Cosmological Effective Hamiltonian from full Loop Quantum Gravity

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Introduction

- The *effective dynamics program* [Ashtekar, Pawlowski, Singh '06] had much success by giving an excellent approximation to the exact quantum corrections of LQC, e.g. big bounce
- Its idea was the following: Take the classical cosmological Hamiltonian

$$H_{\cos}(p,c) \sim \sqrt{p}c^2 \tag{1}$$

where (c,p) are obtained from the Ashetkar-Barbero variables (A,E) of the isotropic spacetime by fixing a fiducial Minkowski metric:

$$A_a^I = c\delta_a^I, \quad E_I^a = p\delta_I^a \tag{2}$$

- By a *loop-inspired quantization* one promotes this to the LQC Hamiltonian constraint operator $\bar{\mu}$-scheme: $\lambda^2 = \ell_P^2 2\sqrt{3}\pi\beta$):

$$\hat{H}_{LQC} \sim \frac{\hat{N}_\lambda - \hat{N}_{-\lambda}}{2\lambda}\hat{\nu}\frac{\hat{N}_\lambda - \hat{N}_{-\lambda}}{2\lambda}\hat{\nu} \tag{3}$$

Introduction

- It was shown that the *LQC coherent states*

$$\psi^{LQC}_{coh}(v) = \int_0^\infty dk \; e^{-\frac{(k-k_o)^2}{2\sigma^2}} e_k(v), \qquad \hat{H}_{LQC} e_k = 12\pi G k^2 e_k \tag{4}$$

  are stable under the evolution produced by $\hat{H}_{LQC}$.
- When evaluating the expectation value of $\hat{H}_{LQC}$ on ψ^{LQC}_{coh}, one obtains the *effective Hamiltonian*:

$$H_{\rm eff}(p,c) \sim \sqrt{p}\frac{\sin(c\bar{\mu})^2}{\bar{\mu}^2}, \qquad \bar{\mu} = \lambda/\sqrt{p} \tag{5}$$

  Quantum evolution of state (4) is well approximated by classical evolution by $H_{\rm eff}$.
- Moreover the framework of QRLG produces the same effective Hamiltonian [Alesci, Cianfrani '14]

Can we repeat this effective dynamics program in the full theory?

Outline of the talk

Can we repeat this effective dynamics program in the full theory?

1. Choose a suitable *full-theory coherent state* $\Psi_{(c,p)}$ which resembles isotropic cosmology peaked on (c, p)
2. Calculate the expectation values of basic polynomials in $(\hat{h}, \hat{E})$ on said $\Psi_{(c,p)}$, e.g. kinematical observables like the volume
3. Plug everything together to obtain the expectation value of the *physical Hamiltonian* $\langle \hat{H} \rangle_{(c,p)}$, i.e. a candidate for the *effective Hamiltonian*
4. Show that $\Psi_{(c,p)}$ also stay peaked under evolution of $\hat{H}$ *(OPEN)*
5. Compare the effective dynamics of LQC with the one of LQG *(or at least of a toy-model)*

(brief) Recap of LQG

- We perform a canonical quantization of general relativity in its formulation of the canonical pair (A_a^I, E_I^a) known as the Ashtekar-Barbero variables
- The kinematical Hilbertspace $\mathcal{H}_{\text{kin}}$ is the set of *all* graphs γ, where on every edge e lives an square integrable function over SU(2), i.e.

$$\mathcal{H}_{\text{kin}} = \bigotimes_{e \in \gamma} \mathcal{H}_e, \qquad \mathcal{H}_e \cong L_2\left(\text{SU}(2), d\mu_H\right) \tag{6}$$

