Spin foams with timelike surfaces

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ILQG seminar
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FC, Jeff Hnybida, arXiv:1002.1959 [gr-qc]
FC, arXiv:1003.5652 [gr-qc]

Outline

Motivation

Main innovations of the last years

EPRL Engle, Livine, Pereira, Rovelli, Nucl.Phys.B799,2008

- master constraint

→ EPRL model

- correct coupling between 4–simplices
- relation to canonical LQG

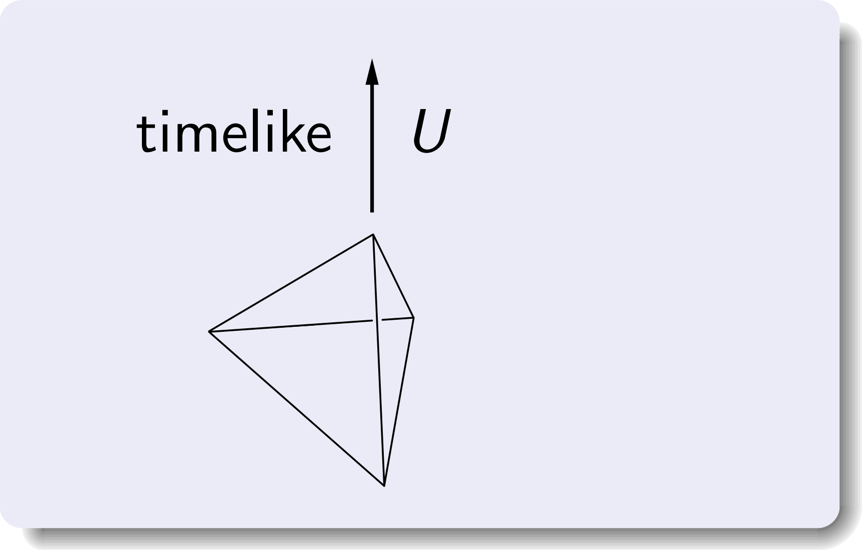
Coherent states Livine, Speziale, Phys.Rev.D76:084028,2007

- simplicity constraints on expectation values
- geometric understanding of intertwiners

→ FK model Freidel, Krasnov, Class.Quant.Grav.25:125018,2008

Restriction of triangulations

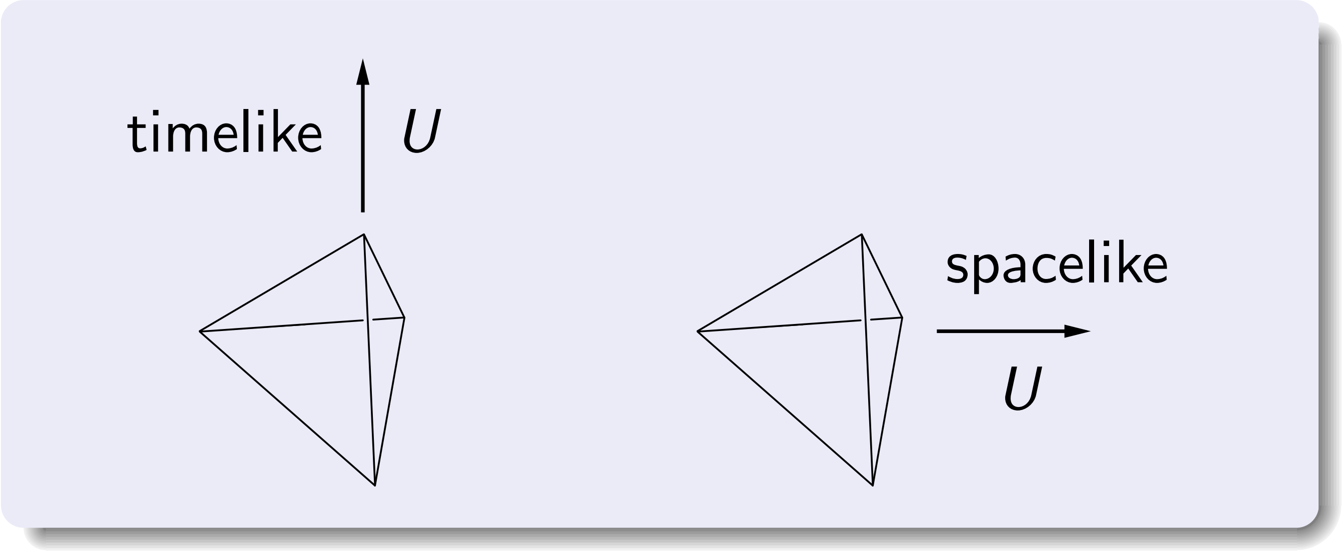
In the Lorentzian EPRL model, normals U of tetrahedra are timelike.



$\Rightarrow$ All tetrahedra are Euclidean and triangles can be only spacelike.

