# Fusion basis for LQG

## Application to coarse-graining and entanglement entropy

Clement Delcamp

Perimeter Institute

October 11, 2016

In collaboration w/ B. Dittrich and A. Riello

arXiv:1606.02384 [CD, Dittrich]
arXiv:1607.08881 [CD, Dittrich, Riello]
arXiv:1609.04806 [CD, Dittrich, Riello]

# Motivations and results

## Motivations

- Construction of a basis stable under coarse-graining
- Definition of a region in a relational way

## Fusion basis

- Shift the focus from the original lattice to the excitations themselves
- Excitations are classified by the irreducible representations of the Drinfel'd double $\mathcal{D}(\mathcal{G})$ of the gauge group $\mathcal{G}$ (finite group)
- Mathematical structure of an extented TQFT:

$$\text{TQFT} + \text{defects} = \underbrace{\text{vacuum}}_{\text{BF}} + \underbrace{\text{excitations}}_{\text{curvature + torsion}}$$

## Results

- Natural notion of local subsystems
- Solve the problem of emerging torsion d.o.f during the coarse-graining
- Identification of a new complete set of mutually independent gauge invariant d.o.f and observables
- Physical states for LQG coupled to point particles

# Outlook

1) BF representation
   - Triangulation based
   - Excitations based
2) Fusion basis
3) Ribbon operators
4) Applications
   - Coarse-graining
   - Entanglement entropy
5) Discussion

# Triangulation based BF representation

[Dittrich, Geiller 15-16]

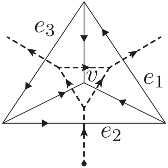
- **BF** representation for $2+1$ LQG
  - Based on an inductive limit of Hilbert spaces
  - 2D hypersurface $\Sigma \to$ triangulation $\Delta$
    $\to$ configuration space $= \mathcal{M}^{\text{flat}}(\Sigma\backslash\Delta_0)$
  - $\mathcal{H}_\Delta$ is given by the gauge inv. functions of the holonomies describing $\mathcal{M}^{\text{flat}}(\Sigma\backslash\Delta_0)$
    $\hookrightarrow$ Inductive limit $\Rightarrow$ continuum Hilbert space $\mathcal{H}_\Sigma$

| | AL | BF |
|---|---|---|
| Background TQFT | $E=0$ | $F(A)=0$ |
| Vacuum state | $\vert\emptyset\rangle =$ nothing | $\vert\emptyset\rangle = \prod_{\text{loops}} \delta(g_\ell, \mathbb{1})$ |
| Measure | Haar | Discrete |
| Refinement | $j=0$ | flatness |
| Kin. $C^\infty$ limit | ✓ | ✓ |
| Operators | Holonomies<br>Fluxes | Holonomies<br>Exponentiated fluxes |

# Coarse-graining of spin networks

[Livine 13][Rovelli et al. 15][Livine, Charles 16][Dittrich, Geiller 16]

- Advantage BF representation:
  - Encodes the interplay between curvature and torsion degrees of freedom
  - $X_{e_1 \circ e_2 \circ e_3} \neq 0$ if $g_v \neq \mathbb{1}$

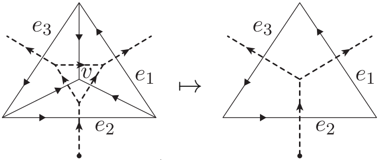



- Coarse-graining of spin networks in terms of **density matrices**:
  - Identification of commuting sets of coarser and finer obsevables
  - Tracing over the finer degrees of freedom

$$\rho^{\mathfrak{c}}\big(\{g^{\mathfrak{c}}\};\{\tilde{g}^{\mathfrak{c}}\}\big) = \int_{\mathcal{G}} \mathbf{d}g^{\mathfrak{f}} \rho^{\mathfrak{f}}\big(\{g^{\mathfrak{f}}\},\{g^{\mathfrak{c}}\};\{g^{\mathfrak{f}}\},\{\tilde{g}^{\mathfrak{c}}\}\big)$$