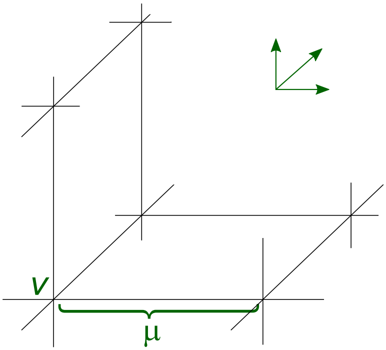
- Holonomy- and flux-operators act on them in the usual way:

$$\hat{h}_{ab}(e) f_e(g) := D_{ab}^{(1/2)}(g) f_e(g) \tag{7}$$

$$\hat{E}^k(S_e) f_e(g) := -\frac{i\hbar\kappa\beta}{4}\frac{d}{ds}\,|_{s=0}\; f_e(e^{-is\sigma_k} g) \tag{8}$$

The graph

To perform the effective dynamics program, we have to pick a *suitable* state:



- A cubic lattice, i.e. every vertex has valence 6, with periodic boundaries, i.e. a 3-torus $T^3 = S^1 \times S^1 \times S^1$
- We embed each edge e along the coordinate axes
- Associate with every edge the *coordinate length* μ
- Finally, the SU(2)-function on each edge is assigned to a semi-classical *coherent state* of the classical variables along this path.

Complexifier coherent states

- The *complexifier coherent states* ψ_{e,h_e} for gauge group SU(2) are given on each edge e by [Thiemann, Winkler '00]

$$\psi^t_{e,h_e}(g) := \frac{1}{||\psi_{e,h_e}||} \sum_{j=0}^{\infty} (2j+1) e^{-j(j+1)t} \sum_{m=-j}^{j} e^{izm} D^{(j)}_{mm}(n_e^\dagger g n_e')$$

with the *semiclassicality parameter* t being any dimensionless number (e.g. $t = \ell_p^2/a^2$) and the SL$(2,\mathbb{C})$ element:

$$h_e = n_e e^{-iz_e\sigma_3/2} {n_e'}^\dagger \tag{9}$$

- For the cosmological coherent state we choose on *every link* $n_e = n_e' \in$ SU(2) and $z = \mu c - i\mu^2 a^{-2}\beta^{-1} p$

$$\Psi_{(c,p)} := \bigotimes_{e\in\mathbb{Z}^3_M} \psi^t_{e,n_e \exp(-iz\sigma_3/2) n_e^\dagger} \tag{10}$$

Calculation of the expectation values

In order to obtain the expectation values one needs knowledge of the $SU(2)$-calculus, e.g.:

$$D^{(j_1)}_{ab}(g)D^{(j_2)}_{cd}(g) = \tag{11}$$
$$= \sum_{j=|j_1-j_2|}^{j_1+j_2} d_j(-1)^{m-n} \begin{pmatrix} j_1 & j_2 & j \\ a & c & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ b & d & n \end{pmatrix} D^{(j)}_{-m-n}(g)$$

and

$$\int d\mu_H(g)\overline{D^{(j)}_{mn}(g)}D^{(j')}_{m'n'}(g) = \frac{1}{d_j}\delta_{jj'}\delta_{mm'}\delta_{nn'} \tag{12}$$

which allows us to calculate $h_e = n\exp(\xi + i\eta)n^\dagger$

$$\langle\psi_{e,h_e}, \hat{h}_{ab}\psi_{e,h_e}\rangle = e^{i\xi c}D^{(1/2)}_{ac}(n)D^{(1/2)}_{cb}(n^\dagger)e^{-\eta(m+m')}\times \tag{13}$$
$$\times \sum_{j,j'} d_j d_{j'} e^{-t[j(j+1)+j'(j'+1)]/2} \begin{pmatrix} 1/2 & j & j' \\ c & m & -m' \end{pmatrix}^2$$

Theorem: (Poisson Summation Formula) Let $f \in L_1(\mathbb{R}, dx)$, then

$$\sum_{n\in\mathbb{Z}} f(ns) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} dx e^{-i2\pi mx} f(sx) \tag{14}$$