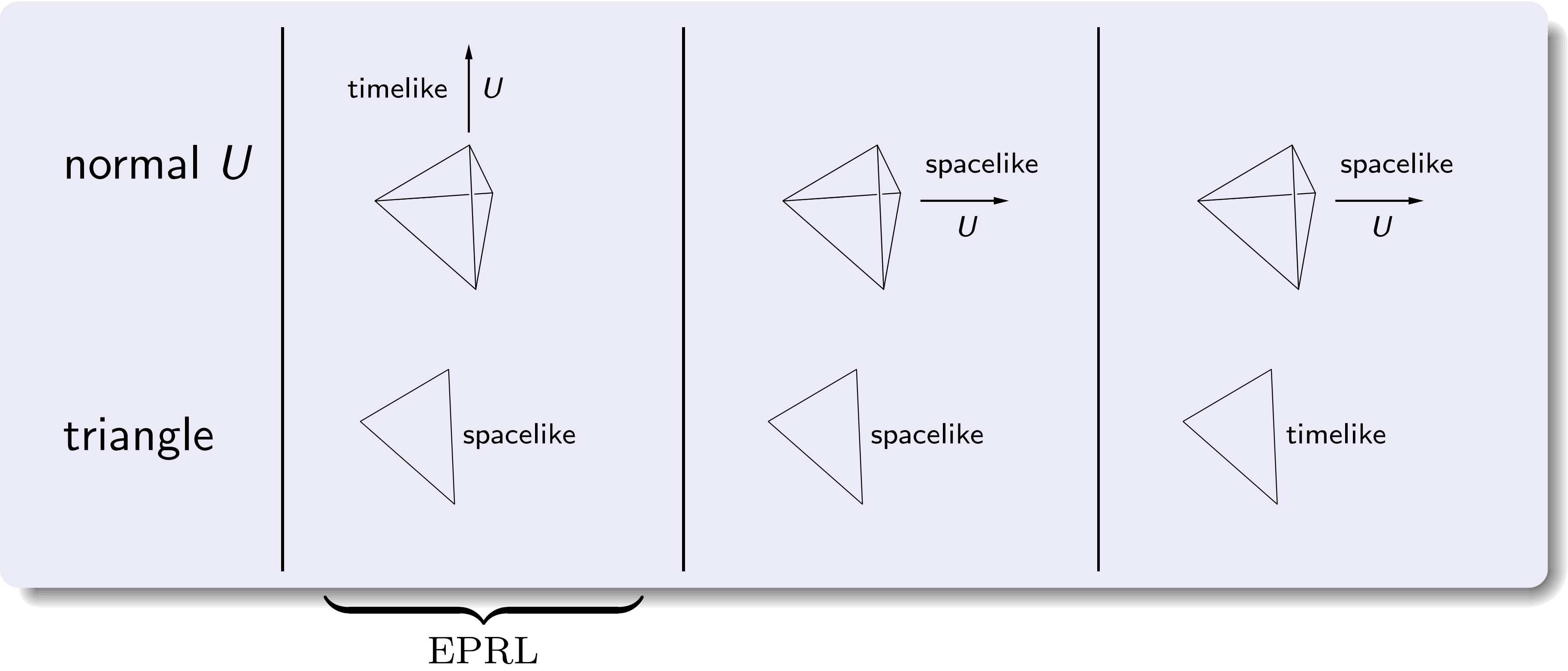
What we did

We extended the EPRL model to also include spacelike normals U.



$\Rightarrow$ Lorentzian tetrahedra are allowed and triangles can be spacelike or timelike.

Three cases



normal U	timelike U	spacelike U	spacelike U
triangle	spacelike	spacelike	timelike

$\underbrace{\hphantom{xxxxxxxxxx}}_{\text{EPRL}}$

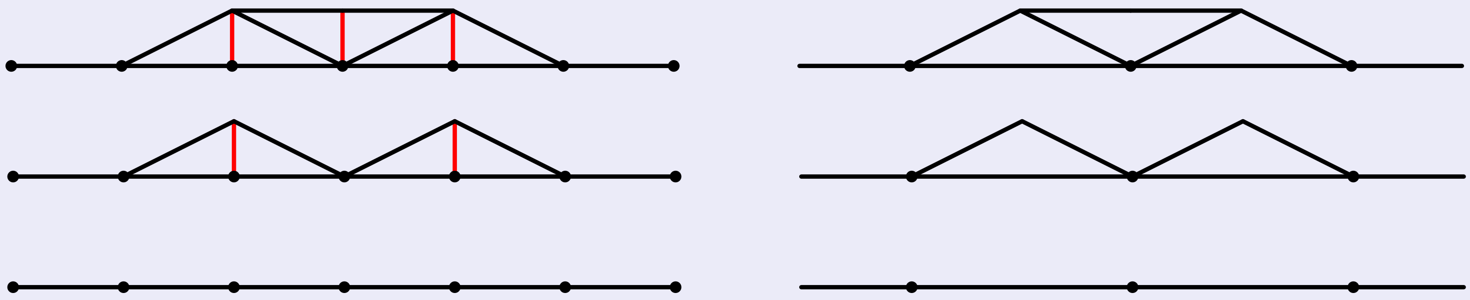
Covariant perspective

- extension natural
- a priori no reason to forbid Lorentzian tetrahedra
- permits timelike boundaries
- restriction could lead to artifacts

Canonical perspective

Are restricted triangulations preferred from a Hamiltonian point of view?

- In the examples I know of the transition from space to spacetime leads to a 4d lattice with timelike (or null) edges.
  - causal dynamical triangulations **Ambjorn, Jurkiewicz, Loll**
  - evolution schemes for Lorentzian Regge calculus **Barrett, Galassi, Miller, Sorkin, Tuckey, Williams**



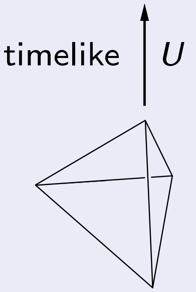
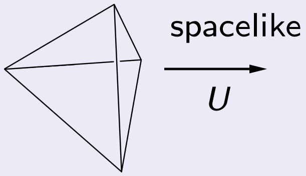
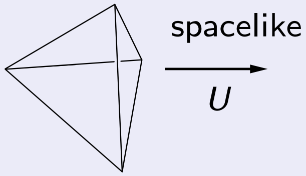
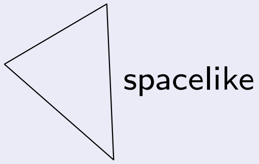
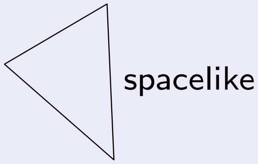
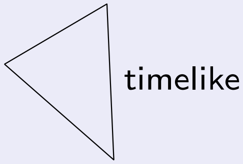
Canonical perspective

Are restricted triangulations preferred from a Hamiltonian point of view?

- In the examples I know of the transition from space to spacetime leads to a 4d lattice with timelike (or null) edges.
    - causal dynamical triangulations **Ambjorn, Jurkiewicz, Loll**
    - evolution schemes for Lorentzian Regge calculus **Barrett, Galassi, Miller, Sorkin, Tuckey, Williams**
- The Hamiltonian approach to Lorentzian spin foams creates a sequence of 3d spatial lattices. **Han, Thiemann**
    - cannot (yet) be directly compared with 4d triangulations discussed here

