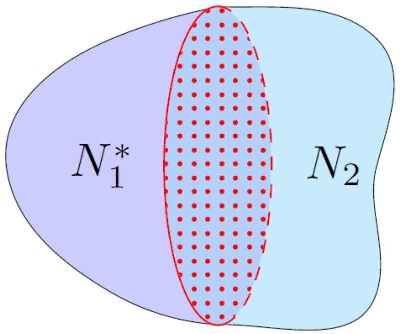
5. Factorization: For closed boundary manifolds M_1, M_2 we have $\zeta(M_1 \sqcup M_2) = \zeta(M_1)\zeta(M_2)$

   Note: if the path integral is equivalent to a **collection of "baby universe superselection sectors"** [Coleman, Giddings, Strominger, Marolf, Maxfield, …] the factorization property holds sector-by-sector, and our analysis applies in that sense.

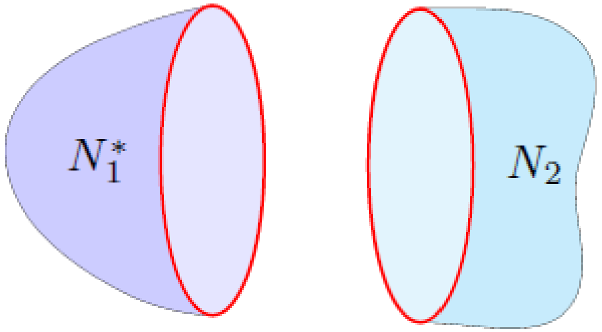
2. Hilbert Space from the Path Integral

Hilbert Space from the Path Integral

When we "cut open a quantum gravity path integral", we cut the closed boundary into two pieces N_1, N_2 with $\partial N_1 = \partial N_2$, then associate states with these two pieces such that



$M_{N_1^* N_2}$

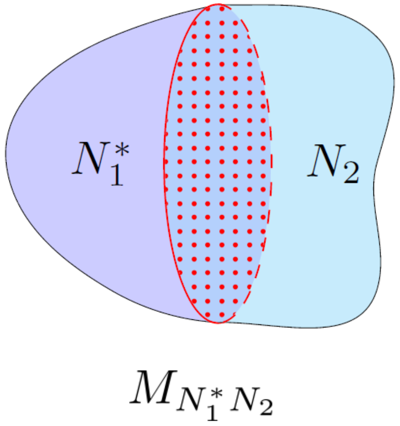


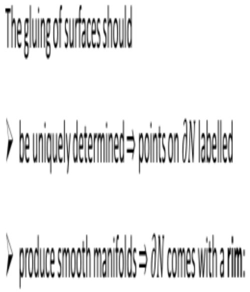
∂N ∂N

$$\langle N_1 | N_2 \rangle := \zeta(M_{N_1^* N_2})$$

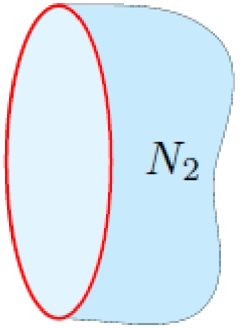
Hilbert Space from the Path Integral

When we “cut open a quantum gravity path integral”, we cut the closed boundary into two pieces N_1, N_2 with $\partial N_1 = \partial N_2$, then associate states with these two pieces such that





∂N

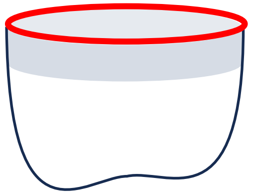


∂N

$$\langle N_1 | N_2 \rangle := \zeta(M_{N_1^* N_2})$$

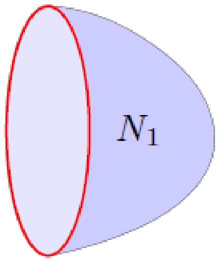
The gluing of surfaces should

- be uniquely determined ⇒ points on ∂N labelled
- produce smooth manifolds ⇒ ∂N comes with a **rim**:



∂N

Hilbert Space from the Path Integral



$Y^d_{\partial N}$

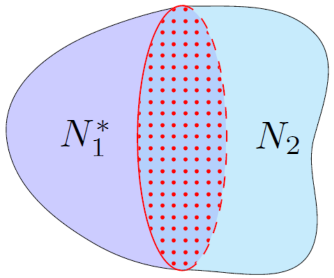
compact d-dim manifolds
with boundary (+rim) ∂N

$H_{\partial N}$

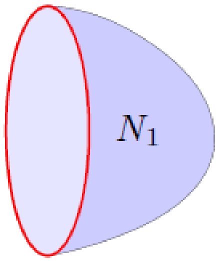
(pre-)Hilbert space

$N \quad \longleftrightarrow \quad |N\rangle$

$$\langle N_1|N_2\rangle := \zeta(M_{N_1^* N_2})$$



Hilbert Space from the Path Integral



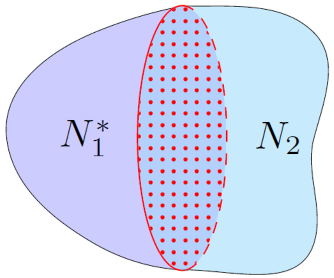
$Y^d_{\partial N}$		$H_{\partial N}$
compact d-dim manifolds with boundary (+rim) ∂N		(pre-)Hilbert space
N	$\longleftrightarrow$	$\lvert N\rangle$