Applying this changes $e^{-tn(n+1)/2} \to e^{-4\pi^2 m^2/t}$ which decays for $t \to 0$ extremly fast unless $m = 0$. We conclude that for $1 \gg t$ the sum over m collapses to only one term!
Evaluating said one yields finally:

$$\begin{aligned} \langle \psi_{e,h_e}, \hat{h}_{ab}\psi_{e,h_e}\rangle = D^{(1/2)}_{ab}(ne^{-i\mu c\sigma_3/2}n^\dagger)\times \\ \times[1 + t\frac{3}{16} - t\frac{a^2\beta}{4\mu^2 p}\tanh(\frac{\mu^2 p}{2a^2\beta}) + \mathcal{O}(t^2)] \end{aligned} \tag{15}$$

Polynomials of the fluxes I

To continue with computing the expectation value of monomials of the fluxes $\hat{E}^k$, one needs two observations:

Theorem: (Cyclicity Property) Be ψ_h as coherent state with $h = e^{-i\mu c\sigma_3/2}$, then

$$\langle \psi_h, \hat{E}^{k_1} \dots \hat{E}^{k_N} \psi_h \rangle = e^{-2k_N \frac{\mu^2 p}{a^2 \beta}} \langle \psi_h \hat{E}^{k_N} \hat{E}^{k_1} \dots \hat{E}^{k_{N-1}} \psi_h \rangle \tag{16}$$

Further, as $\hat{E}^k$ is subject to the algebraic relations ($k_i \neq 0$)

$$[\hat{E}^{k_1}, \hat{E}^{k_2}] \sim -i(k_1 - k_2)\hat{E}^0, \quad [\hat{E}^k, \hat{E}^0] \sim -2ik\hat{E}^k \tag{17}$$

we could also permute the $\hat{E}^{k_N}$ in (16) accordingly!

Polynomials of the fluxes II

Combining (16) and (17) suitably one finds at the end:

$$\langle\psi_{h_e}, \hat{E}^{k_1}\dots\hat{E}^{k_N}\psi_{h_e}\rangle = D^{(1)}_{-k_1 s_1}(n)\dots D^{(1)}_{-k_N s_N}(n)\Big[\delta_0^{s_1}\dots\delta_0^{s_N}(i\partial_\eta)^N -$$

$$-\frac{i}{\sinh(\eta)}\sum_{A<B=1}^{N}\delta_0^{s_1..\not{s}_A..\not{s}_B..s_N}(\delta_1^{s_A}\delta_{-1}^{s_B}e^{\eta}+\delta_{-1}^{s_A}\delta_1^{s_B}e^{-\eta})(i\partial_\eta)^{N-1}\Big]\times$$

$$\times\, 2e^{t/4}\left(-\frac{i\hbar\kappa\beta}{4}\right)^N\sqrt{\frac{\pi}{t^3}}\frac{\eta e^{\eta^2/t}}{\sinh(\eta)}\Big|_{\eta=\frac{\mu^2 p}{a^2\beta}} \tag{18}$$

This gives peakedness on the classical phase space variable plus additional quantum correction for any N. E.g. $N=1$:

$$\langle\psi_{e,h_e}, \hat{E}^k\psi_{e,h_e}\rangle = \mu^2 p D^{(1)}_{-k0}(n)[1+t(\frac{a^2\beta}{\mu^2 p}+\coth(\frac{\mu^2 p}{a^2\beta}))+\mathcal{O}(t^2)] \tag{19}$$

The Volume Operator I

Example: A kinematic operator, which is crucial for defining the dynamics, is the Volume!

- A definition which is consistent with the flux-operators is the *Ashtekar-Lewandowski volume operator* ['98]

$$\hat{V}(R) := \sum_{v \in R} \hat{V}_v^{AL} \sim \sum_{v \in R} \sqrt{|\ \hat{Q}_v\ |} \tag{20}$$

$$\hat{Q}_v := i \sum_{e \cap e' \cap e'' = v} \epsilon(e, e', e'') \epsilon_{ijk} \hat{E}^i(e) \hat{E}^j(e') \hat{E}^k(e'') \tag{21}$$

- The square-root makes analytical treatment nearly impossible as long as we don't know the spectrum of $\hat{Q}_v$!

Is there a work-around?

Maybe a power-series of $\sqrt{|\ .\ |}$?

