Rainbows from Quantum Gravity

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Assanioussi, AD, Lewandowski 2014 [arXiv:1412.6000]
Introduction

**Idea:**
Construct QFT on quantum spacetime and study how modes of the field probe the QG

*What do we expect to find?*

Classical spacetime $\Rightarrow$ "eigenstate of geometry" $g_{\mu\nu}^{\text{class}}$

$\Rightarrow$ every mode lives on $g_{\mu\nu}^{\text{class}}$

Quantum spacetime $\Rightarrow$ semiclasical state $\Psi_o$, superposition of "metric-eigenvalues" peaked on $g_{\mu\nu}^{\text{class}}$

$\Rightarrow$ maybe different modes $\vec{k}$ of the test field live on different components
Outline

1. Classical Theory
2. Quantization
3. Effective Metric
4. Lorentz Violation
5. Conclusion
Spacetime manifold: $M = \mathbb{R} \times \Sigma$. For simplicity in treating quantum fields, $\Sigma$ is topologically a 3-torus.

The theory:

\[
S[g, \phi] = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]
\]

Canonical analysis:
- geometry, $g_{\mu\nu} \rightarrow (q_{ab}(x); \pi^{ab}(x))$
- K-G matter field, $\phi \rightarrow (\phi(x); \pi_\phi(x))$

$\Rightarrow$ Every point $\gamma$ in phase space $\Gamma$ is uniquely defined by coordinates

\[
\gamma = \left( q_{ab}(x), \phi(x); \pi^{ab}(x), \pi_\phi(x) \right)
\]

Not all of $\Gamma$ is physical: 4 constraints per each point $x \in \Sigma$,

\[
C(x), \quad C_a(x)
\]
Coordinates on $\Gamma$ can be split into *homogeneous* and *inhomogeneous*:

\[
q_{ab} = q^{(0)}_{ab} + \delta q_{ab}, \quad \phi = \phi^{(0)} + \delta \phi
\]

(4)

\[
\pi^{ab} = \pi^{(0)}_{ab} + \delta \pi^{ab}, \quad \pi_{\phi} = \pi^{(0)}_{\phi} + \delta \pi_{\phi}
\]

where in particular

(5) \[a^2 := \int_{\Sigma} d^3 x \, \delta^{ab} q_{ab}(x) \quad \text{defines} \quad q^{(0)}_{ab} = a^2 \delta_{ab}\]

Note that $q^{(0)}_{ab}$ is of the Robertson-Walker type, but in general does not satisfy 0th order Einstein equation (i.e., Friedmann equation).
Physical phase space: solve the constraints $C = 0$ and $C_a = 0$. To linear order in the inhomogeneities, one can show [AD, Lewandowski, Puchta 2013] that the only physical degrees of freedom are:

- homogeneous geometry $\rightarrow a$
- tensor modes of geometry (graviton-to-be) $\rightarrow \delta q^+, \delta q^x$
- scalar modes of matter (scalar field) $\rightarrow \delta \phi\hat{k}$

where we already performed spatial Fourier transform, and $\hat{k} \in \mathcal{L} = (2\pi\mathbb{Z})^3 - \{0\}$.

Homogeneous part of Hamiltonian constraint, $\int d^3 x C(x) = 0$, is solved for momentum $\pi^{(0)}_\phi$ of $\phi^{(0)}$. Hence, $\phi^{(0)}$ is physical time, and we obtain a physical Hamiltonian:

\[
\frac{d}{d\phi^{(0)}} F = \{F, h_{\text{phys}}\}
\]

where

\[
h_{\text{phys}} = H_{\text{hom}} - \sum_{\hat{k}} \frac{H_{\text{hom}}^{-1}}{2} \left[ \delta \pi^2_{\hat{k}} + (a^4 k^2 + a^6 m^2) \delta \phi^2_{k} \right] + \text{Hamiltonian for } \delta q^i_{\hat{k}}
\]

and $H_{\text{hom}} = \sqrt{\kappa/6} a\pi a$. 
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Focus on the scalar part:

\( \mathcal{H} = \mathcal{H}_{\text{hom}} \otimes \mathcal{H}_\phi \)  

and quantum dynamics driven by Hamiltonian

\[
\hat{H} = \hat{H}_{\text{hom}} \otimes \hat{1} - \frac{1}{2} \sum_k \left( \hat{H}_{\text{hom}}^{-1} \otimes \delta \hat{\pi}_k^2 + \hat{\Omega}(k, m) \otimes \delta \hat{\phi}_k^2 \right)
\]

where

\[
\hat{\Omega}(k, m) := k^2 \frac{\hat{H}_{\text{hom}}^{-1} \hat{a}^4 + \hat{a}^4 \hat{H}_{\text{hom}}^{-1}}{2} + m^2 \frac{\hat{H}_{\text{hom}}^{-1} \hat{a}^6 + \hat{a}^6 \hat{H}_{\text{hom}}^{-1}}{2}
\]

