Black Hole Entropy and Number Theory

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Jacobo Díaz Polo
Universidad de Valencia

Work in collaboration with I. Agulló, J.F. Barbero G.,
E.F. Borja and E.J.S. Villaseñor.
PLAN OF THE TALK

• Introduction (extra-brief review to BH entropy in LQG)
• Motivation
• Combinatorial problem
• How can Number Theory help?
• Improved computational analysis
• Generating Functions
• Asymptotics revisited
• New windows
  – Asymptotic expansion
  – CFT-related computation
  – Hawking radiation
INTRODUCTION

- Black hole entropy arises, in LQG, from the (isolated) horizon degrees of freedom
- Horizon described by a Chern-Simons theory on a sphere with N distinguishable punctures
- CS states labeled by quantized angle deficits $a_i$
- N spin network edges, with labels $j_i$ and $m_i$, pierce the horizon
  - $j_i$ labels the contribution to area
  - $m_i \in \{-j_i, j_i+1, \ldots, j_i\}$ are the spin projections and match with $a_i$ through IH boundary conditions
INTRODUCTION


• For a given value of area $A$, entropy is given by

$$S(A) = k_B \ln N(A)$$  \hspace{0.5cm} (1)

– $N(A) =$ number of independent CS states

• Horizon states must satisfy:

$$\sum_{i=1}^{N} a_i = 0 \hspace{0.5cm} (2)$$

$$2m_i = -a_i \mod k \hspace{0.5cm} (3)$$

$$m_i \in \{-j_i, -j_i + 1, \ldots, j_i\} \hspace{0.5cm} (4)$$

$$8\pi \gamma \ell_p^2 \sum_{i=1}^{N} \sqrt{j_i(j_i + 1)} = A \hspace{0.5cm} (5)$$

• Goal: count all possible ordered lists of $a_i$ satisfying the above constraints for a given $A$. 
MOTIVATION

• Approximate solution of the problem for the asymptotic limit:

\[
S(A) = \frac{\gamma_0}{\gamma} \frac{A}{4\ell_P^2} - \frac{1}{2} \ln \frac{A}{\ell_P^2}
\]  

  – Linear behavior of entropy with area
  – Choosing the appropriate value of \( \gamma \), obtain the Bekenstein-Hawking law
  – Additional logarithmic correction

• Exact computational solution for microscopic black holes
  – Discretization
  – Band structure

• Questions:
  – Is the discrete behavior a feature only of the low area regime?
  – Can we better understand its origin?
  – Can we extend the exact computations to the large area limit?
COMBINATORIAL PROBLEM
M. Domagala, J. Lewandowski. Class. Quantum Grav. 21 (2004) 5233

- State the combinatorial problem in a precise but simple way
- Express it in terms of only $m_i$ labels

\[ N(A) \text{ is 1 plus the number of all the finite, arbitrarily long, sequences } (m_i, \ldots, m_N) \text{ non-zero half integers, such that} \]

\[ \sum_{i=1}^{N} m_i = 0 \quad (7) \]

\[ \sum_{i=1}^{N} \sqrt{|m_i|(|m_i| + 1)} \leq \frac{A}{8\pi \gamma \ell_P^2} \quad (8) \]

- This is equivalent to compute all CS states satisfying equations (2), (3), (4) and (5)
In order to further simplify the notation, let us work with integer positive numbers, by defining $k_i = 2|m_i|$

Define also the occupancy numbers $n_k$ as the number of punctures in a given state with labels $m$ such that $2|m| = k$.

Use units such that $4\pi\gamma\ell_p^2 = 1$.

A set of $n_k$ numbers characterizes a horizon state up to reorderings and sign assignments.