- An alternative definition instead of $\hat{V}_v^{AL} = \sqrt{|\hat{Q}_v\ |}$ we call the *Giesel-Thiemann volume operator*

$$\hat{V}_{k,v}^{GT} := \langle \hat{Q}_v \rangle^{1/2} \left[\mathbb{1} + \sum_{n=1}^{2k+1} \frac{(-1)^{n+1}}{n!} \frac{1}{4} ... (n - 1 - \frac{1}{4}) (\frac{\hat{Q}_v^2}{\langle \hat{Q}_v \rangle^2} - \mathbb{1})^n \right]$$

- The amazing feature is, that for the expectation values of any coherent state $\Psi_{(c,p)}$ and any polynomial P: [Giesel, Thiemann '06]

$$\langle \Psi_{(c,p)}, P(\hat{V}_v^{AL}, \{\hat{h}\}) \Psi_{(c,p)} \rangle = \langle \Psi_{(c,p)}, P(\hat{V}_{k,v}^{GT}, \{\hat{h}\}) \Psi_{(c,p)} \rangle + \mathcal{O}(t^{k+1})$$

This allows us to replace $\hat{V}_v^{AL}$ with e.g. $\hat{V}_{1,v}^{GT}$ and to compute every expectation value correctly up to $\mathcal{O}(t^2)$!

The Volume Operator III

We are now in the situation that we have on six edges a polynomial of $\hat{E}$ and can thus use (18):

$$\langle \Psi_{(c,p)}, \hat{Q}_v^N \Psi_{(c,p)} \rangle =$$

$$= \langle \Psi_{(c,p)}, i^N \left(6(\hat{E}_1^I + \hat{E}_{-1}^I)(\hat{E}_2^J + \hat{E}_{-2}^J)(\hat{E}_3^K + \hat{E}_{-3}^K)\epsilon_{IJK} \right)^N \Psi_{(c,p)} \rangle =$$

$$\sim (6i)^N \left(\frac{2\eta i}{t} \right)^{3N} [2^N + \frac{t}{2\eta^2}(N^2 + 3N)2^{N-2} - \frac{t}{2\eta} N 2^N \coth(\eta)]^3$$

and finally using the definition of the Giesel-Thiemann volume we approximate the Ashetkar-Lewandowski volume operator (with $\mathcal{N}^3$ being the number of vertices in σ)

$$\langle \Psi_{(c,p)}, \hat{V}(\sigma) \Psi_{(c,p)} \rangle = \mathcal{N}^3 (\mu^2 p)^{3/2} \times$$
$$\times [1 + t \frac{3a^4 \beta^2}{4\mu^4 p^2} \left(\frac{7}{8} - \frac{\mu^2 p}{a^2 \beta} \coth(\frac{\mu^2 p}{a^2 \beta}) \right) + \mathcal{O}(t^2)] \quad (22)$$

Kinematics: Recap

- $\Psi_{(c,p)}$ is a state in the full theory, peaked in holonomy and flux of flat RW metric, (c, p):

$$\langle \hat{h}_{ab} \rangle = D^{(1/2)}_{ab}(n e^{-i\mu c \sigma_3/2} n^\dagger)[1 + \mathcal{O}(t)]$$

$$\langle \hat{E}^k \rangle = \mu^2 p D^{(1)}_{-k0}(n)[1 + \mathcal{O}(t)]$$

- We can compute expectation value of polynomials in holonomy and polynomials in flux. E.g., relative dispersions:

$$\delta h_{ab} := \frac{\langle \hat{h}_{ab} \hat{h}_{ab} \rangle}{\langle \hat{h}_{ab} \rangle^2} - 1 = t \; f(p, c, n)$$

$$\delta E^k := \frac{\langle \hat{E}^k \hat{E}^k \rangle}{\langle \hat{E}^k \rangle^2} - 1 = t \; g(p, n)$$

- We can compute expectation value of (G-T) volume operator:

$$\langle \hat{V}(\sigma) \rangle = \mathcal{N}^3 (\mu^2 p)^{3/2}[1 + \mathcal{O}(t)]$$