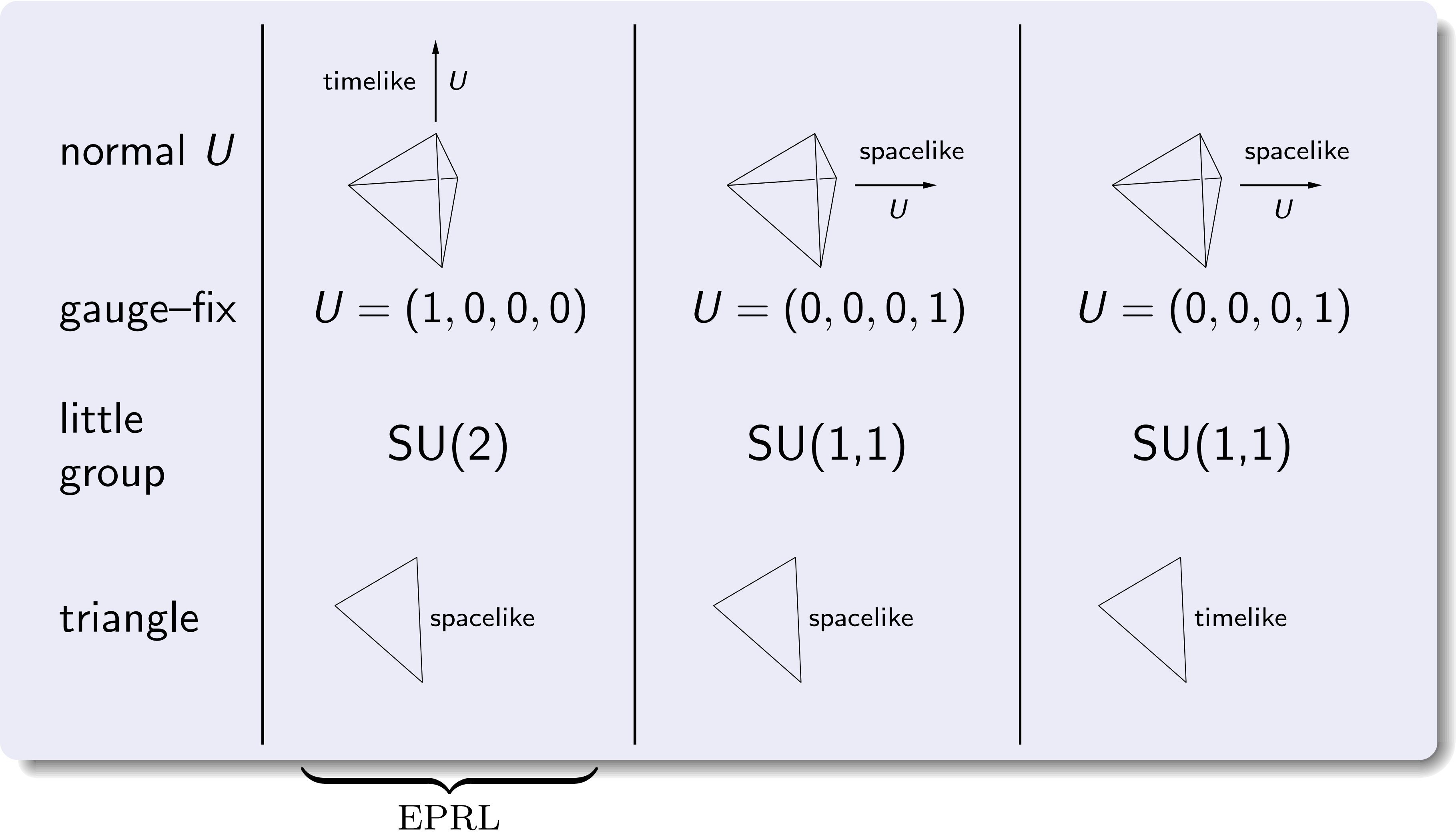
Coherent states

Little groups

normal U			
gauge–fix	$U = (1, 0, 0, 0)$	$U = (0, 0, 0, 1)$	$U = (0, 0, 0, 1)$
little group	SO(3)	SO(1,2)	SO(1,2)
triangle			

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{EPRL}}$$

Little groups



normal U	timelike U	spacelike U	spacelike U
gauge–fix	$U = (1, 0, 0, 0)$	$U = (0, 0, 0, 1)$	$U = (0, 0, 0, 1)$
little group	SU(2)	SU(1,1)	SU(1,1)
triangle	spacelike	spacelike	timelike

EPRL

Representation theory

	$\mathrm{SL}(2,\mathbb{C})$	SU(2)
generators	$J^i,\ K^i$	$J^1,\ J^2,\ J^3$
Casimirs	$C_1 = \vec{J}^2 - \vec{K}^2$ $C_2 = -4\vec{J}\cdot\vec{K}$	$\vec{J}^2$
unitary irreps	$\mathcal{H}_{(\rho,n)}$	$\mathcal{D}_j$
	$\rho \in \mathbb{R},\ n \in \mathbb{Z}_+$	$j \in \mathbb{Z}_+/2$
	$C_1 = \frac{1}{2}(n^2 - \rho^2 - 4)$ $C_2 = \rho n$	$\vec{J}^2 = j(j+1)$

Representation theory

	SU(2)	SU(1,1)	
generators	$J^1,\ J^2,\ J^3$	$J^3,\ K^1,\ K^2$	
Casimirs	$\vec{J}^2$	$Q = (J^3)^2 - (K^1)^2 - (K^2)^2$	
		discrete series	continuous series
unitary irreps	$\mathcal{D}_j$	$\mathcal{D}_j^{\pm}$	$\mathcal{C}_s^{\epsilon}$
	$j \in \mathbb{Z}_+/2$	$j = \frac{1}{2}, 1, \frac{3}{2} \ldots$	$j = -\frac{1}{2} + \mathrm{i}s$, $0 < s < \infty$
	$\vec{J}^2 = j(j+1)$	$Q = j(j-1)$	$Q = -s^2 - \frac{1}{4}$

Notation

$$\vec{J} \equiv \begin{pmatrix} J^1 \\ J^2 \\ J^3 \end{pmatrix} \qquad \text{and} \qquad \vec{K} \equiv \begin{pmatrix} K^1 \\ K^2 \\ K^3 \end{pmatrix}$$

transform like Euclidean 3–vectors under SU(2).

Similarly,

$$\vec{F} \equiv \begin{pmatrix} J^3 \\ K^1 \\ K^2 \end{pmatrix} \qquad \text{and} \qquad \vec{G} \equiv \begin{pmatrix} K^3 \\ -J^1 \\ -J^2 \end{pmatrix}$$

transform like Minkowksi 3–vectors under SU(1,1).

SU(2) decomposition of $\mathrm{SL}(2,\mathbb{C})$ irrep

Canonical basis

$$\mathcal{H}_{(\rho,n)} \simeq \bigoplus_{j=n/2}^{\infty} \mathcal{D}_j$$

$$\mathbb{1}_{(\rho,n)} = \sum_{j=n/2}^{\infty} \sum_{m=-j}^{j} |\Psi_{j\,m}\rangle \langle\Psi_{j\,m}|$$

SU(1,1) decomposition of $\mathrm{SL}(2,\mathbb{C})$ irrep

$$\mathcal{H}_{(\rho,n)} \simeq \left(\bigoplus_{j>1/2}^{n/2} \mathcal{D}_j^+ \oplus \int_0^\infty \mathrm{d}s\, \mathcal{C}_s^\epsilon \right) \oplus \left(\bigoplus_{j>1/2}^{n/2} \mathcal{D}_j^- \oplus \int_0^\infty \mathrm{d}s\, \mathcal{C}_s^\epsilon \right)$$

$$\begin{aligned} \mathbb{1}_{(\rho,n)} &= \sum_{j>1/2}^{n/2} \sum_{m=j}^{\infty} \left|\Psi^+_{j\,m}\right\rangle \left\langle\Psi^+_{j\,m}\right| + \sum_{j>1/2}^{n/2} \sum_{-m=j}^{\infty} \left|\Psi^-_{j\,m}\right\rangle \left\langle\Psi^-_{j\,m}\right| \\ &+ \sum_{\alpha=1,2} \int_0^\infty \mathrm{d}s\, \mu_\epsilon(s) \sum_{\pm m=\epsilon}^{\infty} \left|\Psi^{(\alpha)}_{s\,m}\right\rangle \left\langle\Psi^{(\alpha)}_{s\,m}\right| \end{aligned}$$

(see chapter 7 in Rühl's book)

Non–normalizability

States $|j\,m\rangle$ in the continuous series irrep $\mathcal{C}_s^\epsilon$ are normalizable, but the corresponding states $\left|\Psi^{(\alpha)}_{s\,m}\right\rangle$ in $\mathcal{H}_{(\rho,n)}$ are not.

$$\left\langle\Psi^{(\alpha')}_{s'\,m'}\middle|\Psi^{(\alpha)}_{s\,m}\right\rangle = \frac{\delta(s'-s)}{\mu_\epsilon(s)}\,\delta_{\alpha'\alpha}\,\delta_{m'm}$$