$$\langle N_1 | N_2 \rangle := \zeta(M_{N_1^* N_2})$$

$$+ \quad /\mathcal{N}_{\partial N}$$

$+$ completion

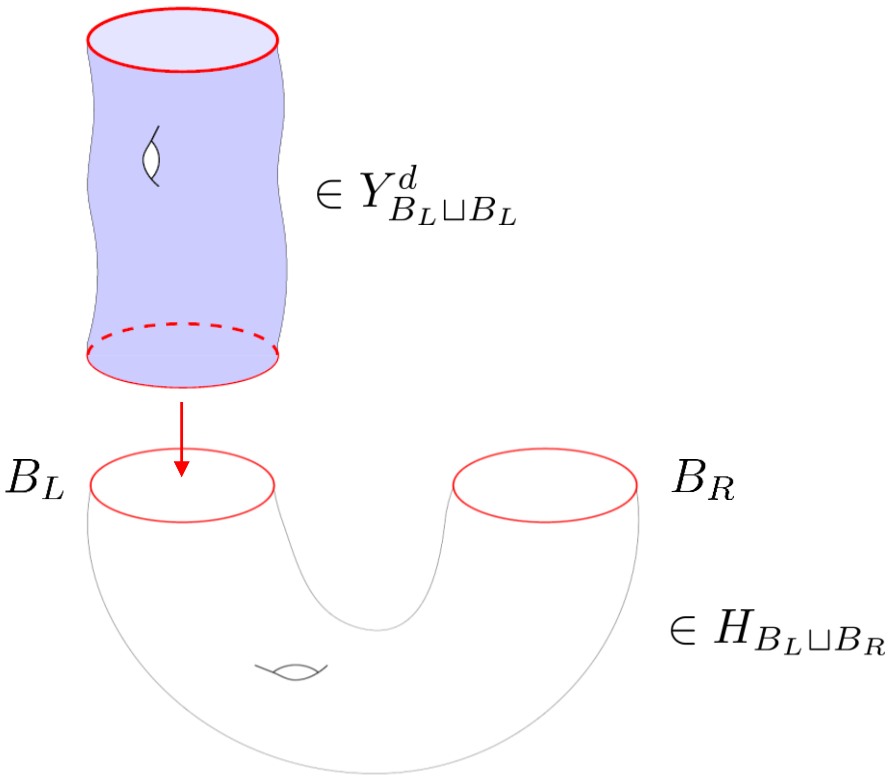
$$= \mathcal{H}_{\partial N}$$



3. Operator Algebras from the Path Integral

Surface Algebra

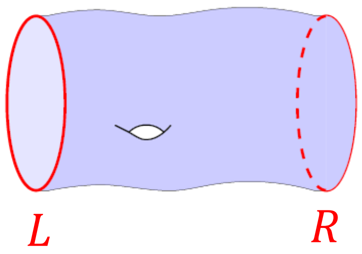
Consider $\partial N = B_L \sqcup B_L$



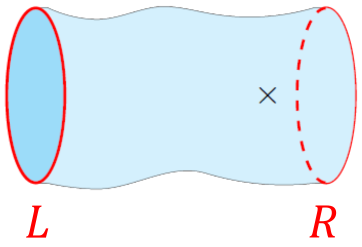
Surface Algebra

- On the set $Y^d_{B\sqcup B}$ we define a left product and a right product:

$a =$

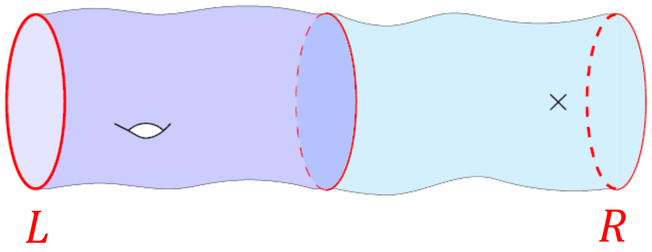


$b =$



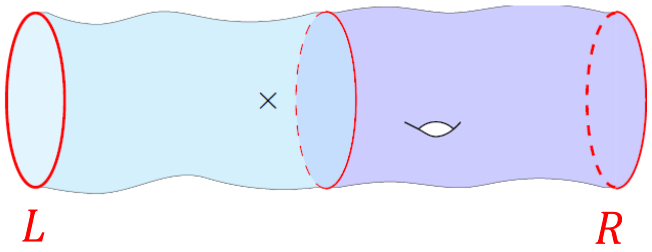
<u>left product:</u> $(\cdot_L)$

$a \cdot_L b =$



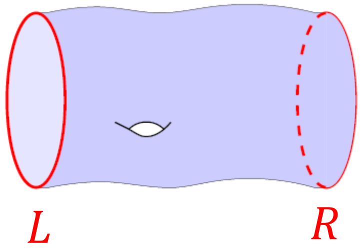
<u>right product:</u> $(\cdot_R)$

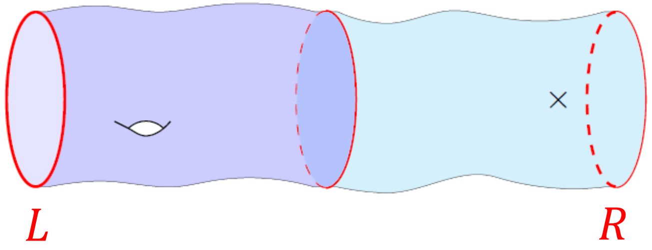
$a \cdot_R b =$

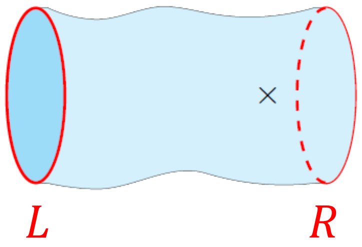


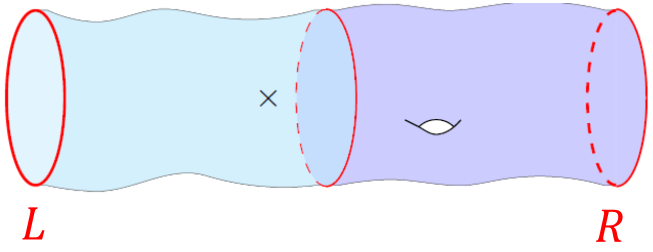
Surface Algebra

- On the set $Y^{d}_{B\sqcup B}$ we define a left product and a right product:

$a =$ 

left product: $(\cdot_L)$ $\quad a \cdot_L b =$ 

$b =$ 

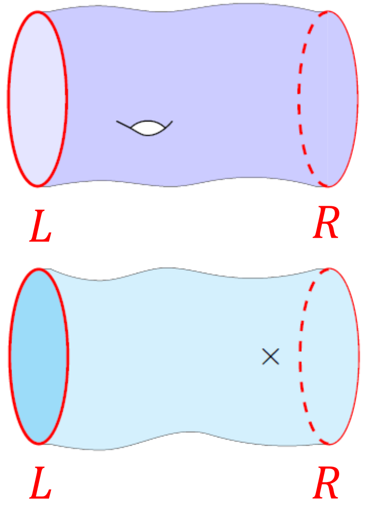
right product: $(\cdot_R)$ $\quad a \cdot_R b =$ 

- The set $Y^{d}_{B\sqcup B}$ equipped with the left/right product defines a left/right surface algebra $A^{B}_{L/R}$

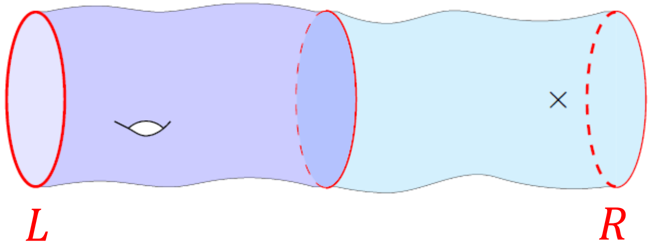
Surface Algebra

- On the set $Y^d_{B\sqcup B}$ we define a left product and a right product:

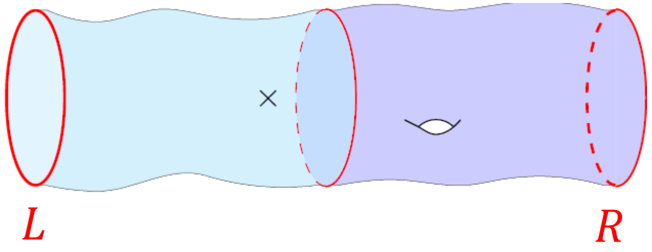
$a =$, $b =$



<u>left product:</u> $(\cdot_L)$ $a \cdot_L b =$



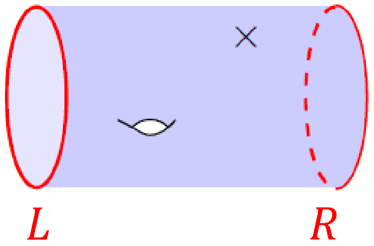
<u>right product:</u> $(\cdot_R)$ $a \cdot_R b =$



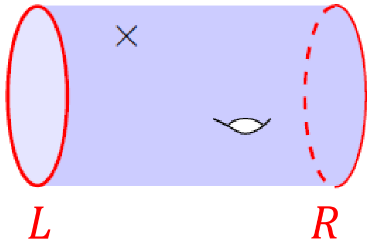
- The set $Y^d_{B\sqcup B}$ equipped with the left/right product defines a left/right surface algebra $A^B_{L/R}$

- Star operation:

$a =$



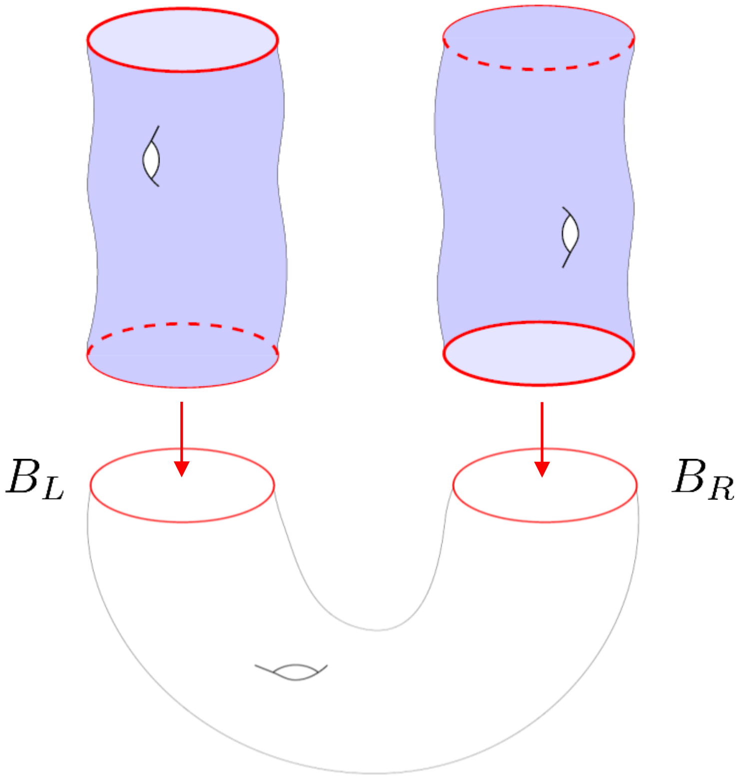
$a^\star =$



Surface Algebra

A_L^B

Left surface algebra



A_R^B

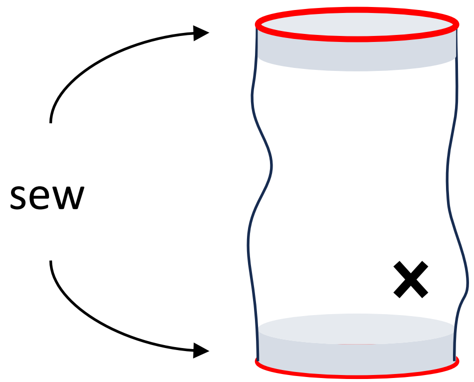
Right surface algebra

Surface Algebra

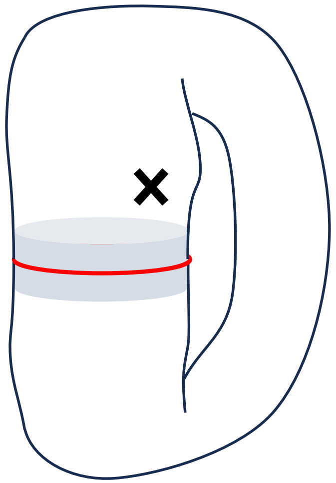
trace and trace inequality

The path integral defines a trace operation on the surface algebras:

$$\mathrm{tr}(a) := \zeta\left(M(a)\right)$$



$= a$



$= M(a)$

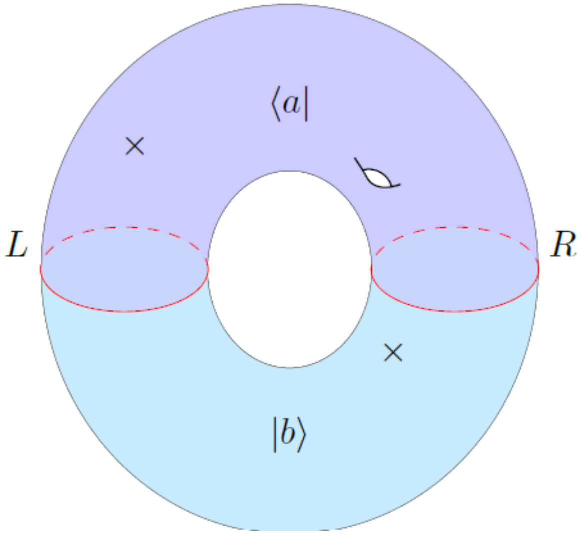
Surface Algebra

trace and trace inequality

The path integral defines a trace operation on the surface algebras:

$$\mathrm{tr}(a) := \zeta\left(M(a)\right)$$

From the dictionary between rimmed surfaces and states:

$$\langle a|b\rangle = \mathrm{tr}(a^{\star}b) = \zeta \Bigg[$$


$$\Bigg]$$

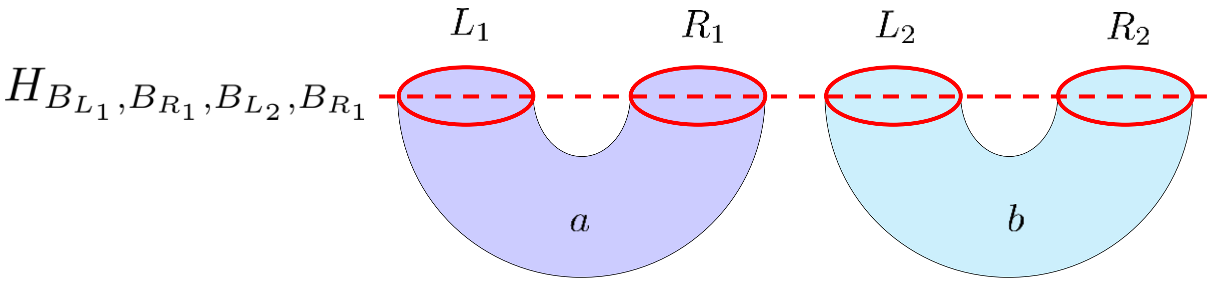
$$\mathrm{tr}(a^{\star}a) = \zeta\left(M(a^{\star}a)\right) = \langle a|a\rangle \geq 0$$

Axiom 3

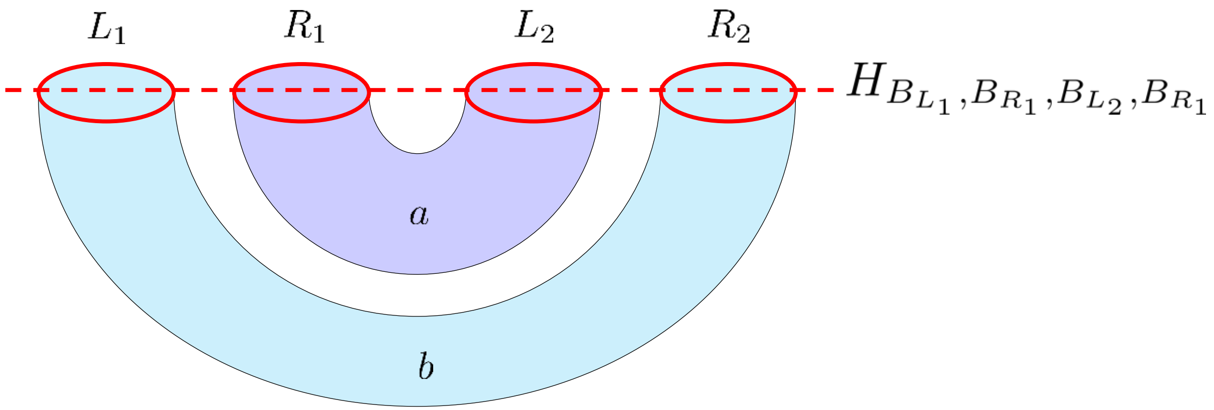
Surface Algebra

trace and trace inequality

We can use $a, b \in Y^d_{B \sqcup B}$ to define elements of $Y^d_{(B \sqcup B) \sqcup (B \sqcup B)}$