  - Example:

$\mathcal{P}_{3\to 1}$ :




$\Rightarrow$ **Violation** of the Gauß constraint (curvature induced torsion)

  - **SNW basis not stable** under coarse-graining

$\hookrightarrow$ Issues when defining coarse-graining algorithms [CD, Dittrich to appear]

$\hookrightarrow$ Fusion basis: a basis adapted to the excitations

# Alternative BF representation

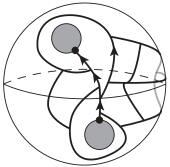
[Konig, Kuperberg, Reichardt 10][Dittrich, Geiller 16][CD, Dittrich, Riello 16]

- Basic idea:
  - Punctures replace the vertices of the triangulation and thus the triangulation
  - Graphs embedded on punctured surfaces

- Punctured surface:
  - Two dimensional hypersurface $\Sigma$ is the 2-sphere $\mathbb{S}$
  - Punctured sphere $\mathbb{S}_p$ is $\mathbb{S}$ with one disc removed for each one of the $p$ punctures
  - Graphs embedded on $\mathbb{S}_p$

    $\hookrightarrow$ open edges = **torsion** d.o.f

    $\hookrightarrow$ non-contractible loops = **curvature** d.o.f



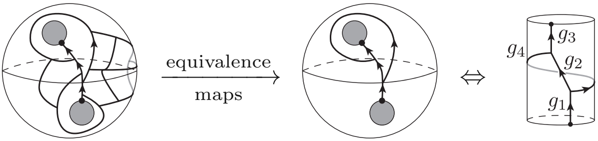
- Hilbert space $\mathcal{H}_p$:
  - Configuration space $\mathcal{M}^{\text{flat}}(\mathbb{S}_p)$ characterized by holonomy along links of $\Gamma$
  - $\mathcal{H}_\Gamma$ =Hilbert space of functions of such holonomies
  - Inductive limit

    $\Rightarrow$ A **minimal graph** on $\mathbb{S}_p$ fully characterizes $\mathcal{M}^{\text{flat}}(\mathbb{S}_p)$

# Graphs on punctured manifolds

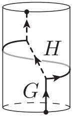
[Konig, Kuperberg, Reichardt 10][Dittrich, Geiller 16][CD, Dittrich, Riello 16]

- $\mathbb{S}_1$ cannot carry any excitations
- 2-punctured sphere $\mathbb{S}_2 \leftrightarrow$ topology of a cylinder




  - A basis for $\mathcal{H}_2$ is given by the gauge fixed states

$$\psi^{\mathbb{S}_2}_{G,H}(g_1, g_4) = |\mathcal{G}|^{3/2}\delta(G, g_1)\delta(H, g_4) \equiv$$




  - BF vacuum on $\mathbb{S}_2$ (given by a superposition of basis states):

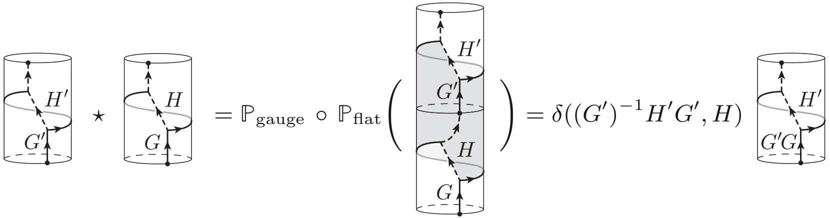
$$\psi^{\mathbb{S}_2}_0 = \delta(\mathbb{1}, g_4 g_2^{-1}) 1(g_1) 1(g_3)$$

# Gluing of manifolds ⇒ Drinfel'd double

[Hu *et al.* 15][Lan, Wen 15][Dittrich, Geiller 15][CD, Dittrich, Riello 16]

- **Gluing** of manifolds along boundaries
  1. Identification of the marked points
  2. Gauge averaging at the new vertices
  3. Imposition of the flatness at the new faces