Dynamics: The Quantum Hamiltonian

We take a *non-graph-changing* regularization [Giesel, Thiemann, '06] of the scalar-constraint as quantized by Thiemann ['98]

$$\hat{H} = \hat{H}_E - (\beta^2 + 1)\hat{H}_L$$

where

$$\begin{aligned} \hat{H}_E(N_v) &\sim \textstyle\sum_{v\in V(\gamma)} N_v \sum_{e\cap e'\cap e''=v} \epsilon(e,e',e'')\times \\ &\times Tr\left((\hat{h}(\Box_{ee'}) - \hat{h}(\Box_{ee'})^{-1})\hat{h}(e'')\left[\hat{h}(e'')^{-1}, \hat{V}^{GT}_{1,v}\right]\right) \end{aligned}$$

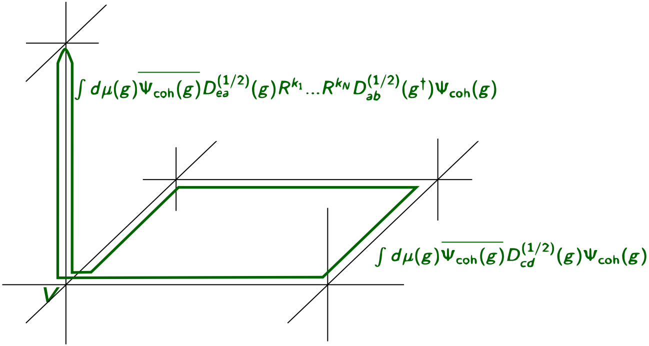
$$\begin{aligned} \hat{H}_L(N_v) &\sim \textstyle\sum_{v\in V(\gamma)} N_v \sum_{e\cap e'\cap e''=v} \epsilon(e,e',e'')\times \\ &\times Tr\left(\hat{K}(e)\hat{K}(e')\hat{h}(e'')[\hat{h}(e'')^{-1}, \hat{V}^{GT}_{1,v}]\right) \end{aligned}$$

with the extrinsic curvature $\hat{K}_{ab}(e)$ given by

$$\hat{K}(e) = \hat{h}(e)[\hat{h}(e)^{-1}, [\hat{H}_E(1), \hat{V}^{GT}_{1,v}]]$$

Action of the Euclidean part

$$\hat{H}_E(N_v) \sim \sum_{v\in V(\gamma)} N_v \sum_{e\cap e'\cap e''=v} \epsilon(e,e',e'')\times$$
$$\times\ Tr\left((\hat{h}(\Box_{ee'}) - \hat{h}(\Box_{ee'})^{-1})\hat{h}(e'')\left[\hat{h}(e'')^{-1}, \hat{V}^{GT}_{1,v}\right]\right)$$



Effective Hamiltonian

$$H_{\text{eff}}(p,c) \quad := \quad \langle \hat{H}_E \rangle \quad - \quad (\beta^2+1)\, \langle \hat{H}_L \rangle \qquad + \mathcal{O}(t)$$

Isotropic cosmology $\Rightarrow$ choose lapse function $N_v = 1$. One finds:

$$\langle \hat{H}_E \rangle = \mathcal{N}^3 \mu \sqrt{p} \sin(c\mu)^2 \left[c_1 + t\left(c_2 \frac{\coth(c_3 \mu^2 p)}{\mu^2 p} - c_4 \frac{1}{(\mu^2 p)^2} \right) + \mathcal{O}(t^2) \right]$$

- Identify the number of vertices with the inverse coordinate volume ($\mathcal{N} = \mu^{-1}$):

$$\langle \hat{H}_E \rangle \sim \sqrt{p} \frac{\sin(c\mu)^2}{\mu^2} \left[1 + \mathcal{O}(t) \right]$$

- In the classical limit $t \to 0$ we obtain a lattice theory, which equals the effective one of LQC (in what scheme? Depends on what μ is...)
- In the classical&continuum-limit $t, \mu \to 0$ we obtain the classical cosmological Hamiltonian $\sqrt{p}c^2$