$$|\cup\cup\rangle := |a_{L_1R_1}, b_{L_2R_2}\rangle$$

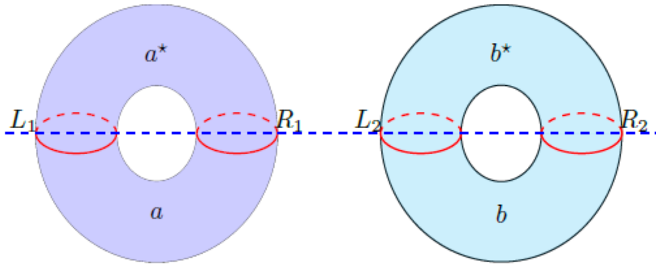


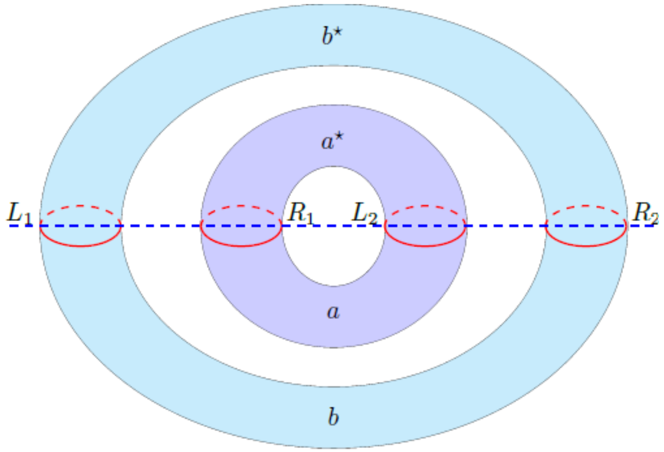
$$|\Cup\rangle := |a_{L_2R_1}, b_{L_1R_2}\rangle$$

Surface Algebra

trace and trace inequality

$$\langle \cup \cup \mid \cup \cup \rangle = \langle \Cup | \Cup \rangle = \langle a|a\rangle \, \langle b|b\rangle = \mathrm{tr}(a^\star a)\mathrm{tr}(b^\star b)$$

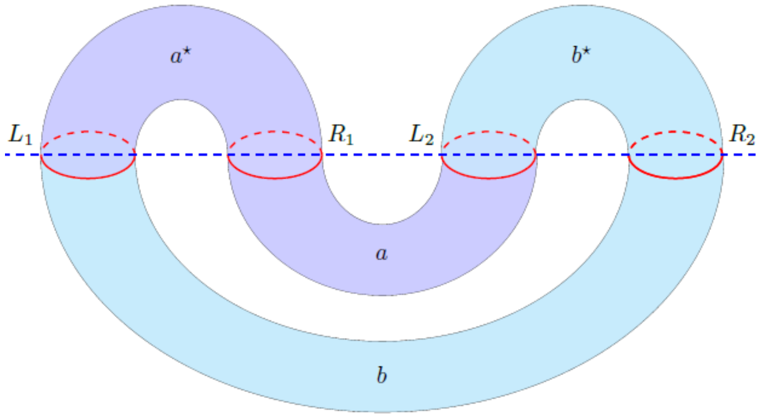




From the Cauchy-Schwarz inequality (consequence of positivity of the inner product on $H_{B_{L_1}, B_{R_1}, B_{L_2}, B_{R_1}}$):

$$\Big| \langle \Cup | \cup \cup \rangle \Big| \le \Big| |\cup\cup\rangle \Big| \Big| |\Cup\rangle \Big|$$

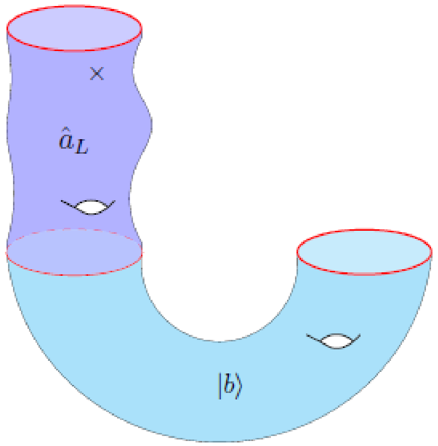
$$\implies \mathrm{tr}(aa^\star bb^\star) \le \mathrm{tr}(a^\star a)\mathrm{tr}(b^\star b)$$



Representation on the Hilbert Space

We define a representation of the left surface algebra on the Hilbert space: given $a \in A_L^B$ there is an associated operator $\hat{a}_L \in \hat{A}_L$ such that

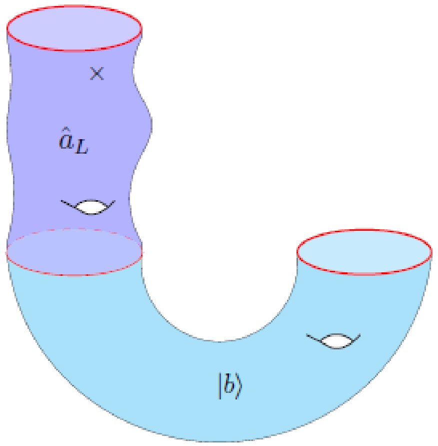
$$\hat{a}_L |b\rangle = |a \cdot_L b\rangle = |ab\rangle$$



Representation on the Hilbert Space

We define a representation of the left surface algebra on the Hilbert space: given $a \in A_L^B$ there is an associated operator $\hat{a}_L \in \hat{A}_L$ such that

$$\hat{a}_L \, |b\rangle = |a \cdot_L b\rangle = |ab\rangle$$



These operators are **bounded**:

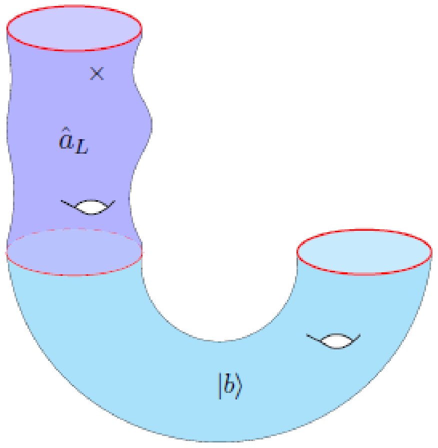
$$|\hat{a}_L|b\rangle|^2 = \langle ab|ab\rangle = \mathrm{tr}(a^\star abb^\star) \leq \mathrm{tr}(a^\star a)\mathrm{tr}(bb^\star) = \mathrm{tr}(a^\star a)\langle b|b\rangle$$

trace inequality

Representation on the Hilbert Space

We define a representation of the left surface algebra on the Hilbert space: given $a \in A_L^B$ there is an associated operator $\hat{a}_L \in \hat{A}_L$ such that

$$\hat{a}_L \left|b\right\rangle = \left|a \cdot_L b\right\rangle = \left|ab\right\rangle$$