SU(2) coherent states

Definition

$$|j\,g\rangle \equiv D^j(g)|j\,j\rangle\,, \qquad g \in \mathrm{SU}(2)$$

$$|j\,\vec{N}\rangle \equiv D^j(g(\vec{N}))|j\,j\rangle\,, \qquad \vec{N} \in S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$$

Completeness relation

$$\mathbb{1}_j \;=\; (2j+1)\int\limits_{\mathrm{SU}(2)} \mathrm{d}g\;|j\,g\rangle\langle j\,g| \;=\; (2j+1)\int\limits_{S^2} \mathrm{d}^2N\;|j\vec{N}\rangle$$

At the level of the $\mathrm{SL}(2,\mathbb{C})$ irrep $\mathcal{H}_{(\rho,n)}$ this becomes

$$P_j \;=\; (2j+1)\int\limits_{\mathrm{SU}(2)} \mathrm{d}g\;|\Psi_{j\,g}\rangle\,\langle\Psi_{j\,g}| \;=\; (2j+1)\int\limits_{S^2} \mathrm{d}^2N\;\left|\Psi_{j\vec{N}}\right\rangle\left\langle\Psi_{j\vec{N}}\right|\,.$$

SU(1,1) coherent states — discrete series

Definition

$$|j\, g\rangle_\pm \equiv D^j(g)|j\, \pm j\rangle\,, \qquad g \in \mathrm{SU}(1,1)$$

$$|j\, \vec{N}\rangle \equiv D^j(g(\vec{N}))|j\, \pm j\rangle\,, \qquad \vec{N} \in \mathbb{H}_\pm \simeq \mathrm{SU}(1,1)/\mathrm{U}(1)$$

Upper/lower hyperboloid

$$\mathbb{H}_\pm = \{\vec{N} \,|\, \vec{N}^2 = 1\,, N^0 \gtrless 0\}$$

SU(1,1) coherent states — discrete series

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$$|j\,g\rangle_\pm \equiv D^j(g)|j\,\pm j\rangle\,, \qquad g\in\mathrm{SU}(1,1)$$

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Completeness relation

$$\mathbb{1}_j^\pm \;=\; (2j-1)\int\limits_{\mathrm{SU(1,1)}}\mathrm{d}g\;|j\,g\rangle_\pm\langle j\,g|_\pm \;=\; (2j-1)\int\limits_{\mathbb{H}_\pm}\mathrm{d}^2N\;|j\,\vec{N}\rangle\langle j\,\vec{N}|$$

At the level of the $\mathrm{SL}(2,\mathbb{C})$ irrep $\mathcal{H}_{(\rho,n)}$ this becomes

$$P_j^\pm \;=\; (2j-1)\int\limits_{\mathrm{SU(1,1)}}\mathrm{d}g\;\left|\Psi^\pm_{j\,g}\right\rangle\left\langle\Psi^\pm_{j\,g}\right| \;=\; (2j-1)\int\limits_{\mathbb{H}_\pm}\mathrm{d}^2N\;\left|\Psi_{j\vec{N}}\right\rangle\left\langle\Psi_{j\vec{N}}\right|\,.$$

Expectation values

So far the coherent states are the ones introduced by Perelomov.

They have the property that

$$\langle j\,\vec{N}|\vec{J}|j\,\vec{N}\rangle = j\,\vec{N}\,, \qquad \vec{N} \in S^2$$

$$\langle j\,\vec{N}|\vec{F}|j\,\vec{N}\rangle = j\,\vec{N}\,, \qquad \vec{N} \in \mathbb{H}_\pm$$

in accordance with the fact that $\vec{J}^2 = j(j+1) \geq 0$ and $\vec{F}^2 = j(j-1) \geq 0$.

SU(1,1) coherent states — continuous series

Question

What are the appropriate coherent states for the continuous series?

In this case $Q = \vec{F}^2 = -(s^2 + 1/4) < 0$, so the classical vector $\vec{N}$ should be spacelike.

Perelomov uses the state $|j\ m = 0\rangle$, resulting in a zero classical vector.

SU(1,1) coherent states — continuous series

Question

What are the appropriate coherent states for the continuous series?

In this case $Q = \vec{F}^2 = -(s^2 + 1/4) < 0$, so the classical vector $\vec{N}$ should be spacelike.

$$\vec{F} \equiv \begin{pmatrix} J^3 \\ K^1 \\ K^2 \end{pmatrix} \rightsquigarrow \text{Build coherent states from eigenstates of } K^1 \text{ or } K^2!$$

Eigenstates of K^1

$$K^1\,|j\,\lambda\,\sigma\rangle = \lambda|j\,\lambda\,\sigma\rangle\,, \qquad -\infty < \lambda < \infty\,, \quad \sigma = \pm$$

Mukunda, Barut and Phillips, Lindblad and Nagel

These states are not normalizable:

$$\langle j\,\lambda'\,\sigma'|j\,\lambda\,\sigma\rangle = \delta(\lambda' - \lambda)\,\delta_{\sigma'\sigma}$$

To obtain finite inner products, one needs a smearing with wavefunctions!

SU(1,1) coherent states — continuous series

Definition

$$|j\,g\rangle_{\mathrm{sp}} \equiv D^j(g)|j\,s+\rangle\,, \qquad g \in \mathrm{SU}(1,1)$$

$$|j\,\vec{N}\rangle \equiv D^j(g(\vec{N}))|j\,s+\rangle\,, \quad \vec{N} \in \mathbb{H}_{\mathrm{sp}} \simeq \mathrm{SU}(1,1)/\mathrm{G}_1$$

$\mathbb{H}_{\mathrm{sp}} = \{\vec{N}\,|\,\vec{N}^2 = -1\}$ = single–sheeted spacelike hyperboloid

G_1 = subgroup generated by K^1

SU(1,1) coherent states — continuous series

Definition

$$|j\,g\rangle_{\mathrm{sp}} \equiv \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{1}{\sqrt{\delta}} f_\delta(\lambda - s)\, D^j(g)|j\,\lambda\,+\rangle\,, \quad g \in \mathrm{SU}(1,1)$$

$$|j\,\vec{N}\,\lambda\rangle \equiv D^j(g(\vec{N}))|j\,\lambda\,+\rangle\,, \qquad \vec{N} \in \mathbb{H}_{\mathrm{sp}} \simeq \mathrm{SU}(1,1)/\mathrm{G}_1$$

$\mathbb{H}_{\mathrm{sp}} = \{\vec{N}\,|\,\vec{N}^2 = -1\}$ = single–sheeted spacelike hyperboloid

G_1 = subgroup generated by K^1

$$f_\delta(x) = \begin{cases} 1\,, & |x| \leq \delta/2 \\ 0\,, & |x| > \delta/2 \end{cases}$$

SU(1,1) coherent states — continuous series

Definition

$$|j\,g\rangle_{\mathrm{sp}} \equiv \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{1}{\sqrt{\delta}} f_\delta(\lambda - s)\, D^j(g)|j\,\lambda\,+\rangle\,, \quad g \in \mathrm{SU}(1,1)$$

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Completeness relation

$$\begin{aligned} \mathbb{1}_j^\epsilon &= \int_{\mathrm{SU(1,1)}} \mathrm{d}g\; |j\,g\,\delta\rangle_{\mathrm{sp}}\langle j\,g\,\delta|_{\mathrm{sp}} \\ &= \int_{\mathbb{H}_{\mathrm{sp}}} \mathrm{d}^2 N \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{1}{\delta} f_\delta(\lambda - s)\, |j\,\vec{N}\,\lambda\rangle\langle j\,\vec{N}\,\lambda| \end{aligned}$$

Smearing in s

At the level of the $\mathrm{SL}(2,\mathbb{C})$ irrep $\mathcal{H}_{(\rho,n)}$, one also needs a smearing in s.