\( \hat{H} \) acts on a state \( |\Psi(t,a,\phi)\rangle \in \mathcal{H} \) via Schroedinger equation:

\[
-i \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle
\]
Test field approximation (0th order Born-Oppenheimer) ⇒ Geometry and matter are disentangled:

\begin{equation}
|\Psi(t, a, \phi)\rangle = |\Psi_o(t, a)\rangle \otimes |\varphi(t, \phi)\rangle
\end{equation}

where

\begin{equation}
-i \frac{d}{dt} |\Psi_o\rangle = \hat{H}_{\text{hom}} |\Psi_o\rangle
\end{equation}

Plugging this in the Schroedinger equation, and projecting on \langle \Psi_o |, gives

\begin{equation}
i \frac{d}{dt} |\varphi\rangle = \frac{1}{2} \sum_k \left[ \langle \Psi_o | \hat{H}_{\text{hom}}^{-1} |\Psi_o\rangle \delta \hat{\pi}^2_k + \langle \Psi_o | \hat{\Omega}(k, m) |\Psi_o\rangle \delta \hat{\phi}^2_k \right] |\varphi\rangle
\end{equation}

Not surprising: a collection of harmonic oscillators. But the parameters of this h.o. are given in terms of expectation values of geometric operators on quantum state of geometry, \( \Psi_o \).
QFT on quantum spacetime sandwitched on $|\Psi_o\rangle \in \mathcal{H}_{\text{hom}}$:

$$i \frac{d}{dt} |\varphi\rangle = \frac{1}{2} \sum_k \left[ \langle \Psi_o | \hat{H}_{\text{hom}}^{-1} | \Psi_o \rangle \delta \hat{\pi}^2_k + \langle \Psi_o | \hat{\Omega}(k, m) | \Psi_o \rangle \delta \hat{\phi}^2_k \right] |\varphi\rangle \quad (15)$$

QFT on classical Robertson-Walker spacetime

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = -\bar{N}^2 dt^2 + \bar{a}^2 (dx^2 + dy^2 + dz^2) \quad (16)$$

$$\Rightarrow$$

$$i \frac{d}{dt} |\varphi\rangle = \frac{1}{2} \sum_k \left[ \frac{\bar{N}}{\bar{a}^3} \delta \hat{\pi}^2_k + \frac{\bar{N}}{\bar{a}^3} (\bar{a}^4 k^2 + \bar{a}^6 m^2) \delta \hat{\phi}^2_k \right] |\varphi\rangle \quad (17)$$

The comparison gives

$$\begin{cases} 
\bar{N}/\bar{a}^3 = \langle \hat{H}_{\text{hom}}^{-1} \rangle \\
\bar{N} (\bar{a}^4 k^2 + \bar{a}^6 m^2) /\bar{a}^3 = \langle \hat{\Omega}(k, m) \rangle 
\end{cases} \quad (18)$$

$$\Rightarrow$$ Only one real and positive solution:

$$\bar{N} = \langle \hat{H}_{\text{hom}}^{-1} \rangle \bar{a}^3, \quad \bar{a} = \bar{a}(k/m)$$
Striking conclusion:

fundam. quantum gravity+matter $\iff$ QFT on effective, $k$-dependent spacetime

The effective scale factor:

\[
\bar{a}(k/m)^2 = \begin{cases} 
    u_+ + u_- - \frac{k^2}{3m^2} & \text{if } k < k_o \\
    \frac{2k^2}{3m^2} \cos \left[ \frac{1}{3} \arccos \left( -1 + \frac{27m^6}{2k^6} \delta \right) \right] - \frac{k^2}{3m^2} & \text{if } k \geq k_o
\end{cases}
\]

where

\[
u_{\pm} := 3 \sqrt{\frac{\delta}{2} - \frac{k^6}{27m^6}} \pm \sqrt{\frac{\delta^2}{4} - \frac{k^6}{27m^6} \delta}, \quad \delta = \frac{\langle \hat{\Omega}(k, m) \rangle}{m^2 \langle \hat{H}_{\hom}^{-1} \rangle}
\]

remark: if we started with massless field, $m = 0$, the solution turns out to be $k$-independent and given by [Ashtekar, Kaminski, Lewandowski 2009]

\[
\bar{a}_{m=0}^2 = \sqrt{\frac{\langle \hat{H}_{\hom}^{-1} \hat{a}^4 + \hat{a}^4 \hat{H}_{\hom}^{-1} \rangle}{2 \langle \hat{H}_{\hom}^{-1} \rangle}}
\]

This is consistent with the "high energy" limit $k \gg m$ of the massive solution (19).
In the “low energy” limit $k \ll m$, we have

$$ (22) \quad \bar{a} \left( \frac{k}{m} \right)^2 \approx \bar{a}_o^2 \left[ 1 + \frac{\beta}{3} \left( \frac{k}{\bar{a}_o} \right)^2 \right] $$

where

$$ \bar{a}_o^2 = \frac{1}{\sqrt[3]{2} \langle \hat{H}_{\text{hom}}^{-1} \hat{a}^6 + \hat{a}^6 \hat{H}_{\text{hom}}^{-1} \rangle}^{\frac{1}{3}}, \quad \beta := \frac{1}{\sqrt[3]{2} \langle \hat{H}_{\text{hom}}^{-1} \hat{a}^4 + \hat{a}^4 \hat{H}_{\text{hom}}^{-1} \rangle} - 1 $$

**Interpretation**

- Scale factor $\bar{a}_o$ defines the low-energy, $k$-independent metric

$$ (23) \quad \bar{g}_o^{\mu \nu} dx^\mu dx^\nu = -\bar{N}_o^2 dt^2 + \bar{a}_o^2 (dx^2 + dy^2 + dz^2) $$

We can think of it as the semiclassical metric seen by an observer performing macroscopic measurements.