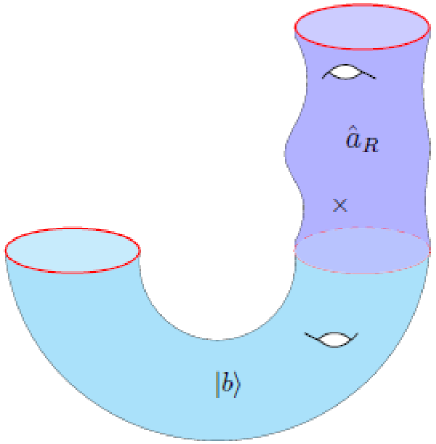


These operators are **bounded**:

$$|\hat{a}_L|b\rangle|^2 = \langle ab|ab\rangle = \mathrm{tr}(a^\star abb^\star) \leq \mathrm{tr}(a^\star a)\mathrm{tr}(bb^\star) = \mathrm{tr}(a^\star a)\langle b|b\rangle$$

trace inequality

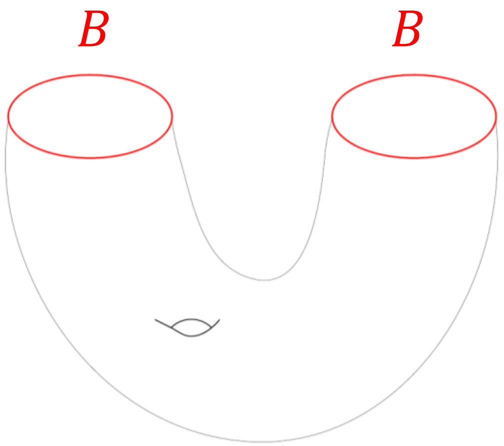
We can similarly define a representation of the right surface algebra:



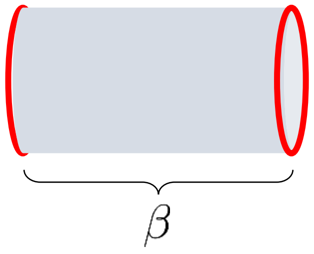
$$\hat{a}_R \left|b\right\rangle = \left|a \cdot_R b\right\rangle = \left|b \cdot_L a\right\rangle = \left|ba\right\rangle$$

Diagonal Sectors

$B_1 = B_2 = B$



$\in \mathcal{H}_{B \sqcup B}$

$C_\beta =$ 

$\in \mathcal{H}_{B \sqcup B}$

We can define a **trace operation** on $\hat{A}_L$ via the path integral trace on the surface algebra thanks to the continuity axiom:

$$\mathrm{tr}(a) = \lim_{\beta \downarrow 0} \langle C_\beta | \hat{a}_L | C_\beta \rangle$$

4. Type I von Neumann Factors

The von Neumann algebras $\mathcal{A}_L^B, \mathcal{A}_R^B$

- We define the von Neumann algebra $\mathcal{A}_L^B$ to be the closure of $\hat{A}_L$ within $\mathcal{B}(\mathcal{H}_{LR})$ in the weak operator topology

The von Neumann algebras $\mathcal{A}_L^B, \mathcal{A}_R^B$

- We define the von Neumann algebra $\mathcal{A}_L^B$ to be the closure of $\hat{A}_L$ within $\mathcal{B}(\mathcal{H}_{LR})$ in the weak operator topology

- We show that the trace defined on $\hat{A}_L$ extends to (all positive elements of) the von Neumann algebra

The von Neumann algebras $\mathcal{A}_L^B, \mathcal{A}_R^B$

- We define the von Neumann algebra $\mathcal{A}_L^B$ to be the closure of $\hat{A}_L$ within $\mathcal{B}(\mathcal{H}_{LR})$ in the weak operator topology

- We show that the trace defined on $\hat{A}_L$ extends to (all positive elements of) the von Neumann algebra

- The trace is
  - Faithful $\mathrm{tr}(a) = 0 \text{ iff } a = 0$
  - Normal for any bounded increasing sequence a_n, $\mathrm{tr}\ sup\ a_n = sup\ \mathrm{tr}\ a_n$
  - Semifinite $\forall a \in \mathcal{A}^+, \exists b < a$ such that $\mathrm{tr}(b) < \infty$

The von Neumann algebras $\mathcal{A}_L^B, \mathcal{A}_R^B$

- We define the von Neumann algebra $\mathcal{A}_L^B$ to be the closure of $\hat{A}_L$ within $\mathcal{B}(\mathcal{H}_{LR})$ in the weak operator topology

- We show that the trace defined on $\hat{A}_L$ extends to (all positive elements of) the von Neumann algebra

- The trace is
  - Faithful $\mathrm{tr}(a) = 0 \text{ iff } a = 0$
  - Normal $\text{for any bounded increasing sequence } a_n, \ \mathrm{tr}\ sup\ a_n = sup\ \mathrm{tr}\ a_n$
  - Semifinite $\forall a \in \mathcal{A}^+, \exists b < a \text{ such that } \mathrm{tr}(b) < \infty$

- The **trace inequality** holds on the von Neumann algebra! (an extension of the 4-boundaries argument applies)

Type I factors

- Trace inequality $\mathrm{tr}(ab) \le \mathrm{tr}(a)\mathrm{tr}(b)$ for $a = b = P \in \mathcal{A}_L^B \implies \mathrm{tr}(P) \ge 1$

Type I factors

- Trace inequality $\mathrm{tr}(ab) \leq \mathrm{tr}(a)\mathrm{tr}(b)$ for $a = b = P \in \mathcal{A}_L^B \implies \mathrm{tr}(P) \geq 1$

Some known results on von Neumann algebras:

- every von Neumann algebra is a direct sum of factors (algebras with trivial center) which can be type I, II or III
- there is no faithful, normal and semifinite trace on type III ⇒ we cannot have type III
- on type II, for any faithful, normal and semifinite trace there are non trivial projections with arbitrarily small trace
  ⇒ we cannot have type II

⇒ we have only type I factors!