- Case of the 2-punctured spheres
  - $\mathbb{S}_2 \star \mathbb{S}_2 = \mathbb{S}_2$ *i.e.* topology unchanged ⇒ defines a multiplication rule




$$\boxed{\psi^{\Sigma_2}_{G',H'} \star \psi^{\Sigma_2}_{G,H} = \delta\big((G')^{-1}H'G',H\big)\psi^{\Sigma_2}_{G'G,H'}}$$

  - Reproduces the **Drinfel'd double** multiplication
    ↪ Appeared before as the algebra describing the constraints
  - $\psi^{\Sigma_2}_{G,H} \leftrightarrow$ Drinfel'd double element

# Drinfel'd double 1/2

[Koornwinder *el al.* 98-99][Dijkgraaf *et al.* 90][Buerschaper *el al.* 09,13]

- Basic properties
  - Drinfel'd doubles are examples of quasi-triangular **Hopf algebras**
  - As a vector space, the Drinfel'd double $\mathcal{D}(\mathcal{G})$ is isomorphic to

$$\mathcal{D}(\mathcal{G}) \simeq \mathbb{C}[\mathcal{G}] \otimes \mathcal{F}(\mathcal{G})$$

  - Basis provided by $[G,H] \equiv G \otimes \delta_H \;\longleftrightarrow\; \psi^{\mathbb{S}_2}_{G,H}$

- As a Hopf algebra, $\mathcal{D}(\mathcal{G})$ is equipped with
  - A multiplication $\star\,:\,([G',H'],[G,H]) \longmapsto \delta\big((G')^{-1}H'G',H\big)[G'G,H']$
  - A comultiplication $\Delta\,:\,[G,H] \longmapsto \sum_{\substack{X,Y\in\mathcal{G}\\ XY=H}} [G,X]\otimes[G,Y]$

- Irreducible representations
  - Complete set of irreducible representations labelled by $\rho \equiv C, R$

    $\hookrightarrow$ $C$: conjugacy class

    $\hookrightarrow$ $R$: irreducible representation of the stabilizer $N_C$
  - Corresponds to the 'observables' **stable under gluing**
  - Orthogonality:

$$\frac{1}{|\mathcal{G}|}\sum_{G,H\in\mathcal{G}} D^{\rho_1}_{I'_1 I_1}([G,H])\overline{D^{\rho_2}_{I'_2 I_2}([G,H])} = \frac{\delta_{\rho_1\,\rho_2}}{d_{\rho_1}}\delta_{I'_1,I'_2}\delta_{I_1,I_2}$$

# Drinfel'd double 2/2

[Koornwinder *el al.* 98-99][Dijkgraaf *et al.* 90][Buerschaper *el al.* 09,13]

- Tensor product of representations
  - Comultiplication $\Rightarrow \exists$ a unitary map $\mathcal{C}^{\rho_1\rho_2} : \bigoplus_{\rho_3 \ni \rho_1 \otimes \rho_2} V_{\rho_3} \to V_{\rho_1} \otimes V_{\rho_2}$:

$$D^{\rho_1}_{I'_1 I_1} \otimes D^{\rho_2}_{I'_2 I_2}(\Delta[G,H]) = \sum_{\rho_3} \sum_{I'_3, I_3} \mathcal{C}^{\rho_1\rho_2\rho_3}_{I'_1 I'_2 I'_2} D^{\rho_3}_{I'_3 I_3}([G,H]) \overline{\mathcal{C}^{\rho_1\rho_2\rho_3}_{I_1 I_2 I_3}}$$

  - Orthogonality of the Clebsch-Gordan coefficients:

$$\sum_{I_1, I_2} \mathcal{C}^{\rho_1\rho_2\rho}_{I_1 I_2 I} \cdot \overline{\mathcal{C}^{\rho_1\rho_2\rho'}_{I_1 I_2 I'}} = \delta_{\rho,\rho'} \delta_{I,I'}$$