The Effective Lorentzian Hamiltonian

Recall that we also have Lorentzian part!

$$H_{\text{eff}}(p,c) \;\sim\; \sqrt{p}\frac{\sin(c\mu)^2}{\mu^2} \quad - \quad (\beta^2+1)\,\langle \hat{H}_L \rangle \qquad + \mathcal{O}(t)$$

Due to limited space, we only present the leading order:

$$\langle \hat{H}_L \rangle \sim \frac{1}{4\beta^2}\sqrt{p}\frac{\sin(2c\mu)^2}{\mu^2} + \mathcal{O}(t)$$

- Again we find a term, that reduce to the classical function for $t, \mu \to 0$
- HOWEVER: it is *different* from the one euclidean one, and not accounted for in LQC!

Full Effective Hamiltonian I

Effective Hamiltonian from full theory:

$$H_{\text{eff}}(p,c) \;\sim\; \sqrt{p}\frac{\sin(c\mu)^2}{\mu^2} \;-\; \frac{\beta^2+1}{4\beta^2}\;\sqrt{p}\frac{\sin(2c\mu)^2}{\mu^2} \;+\; \mathcal{O}(t) \sim$$
$$\sim\; -\frac{1}{\beta^2}\sqrt{p}\frac{\sin(c\mu)^2}{\mu^2}\left[1-(1+\beta^2)\sin(c\mu)^2\right] \qquad (23)$$

Genuinely different from LQC effective dynamics. It can be reproduced within LQC (with $\mu \to \bar{\mu}$) from the following operator:

$$\hat{H}^{new}_{LQC} \sim \;\frac{\hat{N}_\lambda - \hat{N}_{-\lambda}}{2\lambda}\hat{\nu}\frac{\hat{N}_\lambda - \hat{N}_{-\lambda}}{2\lambda}\hat{\nu} - \frac{\beta^2+1}{4}\frac{\hat{N}_{2\lambda} - \hat{N}_{-2\lambda}}{2\lambda}\hat{\nu}\frac{\hat{N}_{2\lambda} - \hat{N}_{-2\lambda}}{2\lambda}\hat{\nu}$$

Full Effective Hamiltonian II

Action of $\hat{H}^{new}_{LQC}$ on $\Psi(\nu) \in \mathcal{H}_{LQC}$ is quartic equation:

$$\begin{aligned}\hat{H}^{new}_{LQC}\Psi(\nu) \sim\ & C_+(\nu)\Psi(\nu+4\lambda) + D_0(\nu)\Psi(\nu) + C_-(\nu)\Psi(\nu-4\lambda)+ \\ & + D_+(\nu)\Psi(\nu+8\lambda) + D_-(\nu)\Psi(\nu-8\lambda)\end{aligned}$$

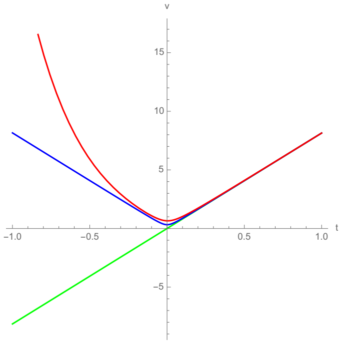
- Similar operator obtained within LQC [Yang, Ding, Ma '09] by a more "Thiemann-like" quantization of H_L.
- $\hat{H}^{new}_{LQC}$ passes von Neumann test on numerical stability $\Rightarrow$ suited for numerical evolution by LQC methods.
- Will the quantum evolution be well-approximated by the effective Hamiltonian $H_{\text{eff}}(p, c)$? We do not know yet.