Type I factors

- Trace inequality $\mathrm{tr}(ab) \leq \mathrm{tr}(a)\mathrm{tr}(b)$ for $a = b = P \in \mathcal{A}_L^B \implies \mathrm{tr}(P) \geq 1$

Some known results on von Neumann algebras:

- every von Neumann algebra is a direct sum of factors (algebras with trivial center) which can be type I, II or III
- there is no faithful, normal and semifinite trace on type III ⇒ we cannot have type III
- on type II, for any faithful, normal and semifinite trace there are non trivial projections with arbitrarily small trace
  ⇒ we cannot have type II

⇒ we have only type I factors!

- The commutation theorem for semifinite traces then tells us that $\mathcal{A}_L^B, \mathcal{A}_R^B$ are commutants on $\mathcal{H}_{B\sqcup B}$

Type I factors

- The central operators of $\mathcal{A}_L^B$ have purely discrete spectrum.
- Simultaneously diagonalise all central operators of $\mathcal{A}_L^B$ yields eigenspaces $\mathcal{H}_{B\sqcup B}^{\mu}$ such that

$$\mathcal{H}_{B\sqcup B} = \bigoplus_{\mu} \mathcal{H}_{B\sqcup B}^{\mu}$$

- As a result, the von Neumann algebras decomposes as

$$\mathcal{A}_L^B = \bigoplus_{\mu} \underbrace{\mathcal{A}_{L,\mu}^B}_{\text{type I factor}}$$

- The algebra $\mathcal{A}_L^{\mu}$ is a type I factor with commutant $\mathcal{A}_R^{\mu}$ on $\mathcal{H}_{B\sqcup B}^{\mu}$, therefore

$$\mathcal{H}_{B\sqcup B} = \bigoplus_{\mu} \mathcal{H}_{B\sqcup B,L}^{\mu} \otimes \mathcal{H}_{B\sqcup B,R}^{\mu}$$

5. Hilbert Space Interpretation of the Ryu-Takayanagi Entropy

Trace Normalization

- Faithful, normal, semifinite traces on type I algebras are unique up to an overall normalization constant, thus

$$\mathcal{C}_\mu \mathrm{tr}(a) = \mathrm{Tr}_\mu(a) := \sum_i {}_L\langle i|a|i\rangle_L,$$

- For $a = P$ one-dimensional projection onto a state in $\mathcal{H}^\mu_{B\sqcup B,L}$ we have $\mathrm{Tr}_\mu(P) = 1$

$$\implies \mathrm{tr}(P) = 1/\mathcal{C}_\mu$$

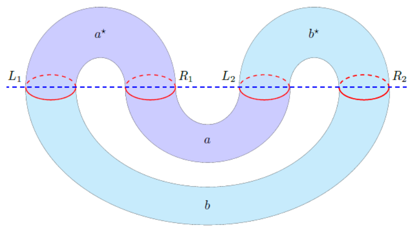
Trace Normalization

- Faithful, normal, semifinite traces on type I algebras are unique up to an overall normalization constant, thus

$$\mathcal{C}_\mu \mathrm{tr}(a) = \mathrm{Tr}_\mu(a) := \sum_i {}_L\langle i|a|i\rangle_L,$$

- For $a = P$ one-dimensional projection onto a state in $\mathcal{H}^\mu_{B\sqcup B,L}$ we have $\mathrm{Tr}_\mu(P) = 1$

$$\underset{\substack{\uparrow \\ \text{trace inequality}}}{1 \leq} \mathrm{tr}(P) = 1/\mathcal{C}_\mu \implies \mathcal{C}_\mu \leq 1$$



positivity of the inner product on $\mathcal{H}_{B\sqcup B\sqcup B\sqcup B}$ $\qquad \mathrm{tr}(P) \geq 1 \qquad \mathrm{tr}(P) = 0$

positivity of the inner product on $\mathcal{H}_{\sqcup_{i=1}^n (B\sqcup B)}$ $\qquad \mathrm{tr}(P) \geq n-1 \qquad \mathrm{tr}(P) = 0, 1, \cdots, n-2$

$\Rightarrow$ $\mathrm{tr}(P)$ is a positive integer!

Hidden Sectors

- For any non-zero finite-dimensional projection $P \in \mathcal{A}_L^B$ the trace $\mathrm{tr}(P)$ is a positive integer.

$$\Longrightarrow \; \mathcal{C}_\mu^{-1} = n_\mu \in \mathbb{Z}^+$$

Hidden Sectors

- For any non-zero finite-dimensional projection $P \in \mathcal{A}_L^B$ the trace $\mathrm{tr}(P)$ is a positive integer.

$$\implies \mathcal{C}_\mu^{-1} = n_\mu \in \mathbb{Z}^+$$

- We define the extended Hilbert space factors:

$$\widetilde{\mathcal{H}}^\mu_{B\sqcup B,L/R} := \underbrace{\mathcal{H}^\mu_{B\sqcup B,L/R} \otimes \mathcal{H}_{n_\mu}}_{\text{"hidden sector"}} \longrightarrow \widetilde{\mathcal{H}}_{B\sqcup B} := \bigoplus_{\mu\in\mathcal{I}} \left(\widetilde{\mathcal{H}}^\mu_{B\sqcup B,L} \otimes \widetilde{\mathcal{H}}^\mu_{B\sqcup B,R}\right)$$

Hidden Sectors

- For any non-zero finite-dimensional projection $P \in \mathcal{A}_L^B$ the trace $\mathrm{tr}(P)$ is a positive integer.