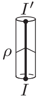
# Fusion basis 1/3

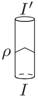
[Konig, Kuperberg, Reichardt 10][CD, Dittrich, Riello 16]

- 2-punctured sphere:
  - Generalized Fourier transform

$$\psi_{\mathfrak{f}}^{\mathbb{S}_2}[\rho, I'I] = \frac{1}{|\mathcal{G}|} \sum_{G,H \in \mathcal{G}} D^{\rho}_{I'I}([G,H]) \sqrt{d_\rho} \psi^{\mathbb{S}_2}_{G,H}$$

  - $\left\{\psi_{\mathfrak{f}}^{\mathbb{S}_2}[\rho, I'I]\right\}$ forms an orthonornal and complete set of states for $\Sigma_2$
    $\Rightarrow$ Fusion basis states for $\Sigma_2$
  - Graphical representation:

$\psi_{\mathfrak{f}}^{\mathbb{S}_2}[\rho, I'I] =$ 

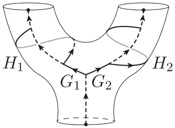
and $\delta_{I'I} =$ 


  - $\psi_{\mathfrak{f}}^{\mathbb{S}_2}$ are states of elementrary excitations $\leftrightarrow$ quasiparticles
  - With SU(2):
    $\hookrightarrow$ Conjugacy class $C$: mass
    $\hookrightarrow$ Representation $R$: spin
    [Noui 06][Noui, Perez 09]

# Fusion basis 2/3

[Konig, Kuperberg, Reichardt 10][CD, Dittrich, Riello 16]

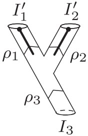
- 3-punctured sphere:
  - Start from the holonomy basis states

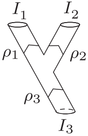
$$\psi^{\mathbb{S}_3}_{\{G,H\}} =$$ 


  - Transformation to the $[\rho, I'I]$-picture for each pair $(G,H) \in \mathcal{G} \times \mathcal{G}$
  - Fusion by introducing Clebsch-Gordan coefficients

$$\psi^{\mathbb{S}_3}_{\mathfrak{f}}[\{\rho, I'I\}] = \mathcal{C}^{\rho_1\rho_2\rho_3}_{I_1 I_2 I'_3}\, \psi^{\mathbb{S}_2}_{\mathfrak{f}}[\rho_1, I'_1 I_1] \otimes \psi^{\mathbb{S}_2}_{\mathfrak{f}}[\rho_2, I'_2 I_2]$$

  - Graphically:

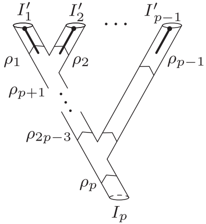
$$\psi^{\mathbb{S}_3}_{\mathfrak{f}}[\{\rho, I'I\}] =$$ 

and $\mathcal{C}^{\rho_1\rho_2\rho_3}_{I_1 I_2 I'_3} =$ 


  - The orthonormality follows from orthonormality of states $\psi^{\mathbb{S}_2}_{\mathfrak{f}}$ and orthogonality of the Clebsch-Gordan coefficients

# Fusion basis 3/3

[Konig, Kuperberg, Reichardt 10][CD, Dittrich, Riello 16]

- General case: $p$-punctured sphere:
  - $$\psi_{\mathfrak{f}}^{\mathbb{S}_p}[\{\rho, I'I\}] =$$ 

  - Graph independent basis
  - Can be obtained via a pants decomposition
  - $\neq$ choice of fusion tree $\Rightarrow$ $\neq$ fusion basis states

# Open ribbon operators

[Kitaev 06][Hu *et al.* 15][Lan, Wen 15][Dittrich, Geiller 16][CD, Dittrich 16]

- Basic operators
  - Configuration space: $\mathcal{M}^{\text{flat}}(\mathbb{S}_p)$
  - Wilson loop operators

$$(W^f_\gamma \psi)(\{g\}) = f(h_\gamma)\psi(\{g\}) \quad \text{w/} \quad \gamma = l_n \circ \cdots \circ l_1$$