We can solve the combinatorial problem in 4 steps:

- Compute all $n_k$ sets that satisfy $\sum_{k=1}^{k_{\text{max}}} n_k \sqrt{k(k+2)} = A$
- Compute, for each of them, all possible ordered lists of $k$
- Compute, for each of them, all possible ordered sign assignments
- Sum the resulting degeneracy for all values of area lower than $A$
STEP 1: CHARACTERIZING THE AREA SPECTRUM

- Solve the equation
  \[ \sum_{k=1}^{k_{\text{max}}} n_k \sqrt{k(k+2)} = A \]  (9)

- Any value \( \sqrt{k(k+2)} \) can be written as a linear combination of square roots of square-free (SQSF) numbers \( p_i \):
  \[ \sum q_i \sqrt{p_i} \]

- Equation (9) can be rewritten:
  \[ \sum_{k=1}^{k_{\text{max}}} n_k \sqrt{(k+1)^2 - 1} = \sum_{i=1}^{r} q_i \sqrt{p_i} \]  (10)

- For a given value of \( p \), which values of \( k \) can contribute?
  - Solve:
    \[ \sqrt{(k+1)^2 - 1} = y \sqrt{p_i} \]  (11)
  - Pell equation \( x^2 - p_i y^2 = 1 \) known in number theory, for a given \( p_i \) infinite known solutions \((x^i_m, y^i_m)\):
    \[ x^i_m + y^i_m \sqrt{p_i} = (x^i_1 + y^i_1 \sqrt{p_i})^m \]

- Then, equation (9) can be rewritten:
  \[ \sum_{i=1}^{r} \sum_{m=1}^{\infty} n_{k_i m} y^i_m \sqrt{p_i} = \sum_{i=1}^{r} q_i \sqrt{p_i} \]  (12)
NUMBER THEORY SOLUTION

Taking into account the linear independence of SQSF numbers with integer coefficients, we get:

\[ \sum_{m=1}^{\infty} y_m^i n_{k_m}^i = q_i, \quad i = 1, 2, \ldots, r \]  

This set of uncoupled diophantine equations can be solved by standard methods.

In practice, all sums are finite.

For a given value of area \((q_1, \ldots, q_r)\):

- Either we get no solutions \(\rightarrow\) area does not belong to the spectrum
- Or we get solutions of the kind \(\{(k_m^i, n_{k_m}^i)\}\) \(\rightarrow\) all \(k\) that contribute to this area and the corresponding occupancy numbers \(\rightarrow\) all \(n_k\) sets we were looking for
NUMBER THEORY SOLUTION


STEP 2: REORDERING DEGENERACY

• Given a \( \{n_k\} \) set, the number of different ordered \( k \) sequences can be obtained from basic combinatorics:

\[
R(\{n_k\}) = \frac{\left(\sum_{k=1}^{k_{\text{max}}} n_k\right)!}{\prod_{k=1}^{k_{\text{max}}} n_k!}
\]  

(14)
STEP 3: SIGN ASSIGNMENTS

• Given an ordered sequence of $k_i$, look for all choices of signs for the corresponding $m_i$, such that the projection constraint (7) is satisfied

• This problem is analogous to the “partition problem”:
  – Given a set of $N$ natural numbers, which are the possible ways of splitting it into two disjoint subsets, such that the elements of both subsets sum up to the same value?
  – To find all solutions in general is a very hard problem in number theory
  – However, we just need the number of solutions. The answer to this question is known in general, and is given by:

$$P(\{n_k\}) = P(\{k_i\}) = \frac{2^N}{M} \sum_{s=0}^{M-1} \prod_{i=1}^{N} \cos \left( \frac{2\pi s k_i}{M} \right)$$  \hspace{1cm} (15)$$

$$M = 1 + \sum_{i=1}^{N} k_i$$
With this three steps, we obtain the “degeneracy spectrum” of the black hole:

STEP 4: INTRODUCING THE INEQUALITY

• Summing for all the lower values of area we get the (exponentiated) stair behavior:
IMPROVED COMPUTATIONAL ANALYSIS

- This 4-step procedure can be implemented in a computer
- Improve the existing results by allowing to:
  - Reach higher values of area (seen in the previous slide)
  - Weight the contribution of the different “spin” values k
  - Make a separate study of the different sources of degeneracy
  - Test the previous models that account for the origin of bands
- Contribution of different k values
  - Just by choosing the square free numbers p_i appearing in the values of area we control the values of k that can contribute
  - A detailed analysis shows that only the lower values of k are responsible for the highest degenerate states (the peaks)
IMPROVED COMPUTATIONAL ANALYSIS
I. Agulló, J.F. Barbero G., E.F. Borja, J. Díaz-Polo, E.J.S. Villaseñor. *In progress*