We define a smeared projector onto the irrep with spin $j = -1/2 + \mathrm{i}s$:

$$P_s^\epsilon(\delta) \equiv \sum_{\alpha=1,2} \sum_{\pm m=\epsilon} \int_0^\infty \mathrm{d}s' \; \mu_\epsilon(s') \, f_\delta(s' - s) \left|\Psi^{(\alpha)}_{s'\,m}\right\rangle \left\langle\Psi^{(\alpha)}_{s'\,m}\right|$$

SU(1,1) coherent states — continuous series

Definition

$$\left|\Psi^{(\alpha)}_{s\,g\,\delta}\right\rangle \equiv \int\limits_0^\infty \mathrm{d}s'\; \mu_\epsilon(s')\, f_\delta(s'-s) \int\limits_{-\infty}^{\infty} \mathrm{d}\lambda\; \frac{1}{\sqrt{\delta}} f_\delta(\lambda - s)\, D^{(\rho,n)}(g) \left|\Psi^{(\alpha)}_{s'\,\lambda\,+}\right\rangle$$

Completeness relation

$$P^\epsilon_s(\delta) = \sum_{\alpha=1,2} \int\limits_{\mathrm{SU}(1,1)} \mathrm{d}g\; \left|\Psi^{(\alpha)}_{s\,g\,\delta}\right\rangle \left\langle\Psi^{(\alpha)}_{s\,g\,\delta}\right|$$

Three ways to simplicity

Three ways to simplicity

Derivation of extended simplicity constraints by three methods

1. weak imposition of constraints
2. master constraint

$\}$ (as advocated by EPRL)

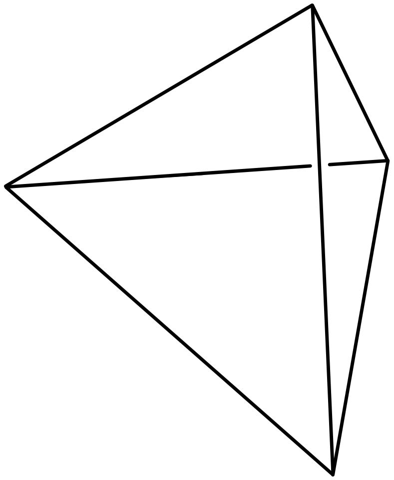
3. restriction of coherent state basis (inspired by FK model)

Classical tetrahedron

Describe a tetrahedron by four bivectors

$$J = B + \frac{1}{\gamma} \star B \,,$$

where B is constrained to be simple.



Simplicity constraint: $\exists$ unit four–vector U such that

$$U \cdot \star B = 0 \,.$$

Classical simplicity constraints

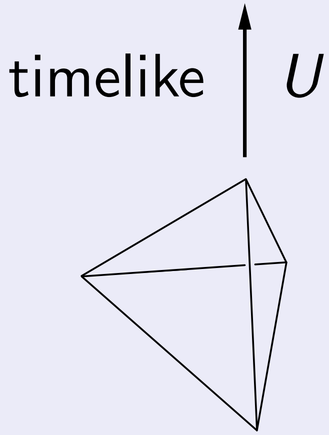
Express B in terms of the total bivector J:

$$B = \frac{\gamma^2}{\gamma^2 + 1} \left(J - \frac{1}{\gamma} \star J \right)$$

Starting point for quantization:

$$\boxed{U \cdot \left(J - \frac{1}{\gamma} \star J \right) = 0}$$

First case: normal U timelike

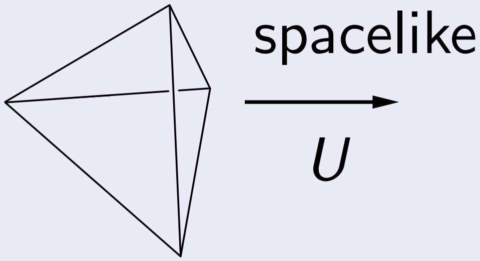


In the gauge $U = (1, 0, 0, 0)$, the simplicity constraint takes the form

$$\vec{J} + \frac{1}{\gamma}\vec{K} = 0$$

The little group is SU(2), so we use states of the SU(2) decomposition!

Spacelike U



In the gauge $U = (0, 0, 0, 1)$, the simplicity constraint becomes

$$\vec{F} + \frac{1}{\gamma}\vec{G} = 0$$

The little group is SU(1,1), so we use states of the SU(1,1) decomposition!

Basic example of second–class constraints

Phase space and constraints

$$(q_i, p_i)\,, \;\; i = 1, 2 \qquad\qquad q_1 - q_2 = 0$$

$$\{q_i, p_i\} = \delta_{ij} \qquad\qquad p_1 - p_2 = 0$$

Change of variables

$$q_\pm = \tfrac{1}{2}\left(q_1 \pm q_2\right) \qquad\qquad q_- = p_- = 0$$

$$p_\pm = \tfrac{1}{2}\left(p_1 \pm p_2\right)$$

$$a_\pm = \frac{1}{\sqrt{2}}\left(p_\pm - \mathrm{i} q_\pm\right) \qquad\qquad a_- = 0$$

Weak imposition of constraints

Impose $a_-|\psi\rangle = 0$ on physical states, implying

$$\langle\varphi|a_-|\psi\rangle = \langle\varphi|a_-^\dagger|\psi\rangle = 0 \qquad \forall\ |\varphi\rangle, |\psi\rangle \in \mathcal{H}_{\text{phys}}$$

$\Rightarrow$ Physical Hilbert $\mathcal{H}_{\text{phys}}$ space is spanned by $|n_+\rangle \otimes |0\rangle$, $n_+ \in \mathbb{N}_0$.

Restriction of coherent state basis

Overcomplete basis of coherent states $|\alpha_+\rangle \otimes |\alpha_-\rangle$, $a_\pm|\alpha_\pm\rangle = \alpha_\pm|\alpha_\pm\rangle$:

$$\mathbb{1}_{\mathcal{H}} = \frac{1}{\pi^2}\int \mathrm{d}\alpha_+ \int \mathrm{d}\alpha_-\ |\alpha_+\rangle\langle\alpha_+| \otimes |\alpha_-\rangle\langle\alpha_-|$$

Restrict basis to states whose expectation values satisfy the constraint, i.e. to labels $\alpha_- = 0$.