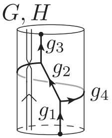
  - Translation operators

$$(T_{k,\gamma}[H]\psi)(g_1, \cdots, g_L) = \psi(g_1, \cdots, h_\gamma^{-1} H^{-1} h_\gamma g_k, \cdots, g_L)$$

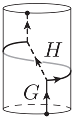
$\Rightarrow T_{k,\gamma}[H]$ induces a violation of the Gauß constraint parallel transported at the desired node

- Ribbon operators
  - On $\mathbb{S}_2$, we define the ribbon operators as $\mathcal{R}[G,H] = W_{321}[G] \circ T_{4,3}[H]$
  - Acting on the cylinder (BF) vacuum state

$$(\mathcal{R}[G,H]\psi_0^{\Sigma_2}) =$$




$$=$$




$$= \psi^{\Sigma_2}_{G,H}$$

  - Ribbon operators only **generate excitations** at their ends

# Closed ribbon operators

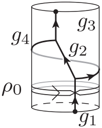
- Construction
  - Expected to **measure the excitations** (not to induce them)
  - Close an open ribbon and prevent excitations $\Rightarrow \mathcal{K}[C, D]$

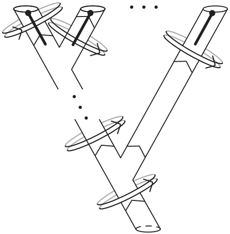
    $\hookrightarrow C$: conjugacy class of $\mathcal{G}$

    $\hookrightarrow D$: conjugacy class of $N_C$
  - Fourier transform $\Rightarrow \mathcal{K}[C, R]$

- Properties
  - Project onto the basis states

$$\mathcal{K}[\rho_0]\psi_{\mathfrak{f}}^{\mathbb{S}_2}[\rho, I'I] = \delta_{\rho_0\rho}\psi_{\mathfrak{f}}^{\mathbb{S}_2}[\rho, I'I] =$$



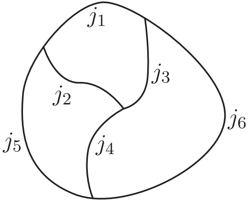
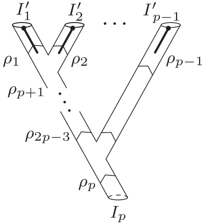



  - Defining property of projectors

$$\mathcal{K}[C, R]\mathcal{K}[C', R'] = \delta_{CC'}\delta_{RR'}\mathcal{K}[C, R]$$

  - Hierarchically ordered set of closed ribbon operators
  - Provides a complete basis of **Dirac observables**

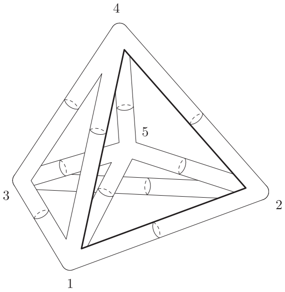
# SNW basis vs Fusion basis

| | SNW Basis | Fusion basis |
|---|---|---|
| Vacuum | AL | BF |
| Diagonalized operators | Casimir operators $\sum_i X^i_\ell X^i_\ell$ | Closed ribbon operators $\mathcal{K}[C,R]$ |
| Graphical representation |   |   |
| Graph independence | ✗ | ✓ |
| Inbuilt coarse-graining scheme (multiscale states) | ✗ | ✓ |

# Generalization to 4D

[CD, Dittrich 16]

- Strategy
  - Perform a Heegaard splitting of the 3D compact manifold $\mathcal{M}$
    $\hookrightarrow \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ and $\partial\mathcal{M}_1 = \partial\mathcal{M}_1 = \Sigma$ with $\Sigma$ the Heegaard surface
  - Can be obtained by blowing-up the one-skeleton $\Delta_1$ of a triangulation $\Delta$
  - Map the space of flat connections on $\mathcal{M}\backslash\Delta_1$ to the space of flat connections on $\Sigma(\Delta)$