- Plot separately the two degeneracies $R$ and $P$

  - The band structure comes entirely from the reordering degeneracy
  - The sign configurations solving the projection constraint give an overall exponential contribution.
• This characterization allowed to compute with good accuracy the area spacing between peaks
• We can test this analysis by plotting all states with a given value of that parameter $K$

![Graph showing data points]

• We observe that this $K$ is a good parameter to characterize a given band
• This supports the reliability of the results in that paper
GENERATING FUNCTIONS

• It is possible to push one more step forward in this analytical solution of the combinatorial problem

• A generating function contains all the information about the solution to a combinatorial problem

• It allows to obtain closed analytical expressions for the solution, that are the starting point for an asymptotic analysis

• Can we find a generating function for our combinatorial problem?

YES!

• We will use functions of the variables $x_i$, such that the coefficient of the $(x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r})$ term in the Taylor expansion is the solution for the value of area $q_1 \sqrt{p_1} + q_2 \sqrt{p_2} + \cdots + q_r \sqrt{p_r}$
Generating functions for the number of (positive) solutions for diophantine equations like (13) are given in text books

\[ G^{\#sol}(x_1, x_2, \ldots) = \prod_{i=1}^{\infty} \prod_{m=1}^{\infty} \frac{1}{1 - x_i^m} \]  

(16)

If we want the solution to include the reordering degeneracy, we get

\[ G^R(x_1, x_2, \ldots) = \left(1 - \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} x_i^m\right)^{-1} \]  

(17)

We can also give a generating function for the solutions of the projection constraint (given a list of \(k_i\))

\[ G^\pm(z) = \prod_{i=1}^{N} (z^{k_i} + z^{-k_i}) \]  

(18)

- This time one has to compute the Laurent expansion
GENERATING FUNCTIONS


• Putting all together, we get the generating function for the whole combinatorial problem

\[ G^d(z, x_1, x_2, \ldots) = \left( \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} (z^{k^i_m} + z^{-k^i_m}) x_i^{y^i_m} \right)^{-1} \]  \hspace{1cm} (19)

where the coefficient of the term \( z^n x_1^{q_1} \cdots x_r^{q_r} \) in the expansion gives the number of solutions such that

\[ \sum_i m_i = n \]

• Closed integral expressions can be obtained for the general form of the coefficients in the expansions, using Cauchy’s theorem

\[ N(A) = N(q_1, \ldots, q_r) = \frac{1}{(2\pi i)^r} \oint_{\gamma_1} \frac{d\zeta_1}{\zeta_1^{q_1+1}} \cdots \oint_{\gamma_r} \frac{d\zeta_r}{\zeta_r^{q_r+1}} G^d(1; \zeta_1, \cdots, \zeta_r, 0, \ldots) \]  \hspace{1cm} (20)
We come back to the question: is the discrete behavior also present in the large area regime?

The new available tools allow to explore this question.

Barbero and Villaseñor found an alternative way to find Meissner’s result, with some subtleties:

- There is a subtle point missing in the asymptotic analysis.
- The real pole is an accumulation point for the real parts of the poles of the Laplace transform:

\[
P(s) = \frac{2 \sum_{k=1}^{\infty} e^{-s \sqrt{k+2}/2}}{s(1 - 2 \sum_{k=1}^{\infty} e^{-s \sqrt{k+2}/2})}
\]

- While still being true that the real pole drives the asymptotic behavior, one cannot exclude a contribution from the infinite many infinitely close (in real part) poles.

There is room for the discrete behavior to appear in the large area limit!
NEW WINDOWS

The powerful techniques presented here allow to address some problems from a new perspective:

- The asymptotic analysis that can determine the presence or not of the discrete behavior of entropy in macroscopic black holes (Agullo, Barbero, Borja, Diaz-Polo, Villaseñor, in progress)

- A new way of computing black hole entropy using CFT-related techniques (Agullo, Borja, Diaz-Polo, in progress)