$$\Longrightarrow \ \mathcal{C}_\mu^{-1} = n_\mu \in \mathbb{Z}^+$$

- We define the extended Hilbert space factors:

$$\underbrace{\widetilde{\mathcal{H}}^\mu_{B\sqcup B,L/R} := \mathcal{H}^\mu_{B\sqcup B,L/R} \otimes \mathcal{H}_{n_\mu}}_{} \longrightarrow \ \widetilde{\mathcal{H}}_{B\sqcup B} := \bigoplus_{\mu\in\mathcal{I}} \left(\widetilde{\mathcal{H}}^\mu_{B\sqcup B,L} \otimes \widetilde{\mathcal{H}}^\mu_{B\sqcup B,R}\right)$$

"hidden sector"

- Then

$$\mathrm{tr}(a) = \tilde{\mathrm{Tr}}_\mu(\tilde{a}) \text{ where } a \in \mathcal{A}_L^\mu \text{ and } \tilde{a} := \left(a \otimes 1_{n_\mu}\right)$$

⇒ The hidden sectors allow to interpret the path integral trace as a Hilbert space trace

State-counting Entropy

- the trace tr can be used to define a notion of left entropy on pure states $|\psi\rangle \in \mathcal{H}_{B\sqcup B}$

$$\tilde{\rho}_\psi = \oplus_\mu \left(p_\mu \, \tilde{\rho}_\psi^\mu \right) \quad \longrightarrow \quad S_{vN}^L(\psi) = \mathrm{tr}(-\tilde{\rho}_\psi \ln \tilde{\rho}_\psi)$$

State-counting Entropy

- the trace tr can be used to define a notion of left entropy on pure states $|\psi\rangle \in \mathcal{H}_{B\sqcup B}$

$$\tilde{\rho}_\psi = \oplus_\mu \left(p_\mu \, \tilde{\rho}^\mu_\psi\right) \quad \longrightarrow \quad S^L_{vN}(\psi) = \mathrm{tr}(-\tilde{\rho}_\psi \ln \tilde{\rho}_\psi)$$

- Thanks to the relation between tr and $\tilde{\mathrm{Tr}}$ this entropy has a Hilbert space interpretation:

$$S^L_{vN}(\psi) = \widetilde{\mathrm{Tr}}(-\tilde{\rho}_\psi \ln \tilde{\rho}_\psi) = \sum_{\mu \in \mathcal{I}} p_\mu \, \widetilde{\mathrm{Tr}}(-\tilde{\rho}^\mu_\psi \ln \tilde{\rho}^\mu_\psi) - \sum_{\mu \in \mathcal{I}} p_\mu \ln p_\mu$$

State-counting Entropy

- the trace tr can be used to define a notion of left entropy on pure states $|\psi\rangle \in \mathcal{H}_{B\sqcup B}$

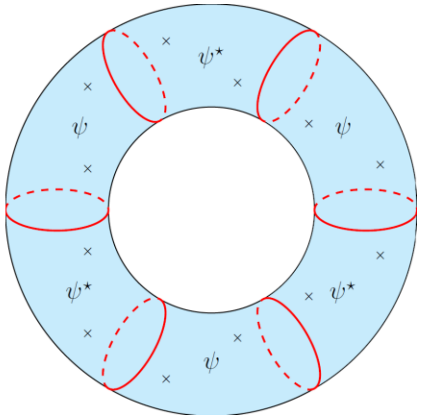
$$\tilde{\rho}_\psi = \oplus_\mu \left(p_\mu \, \tilde{\rho}^\mu_\psi\right) \longrightarrow S^L_{vN}(\psi) = \mathrm{tr}(-\tilde{\rho}_\psi \ln \tilde{\rho}_\psi)$$

- Thanks to the relation between tr and $\tilde{\mathrm{Tr}}$ this entropy has a Hilbert space interpretation:

$$S^L_{vN}(\psi) = \widetilde{\mathrm{Tr}}(-\tilde{\rho}_\psi \ln \tilde{\rho}_\psi) = \sum_{\mu \in \mathcal{I}} p_\mu \, \widetilde{\mathrm{Tr}}(-\tilde{\rho}^\mu_\psi \ln \tilde{\rho}^\mu_\psi) - \sum_{\mu \in \mathcal{I}} p_\mu \ln p_\mu$$

- We can compute this entropy via the replica trick:

$$\mathrm{tr}(\tilde{\rho}^n_\psi) = \zeta\left(M([\psi\psi^\star]^n)\right)$$



- If the theory admits a semiclassical limit described by Einstein-Hilbert or JT gravity, then we can argue (by following Lewkowycz-Maldacena) that in such a limit the entropy is given by the Ryu-Takayanagi entropy

Conclusions

- Any gravitational path integral satisfying the axioms defines von Neumann algebras of observables associated with codimension-2 boundaries: $\mathcal{A}_L^B, \mathcal{A}_R^B$
- These algebras contain only **type I** factors.
- The Hilbert space on which the algebras act decomposes as

$$\mathcal{H}_{B\sqcup B} = \bigoplus_{\mu} \mathcal{H}^{\mu}_{B\sqcup B,L} \otimes \mathcal{H}^{\mu}_{B\sqcup B,R}$$

- The path integral trace is equivalent to a standard trace on an extended Hilbert space, such that

$$\mathrm{tr}(a) = \tilde{\mathrm{Tr}}_{\mu}(\tilde{a}) \text{ where } \tilde{a} := \big(a \otimes 1_{n_\mu}\big)$$

- This provides a Hilbert space interpretation of the entropy, even when the gravitational theory is not known to have a holographic dual:

$$S^L_{vN}(\psi) = \widetilde{\mathrm{Tr}}(-\tilde{\rho}_\psi \ln \tilde{\rho}_\psi) = \sum_{\mu\in\mathcal{I}} p_\mu \, \widetilde{\mathrm{Tr}}(-\tilde{\rho}^\mu_\psi \ln \tilde{\rho}^\mu_\psi) - \sum_{\mu\in\mathcal{I}} p_\mu \ln p_\mu$$

- In the semiclassical limit, the entropy is given by the Ryu-Takayanagi formula.

Thanks for the attention!