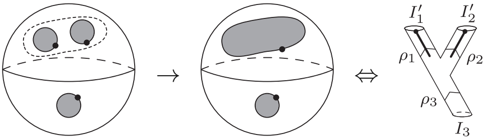



- Results
  - The 2D theory equipped with additional constraints describes the theory of flat connections on the 3D manifold
  - Ribbon operators preserving the 2D flatness constraints $\Rightarrow$ Operators generating curvarture defects for the 3D theory
  - Defects located on the edges of the triangulation

# Coarse-graining w/ the fusion basis

[CD, Dittrich, Riello 16]

- Coarse-graining of fusion basis states in terms of **density matrices**:
  - Merging of defects/particles
  - Fusion of the corresponding irreducible representations

- Example
  - 

  - Definition of the fusion basis state on $\Sigma_3$
  - Coarse-graining obtained by tracing over the irreducible labels associated to the particles which are fused

- Advantages
  - **Stability** of the states under coarse-graining
  - Control the behaviour of the states at all 'scales' at once (MERA style [Vidal 10])
  - Immediate access to coarse-grained observables

# Entanglement entropy 1/2

[Casini 09][Donnelly 08,12,14][Soni, Trivedi 16][CD, Dittrich, Riello 16]

- Requirements
  - For a subregion $A$ of $\Sigma$, the computation of the entanglement entropy $S(A)$ requires:

    $\hookrightarrow$ To associate the theory's d.o.f to $A$

    $\hookrightarrow$ A factorization of the Hilbert space $\mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$
  - Constraints make the d.o.f non local and leads to ambiguities
  - More generally: How to define subsystems in quantum gravity?

    [Giddings 15][Giddings, Donnelly 16][Freidel, Donnelly 16]

- Extended Hilbert space procedure
  - Relax the constraints at the boundary: $\mathcal{H} \to \mathcal{H}_{\text{ext}} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$
  - Embed the states into the extended Hilbert space via $\mathcal{E} : \mathcal{H} \to \mathcal{H}_{\text{ext}}$
  - Compute the entanglement entropy of $\mathcal{E}(\psi)$

    $\hookrightarrow$ Can be employed with both SNW basis and fusion basis

# Entanglement entropy 2/2

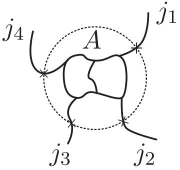
[Casini 09][Donnelly 08,12,14][Soni, Trivedi 16][CD, Dittrich, Riello 16]

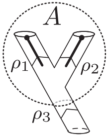
- Subsystems
  - Fusion basis construction clarifies differences between the ≠ definitions of subsystems

    ↪ Characterized by a choice of **boundary conditions** which depend on the choice of vacuum
  - Extended Hilbert space approach relies on extra frames information at the boundary
- SNW basis
  - Local frame for each link cut
  - Region is a subgraph

    → $S(A)$ depends on the choice of graph
- Fusion basis
  - Only one global frame
  - **Region is defined w.r.t the excitation content**

    → unique $S(A)$ associated to $A$




⇒ Results in agreement with replica trick calculations with Chern-Simons theory [Wen, Matsuura, Ryu 16 ]

# Conclusion

- **Summary:**
  - New basis for Lattice gauge theories and 2+1 LQG
    ↪ Shift the focus from the original lattice to the excitations (comes from using a different vacuum)
    ↪ Ribbon operators generate excitations
    ↪ Closed ribbon operators provide basis for Dirac observables
  - Results
    ↪ Non-local basis (multi-scale)
    ↪ Basis ideally suited for coarse graining and designing useful geometrical states
    ↪ Operational definition for entanglement entropy in (2+1) gravity
    ↪ Results in agreement with replica calculations for Chern-Simons theory
- **Future directions**:
  - Generalization to Lie groups
  - Complete the generalization to 3+1 case
  - Homogeneous curvature phase