Projector on $\mathcal{H}_{\mathrm{phys}}$

$$P_{\mathrm{phys}} \equiv \frac{1}{\pi}\int \mathrm{d}\alpha_+\ |\alpha_+\rangle\langle\alpha_+| \otimes |0\rangle\langle 0|$$

Master constraint

Master constraint operator

$$M = a_-^\dagger a_- = \frac{1}{2}\left(p_-^2 + q_-^2\right) + \frac{1}{2}$$

Define $\mathcal{H}_{\mathrm{phys}}$ as the subspace of states with minimal eigenvalue w.r.t. M.

$\Rightarrow \mathcal{H}_{\mathrm{phys}}$ spanned by $|n_+\rangle \otimes |0\rangle$, $n_+ \in \mathbb{N}_0$.

From classical to quantum simplicity

$$\left.\begin{array}{l}\vec{J}+\frac{1}{\gamma}\vec{K}=0\\[2em]\vec{F}+\frac{1}{\gamma}\vec{G}=0\end{array}\right\}\quad\longrightarrow\quad 4\gamma C_3=\rho n$$

$$B^{IJ}(\star B)_{IJ}=0\quad\longrightarrow\quad(\rho-\gamma n)\left(\rho+\frac{n}{\gamma}\right)=0$$

$B^{IJ}(\star B)_{IJ}=0$ is the diagonal simplicity constraint.

C_3 is the Casimir of the little group determined by the normal U.

Weak imposition of constraints

Consider the case $U = (1,0,0,0)$ and suppose that

$$\mathcal{H}_{\text{phys}} = \bigoplus_{j\in\mathcal{J}} \mathcal{D}_j \subset \mathcal{H}_{(\rho,n)}\,,$$

where $\mathcal{J}$ is a subset of the total set of spins $\{j\,|\,j \geq n/2\}$.
Require that

$$\langle\varphi|\vec{C}|\psi\rangle = 0 \qquad \forall\ |\varphi\rangle, |\psi\rangle \in \mathcal{H}_{\text{phys}}.$$

Unless $\mathcal{H}_{\text{phys}}$ is trivial, this implies, that for some $j \geq n/2$,

$$\left\langle j\,m' \left| J^3 + \frac{1}{\gamma}K^3 \right| j\,m \right\rangle = \left\langle j\,m' \left| J^\pm + \frac{1}{\gamma}K^\pm \right| j\,m \right\rangle = 0$$

for all admissible m, m'.

Weak imposition of constraints

By using $K^{\pm} = \pm[K^3, J^{\pm}]$ and

$$K^3|j\,m\rangle = (\ldots)|j+1\,m\rangle - mA_j\,|j\,m\rangle + (\ldots)|j-1\,m\rangle\,, \qquad A_j = \frac{\rho\,n}{4j(j+1)}\,,$$

one obtains

$$A_j = \gamma \qquad \text{or} \qquad 4\gamma C_3 = 4\gamma j(j+1) = \rho n\,.$$

In conjunction with the constraint $B \cdot \star B = 0$, this gives $4j(j+1) = n^2$ if $\rho = \gamma n$ and $4j(j+1) = -\rho^2$ if $n = -\gamma\rho$.

Approximate solution

$$\rho = \gamma n\,, \qquad j = n/2$$

$$\mathcal{H}_{\mathrm{phys}} = \mathcal{D}_{n/2} \subset \mathcal{H}_{(\gamma n, n)}$$

Weak imposition of constraints

Next consider $U = (0,0,0,1)$. Suppose first that the weak constraint holds for some irrep $\mathcal{D}_j^\pm$ of the discrete series:

$$\left\langle j\, m' \left| F^0 + \frac{1}{\gamma} G^0 \right| j\, m \right\rangle = \left\langle j\, m' \left| F^\pm + \frac{1}{\gamma} G^\pm \right| j\, m \right\rangle = 0 \qquad \forall m, m' ,$$

with $F^\pm \equiv F^2 \mp \mathrm{i} F^1$ and $G^\pm \equiv G^2 \mp \mathrm{i} G^1$. According to Mukunda

$$K^3 |j\, m\rangle = (\ldots)|j+1\, m\rangle - m\tilde{A}_j\, |jm\rangle + (\ldots)|j-1\, m\rangle\,, \qquad \tilde{A}_j = \frac{\rho\, n}{4j(j-1)}\,,$$

and $G^\pm = \pm[G^0, F^\pm]$, which leads to $\tilde{A}_j = \gamma$ or $4\gamma j(j-1) = \rho n$.

Approximate solution

$$\rho = \gamma n\,,\; n \geq 2\,, \qquad j = n/2$$

Weak imposition of constraints

For an irrep $\mathcal{C}_s^\epsilon$ of the continuous series, the equations are the same except that $\tilde{A}_j$ is replaced by

$$A_j = \frac{\rho\, n}{4j(j+1)} = -\frac{\rho\, n}{4(s^2+1/4)}\,.$$

A solution exists only when $\rho = -n/\gamma < -1$ and then

$$s^2 + 1/4 = \frac{\rho^2}{4} = \frac{n^2}{4\gamma^2}\,.$$

Overall result for $U = (0,0,0,1)$

$\rho = \gamma n\,,\; n \geq 2$ or $\rho = -n/\gamma < -1$

$\mathcal{H}_{\mathrm{phys}} = \mathcal{D}^+_{n/2} \oplus \mathcal{D}^-_{n/2}$ $\mathcal{H}_{\mathrm{phys}} = \mathcal{C}^\epsilon_{\frac{1}{2}\sqrt{n^2/\gamma^2-1}} \oplus \mathcal{C}^\epsilon_{\frac{1}{2}\sqrt{n^2/\gamma^2-1}}$

Restriction of coherent state basis

Case $U = (1,0,0,0)$.

Resolve the identity on $\mathcal{H}_{(\rho,n)}$ in terms of SU(2) coherent states,

$$\mathbb{1}_{(\rho,n)} = \sum_{j=n/2}^{\infty} (2j+1) \int_{S^2} \mathrm{d}^2 N \, \left|\Psi_{j\vec{N}}\right\rangle \left\langle \Psi_{j\vec{N}}\right| \, ,$$

and require that

$$\left\langle \Psi_{j\vec{N}} \right| \vec{J} + \frac{1}{\gamma}\vec{K} \left| \Psi_{j\vec{N}} \right\rangle = 0 \, .$$

This implies $A_j = \gamma$, that is, $4\gamma j(j+1) = \rho n$.

Restriction of coherent state basis

The second condition is either obtained from $B \cdot \star B = 0$ or alternatively from the requirement of minimal uncertainty in $\vec{K}$.