- Parameter $\beta$ encodes the quantum nature of spacetime: if product of expectation values $= \text{expectation value of products}$, then $\beta = 0$. 
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A cosmological observer (4-velocity $u^\mu$, metric $\bar{g}^\mu_\nu$) measures a particle of wave-vector $k_\mu$ passing through the lab:

- **Energy:** $E := u^\mu k_\mu = k_0/\bar{N}_o$
- **Momentum:** $p^2 := (\bar{g}^\mu_\nu + u^\mu u^\nu) k_\mu k_\nu = k^2/\bar{a}_o^2$

But the particle satisfies the mass-shell relation wrt metric $\bar{g}^\mu_\nu(k/m)$:

\begin{equation}
-m^2 = \bar{g}^\mu_\nu k_\mu k_\nu = -\frac{k_0^2}{\bar{N}^2} + \frac{k^2}{\bar{a}^2} = -f^2 E^2 + g^2 p^2
\end{equation}

where

\begin{equation}
f := \frac{\bar{N}_o}{\bar{N}}, \quad g := \frac{\bar{a}_o}{\bar{a}}
\end{equation}

are the so-called *rainbow functions* [Magueijo, Smolin 2004].

⇒ Modified dispersion relation:

\begin{equation}
E^2 = \frac{1}{f^2} \left(g^2 p^2 + m^2 \right) = m^2 + (1 + \beta)p^2 + O(p^4)
\end{equation}
The standard dispersion relation is recovered two independent limits:

- **semiclassical matter** (i.e., modes with $p \ll m$): in this case $E \approx m$ (the most famous formula of physics!)
- **semiclassical gravity** (i.e., $|\Psi_0\rangle$ such that $\beta \ll 1$): in this case, $E \approx \sqrt{m^2 + p^2}$

**remark:** No particular role is played by $E_{\text{Planck}} \approx 10^{19}$ GeV. Indeed, if $\beta \approx 1$ (i.e., $|\Psi_0\rangle$ is a very non-classical state), then modifications are present for $p \approx m$, which for a proton would be around 1 GeV. We do not see Lorentz-violations in accelerators because $|\Psi_0\rangle$ is extremely classical today!
\[
\begin{align*}
    v &= \frac{dE}{dp} = \frac{1 + \beta}{\sqrt{m^2 + (1 + \beta)p^2}} p \\
\end{align*}
\]

**Remark:** For massless particles, \( m \ll p \), we do not get 1 but rather \( \sqrt{1 + \beta} \). Hence, we have a modified velocity of light. Just a shift by \( \beta \)! Where is the big deal? Well...

1) \( \beta \) is a function of expectation values of geometric operators on \( |\Psi_o\rangle \), and as such it depends on time.

2) This simple form for \( v_{m=0} \) is due to the approximation we considered. For the exact \( \bar{a}^2 \) of equation (19) above, numerics give

![Graph](image)

Green = semiclassical spacetime \((\beta \approx 0)\); Blue = quantum spacetime \((\beta \approx 0.2)\); Dashed = light
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Main result

quantum gravity+matter ⇐⇒ QFT on effective, $k$-dependent spacetime $\bar{g}_{\mu\nu}$

This is true in any theory of quantum gravity based on a Hamiltonian formulation, if the following approximations hold:

- linearized inhomogeneities around homogeneous isotropic background (e.g., LQC)
- test-field approximation: $|\Psi(t, a, \phi)\rangle = |\Psi_o(t, a)\rangle \otimes |\varphi(t, \phi)\rangle$

What is the effect of this result?

$k$-dependence of $\bar{g}_{\mu\nu}$ implies a modified dispersion relation controlled by the scale

$$\beta = \frac{1}{\sqrt{2} \langle \hat{H}_{\text{hom}}^{-1} \rangle} \frac{\langle \hat{H}_{\text{hom}}^{-1} a^4 + a^4 \hat{H}_{\text{hom}}^{-1} \rangle}{\langle \hat{H}_{\text{hom}}^{-1} a^6 + a^6 \hat{H}_{\text{hom}}^{-1} \rangle^{2/3}} - 1 \quad (29)$$

Only one parameter, in spite of the microscopic structure of quantum spacetime $|\Psi_o\rangle$!

⇒ compare with crystals’ refractive properties: described uniquely by refractive index $n$

Can we test this result?

Today the geometry is classical, so $\beta \ll 1$: no Lorentz-violation today :( 

⇒ However, in the primordial Universe the geometry is expected to be "very quantum", in which case $\beta \approx 1$, and hence the Lorentz-violation is present even for $p \approx m$ :o
thank you