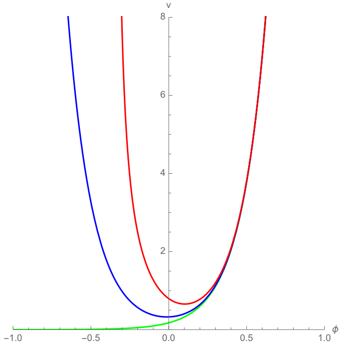
Toy Model

Take seriously this effective Hamiltonian: what evolution does it generate? Add a scalar field ϕ:

$$H = H_\phi + H_{\text{eff}} = \frac{p_\phi^2}{2p^{3/2}} - \frac{3}{8\pi\beta^2}\sqrt{p}\frac{\sin(c\bar{\mu})^2}{\bar{\mu}^2}\left[1-(1+\beta^2)\sin(c\bar{\mu})^2\right] \approx 0$$

Hamilton equations solved numerically (plot for $\beta = 0.12, p_\phi = 1$):





(red = toy model; blue = LQC; green = GR)

Conclusions: what was shown

- Tools to compute expectation values on complexifier CS's
- Proposal for such CS's representing Robertson-Walker geometry (with $k = 0$)
- Application of tools to geometric operators (e.g., Volume)
- Application of tools to scalar constraint
  $\Rightarrow$ cosmological effective Hamiltonian

$$H_{\text{eff}}(p, c) = c_1 \sqrt{p}\frac{\sin(c\mu)^2}{\mu^2} - c_2 \sqrt{p}\frac{\sin(2c\mu)^2}{\mu^2} + \mathcal{O}(t)$$

- If lattice length $\mu \to 0$, $H_{\text{eff}} = H_{\text{class}}$. If instead $\mu = \bar{\mu}$, then $H_{\text{eff}} = H_{\text{LQC}}[1 - (1 + \beta^2)\sin(c\bar{\mu})^2]$.
- Proposal of operator $\hat{H}^{new}_{LQC}$ on $\mathcal{H}_{LQC}$ that corresponds to H_{eff}. It produces a numerically stable 4th order difference equation.
- The evolution generated by H_{eff} coincides with effective LQC (and GR) today, but in the early universe we have difference $\Rightarrow$ big bounce scenario is modified.

Conclusions: what to do next

- As for volume: ideas to obtain matrix elements from expectation values.
- As for Hamiltonian: modification to LQC effective dynamics is due to the Lorentzian term, which we regularize à la Thiemann. Different regularization (i.e., Warsaw Hamiltonian) will lead to different result, even at leading order!
  $\Rightarrow$ Must fix this arbitrariness (e.g., by renormalization group).
- Study quantum evolution generated by $\hat{H}^{new}_{LQC}$.
- Extend to: $k \neq 0$; anisotropic cases.
- Extend beyond cosmology (e.g., spherical symmetry).

THANK YOU!

Why $\sin(2c\mu)^2$?

The extrinsic curvature

$$\hat{K}(e) = \hat{h}(e)[\hat{h}(e)^{-1}, [\hat{H}_E(1), \hat{V}_v]] \longrightarrow e^{c\mu\tau}\{e^{-c\mu\tau}, \{H_E(1), p^{3/2}\}\}$$

Since $H_E(1) \sim \sqrt{p}\sin(c\mu)^2/\mu^2$, one finds

$$K(e) = e^{c\mu\tau}\{e^{-c\mu\tau}, \{H_E(1), p^{3/2}\}\} \sim \frac{3}{2\mu^2} e^{c\mu\tau}\{e^{-c\mu\tau}, p\}\{\sin(c\mu)^2, p\}$$

$$\sim -\frac{3}{2} 2\sin(c\mu)\cos(c\mu)\tau$$

$$\sim \sin(2c\mu)\tau$$

So

$$H_L = \sum_v \sum_{e\cap e'\cap e''} \epsilon(e, e', e'')\, Tr\left(K(e)K(e')h(e'')\{h(e'')^{-1}, V_v\}\right)$$

$$\sim \mathcal{N}^3 \mu\sqrt{p}\sin(2c\mu)^2 \sum_{e\cap e'\cap e''} \epsilon(e, e', e'')\, Tr(\tau_e\tau_{e'}\tau_{e''})$$

$$\sim \sqrt{p}\frac{\sin(2c\mu)^2}{\mu^2}$$