From $\gamma = A_j$ it follows that

$$\langle \vec{J}^2 \rangle = \frac{1}{\gamma^2} \langle \vec{K} \rangle^2 + O(|\vec{J}|) \qquad \text{and} \qquad \langle \vec{J}^2 \rangle = -\frac{1}{\gamma} \langle \vec{J} \cdot \vec{K} \rangle ,$$

so that

$$\begin{aligned} (\Delta K)^2 &= -\frac{1}{\gamma}(1 - \gamma^2) \vec{J} \cdot \vec{K} - \frac{1}{2} C_1 + O(|\vec{J}|) \\ &= -\frac{\gamma}{4} \left[\left(1 - \frac{1}{\gamma^2} \right) C_2 + \frac{2}{\gamma} C_1 \right] + O(|\vec{J}|) \\ &= -\frac{\gamma}{4} B \cdot \star B + O(|\vec{J}|) . \end{aligned}$$

Restriction of coherent state basis

Projector on $\mathcal{H}_{\text{phys}}$

$$P_{\text{phys}} = (n+1) \int\limits_{S^2} \mathrm{d}^2 N \; \left|\Psi_{n/2\vec{N}}\right\rangle \left\langle\Psi_{n/2\vec{N}}\right|$$

Master constraint

For the case $U = (0, 0, 0, 1)$ the Master constraint reads

$$M = \left(\vec{F} + \frac{1}{\gamma}\vec{G}\right)^2 = \left(1 + \frac{1}{\gamma^2}\right)\vec{F}^2 - \frac{1}{2\gamma^2}C_1 - \frac{1}{2\gamma}C_2 = 0$$

The diagonal constraint $B \cdot \star B = 0$ is equivalent to

$$\left(1 - \frac{1}{\gamma^2}\right)C_1 + \frac{2}{\gamma}C_2 = 0\,.$$

By combining the two one arrives at the desired second condition

$$4\gamma\vec{F}^2 = \rho n\,.$$

Master constraint

In the case of the discrete series, the constraints are therefore

$$\left(\rho - \gamma n\right)\left(\rho + \frac{n}{\gamma}\right) = 0$$

$$4\gamma j(j-1) = \rho\, n$$

Approximate solution

$$\rho = \gamma n\,,\; n \geq 2 \qquad\qquad j = n/2$$

Master constraint

For states of the continuous series,

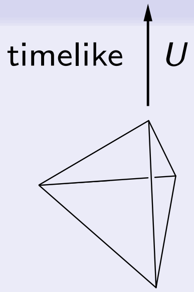
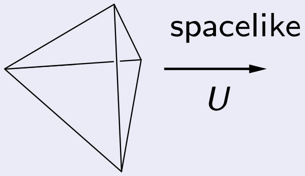
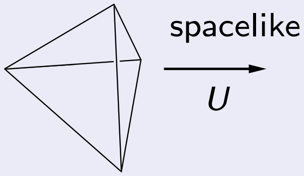
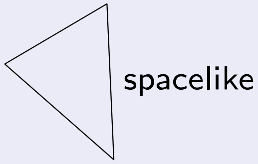
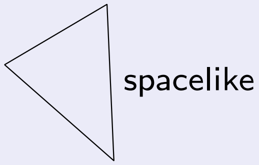
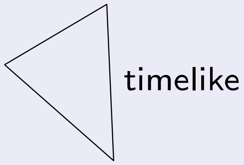
$$\left(\rho - \gamma n\right)\left(\rho + \frac{n}{\gamma}\right) = 0$$

$$-4\gamma \left(s^2 + \frac{1}{4}\right) = \rho\, n$$

Solution

$$\rho = -\frac{n}{\gamma} < -\gamma \qquad s^2 = \frac{1}{4}\left(\frac{n^2}{\gamma^2} - 1\right)$$

Table of constraints

normal U			
triangle			
little group	SU(2)	SU(1,1)	SU(1,1)
relevant irreps	$\mathcal{D}_j$	$\mathcal{D}_j^\pm$	$\mathcal{C}_s^\epsilon$
constr. on (ρ, n)	$\rho = \gamma n$	$\rho = \gamma n,\ n \geq 2$	$n = -\gamma\rho > \gamma$
constr. on irreps	$j = n/2$	$j = n/2$	$s^2 + 1/4 = \rho^2/4$
area spectrum	$\gamma\sqrt{j(j+1)}$	$\gamma\sqrt{j(j-1)}$	$\gamma\sqrt{s^2 + 1/4} = n/2$

EPRL

Spin foams

Spin foam theory

Complex:

- simplicial complex Δ: 4–simplex σ, tetrahedron τ, triangles t, ...
- dual complex Δ^*: vertex v, edge e, face f, ...

Variables (same as in EPRL):

- connection $g_e \in \mathrm{SL}(2,\mathbb{C})$
- irrep label $n_f \in \mathbb{Z}_+$

Additional variables:

- $U_e = (1,0,0,0)$ or $(0,0,0,1)$: normal of tetrahedron dual to e
- $\zeta_f = \pm 1$: spacelike/timelike triangle dual to f

Uniform notation

To cover the different cases we introduce a uniform notation.

Little group

$$H(\zeta, U) \equiv \begin{cases} \mathrm{SU}(2)\,, & \text{if} \quad \zeta = 1\,, \quad U = (1,0,0,0)\,, \\ \mathrm{SU}(1,1)\,, & \text{if} \quad \zeta = \pm 1\,, \quad U = (0,0,0,1)\,, \\ \varnothing\,, & \text{if} \quad \zeta = -1\,, \quad U = (1,0,0,0)\,. \end{cases}$$

Spin

$$j = \begin{cases} n/2\,, & \text{if} \quad \zeta = 1\,, \quad U = (1,0,0,0)\,, \\ n/2\,, & \text{if} \quad \zeta = 1\,, \quad U = (0,0,0,1)\,, \\ -\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{n^2/\gamma^2 - 1}\,, & \text{if} \quad \zeta = -1\,, \quad U = (0,0,0,1)\,. \end{cases}$$

Uniform notation

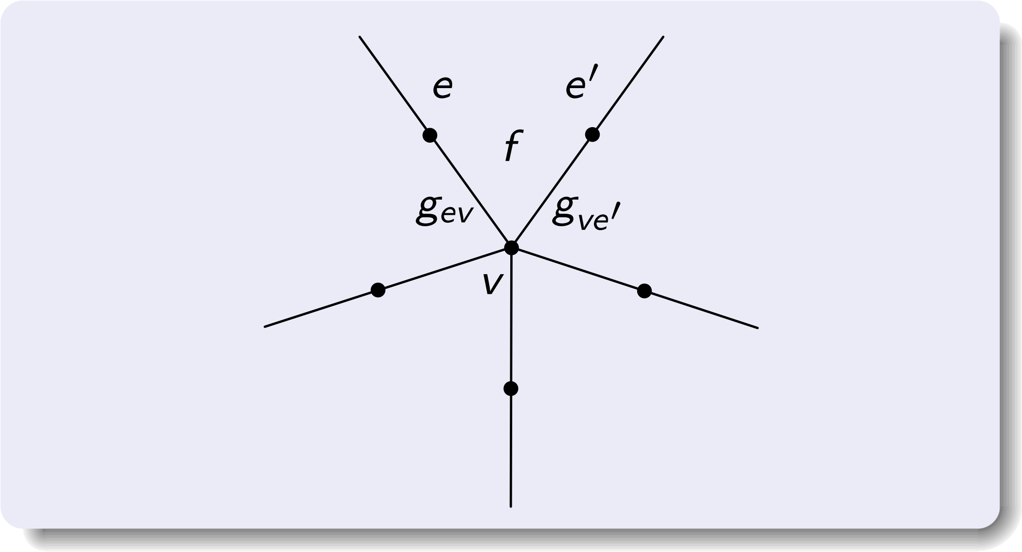
Coherent states

$$|\Psi^{(\alpha)}_{j\,h\,\delta}\rangle = \begin{cases} |\Psi_{j\,h}\rangle\,, & \text{if} \quad \zeta = 1\,, \quad U = (1,0,0,0)\,, \\ |\Psi^{\pm}_{j\,h}\rangle\,, & \text{if} \quad \zeta = 1\,, \quad U = (0,0,0,1)\,, \\ |\Psi^{(\alpha)}_{j\,h\,\delta}\rangle\,, & \text{if} \quad \zeta = -1\,, \quad U = (0,0,0,1)\,. \end{cases}$$

Projector

$$P_j(\zeta, U, \delta) = d_j(\zeta, U) \sum_{\alpha} \int_{H(\zeta,U)} \mathrm{d}h \, \left|\Psi^{(\alpha)}_{j\,h\,\delta}\right\rangle \left\langle \Psi^{(\alpha)}_{j\,h\,\delta}\right|$$

Vertex amplitude



$$A_v((\rho_f, n_f); h_{ef}, \alpha_{ef}, \delta)$$
$$= \int_{\mathrm{SL}(2,\mathbb{C})} \prod_e \mathrm{d}g_{ev} \prod_f \left\langle \Psi^{(\alpha_{ef})}_{j_{ef} h_{ef} \delta} \right| D^{(\rho_f, n_f)}(g_{ev} g_{ve'}) \left| \Psi^{(\alpha_{e'f})}_{j_{e'f} h_{e'f} \delta} \right\rangle$$

Spin foam sum

Partition function

$$Z = \sum_{n_f} \sum_{\zeta_f = \pm 1} \sum_{U_e} \sum_{\alpha_{ef}} \int_{H(U_e, \zeta_f)} \mathrm{d}h_{ef} \, d_{j_f}(U_e, \zeta_f)$$

$$\times \prod_f (1 + \gamma^{2\zeta_f}) n_f^2 \lim_{\delta \to 0} \prod_v A_v \Big((\zeta_f \gamma^{\zeta_f} n_f, n_f); h_{ef}, \alpha_{ef}, \delta \Big)$$

Compatibility of path integral and operator formalism

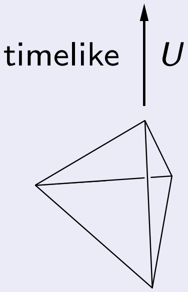
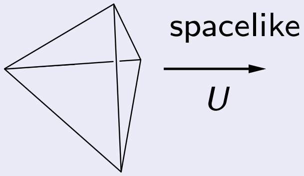
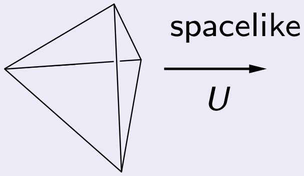
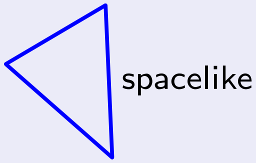
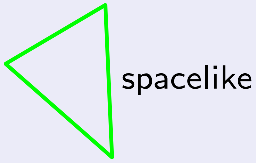
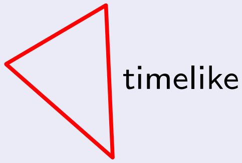
So far we were mainly guided by the path integral picture and defined the spin foam sum by simply summing over all choices of the normal U.

From the canonical perspective, however, one would also require that the sum over intermediate states on a triangulation's boundary is a projector.

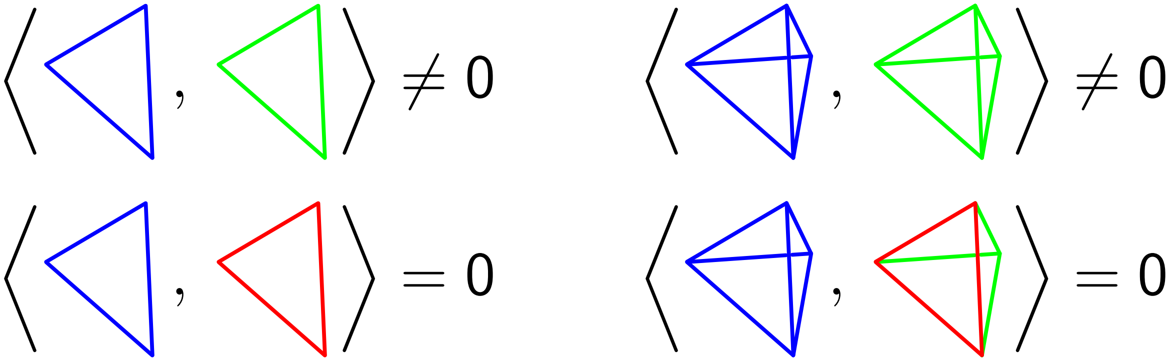
For a give choice of U, this is certainly the case.

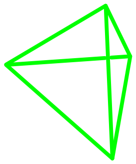
Is it also true when we sum over U?

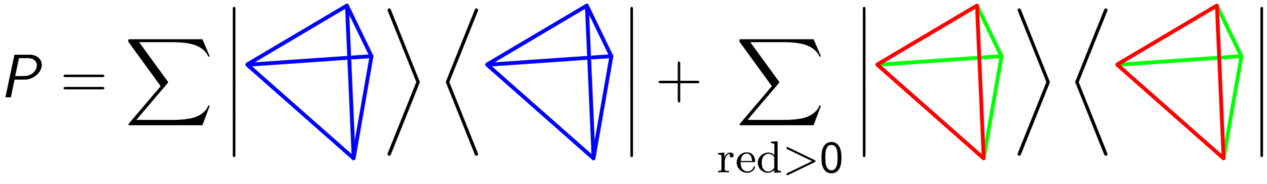
Compatibility of path integral and operator formalism

normal U			
triangle			
(ρ, n)	$\rho = \gamma n$	$\rho = \gamma n,\ n \geq 2$	$n = -\gamma\rho > \gamma$
	EPRL		

Compatibility of path integral and operator formalism



The sum over tetrahedral states is a projector if we exclude .



The boundary Hilbert space is genuinely larger than the EPRL one.

Summary

Summary

- extension of EPRL model
  - tetrahedra can be Euclidean and Lorentzian
  - triangles can be spacelike and timelike
  - larger boundary Hilbert space
- discrete area spectrum of timelike surfaces
- definition of associated spin foam model
- coherent states for timelike triangles

Outlook

- regions with timelike boundaries! **Rovelli et al., Oeckl**
- extension of results on EPRL model?
  - asymptotics, graviton propagator . . . ?
- canonical LQG on general boundaries?
- comparison with previous work on timelike surfaces

Perez, Rovelli
Alexandrov, Vassilevich
Alexandrov, Kadar