Self-dual quantum geometries and four-dimensional TQFTs with defects

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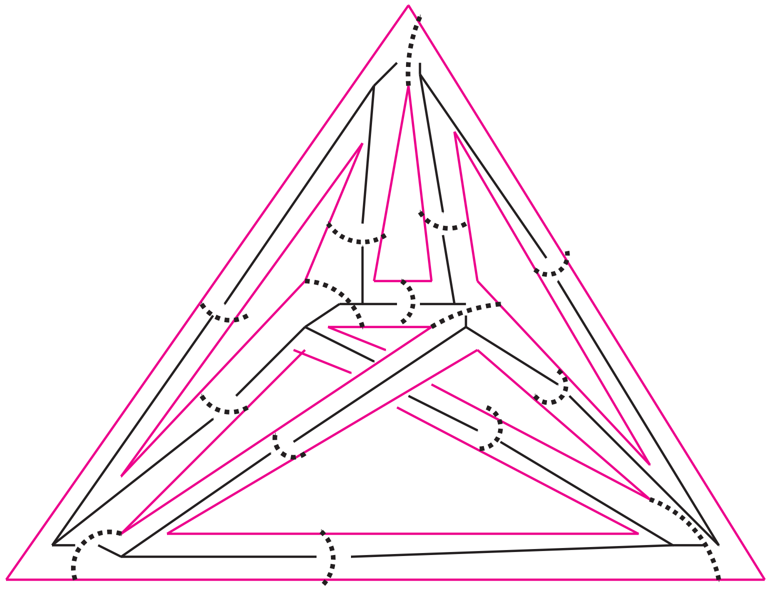


[BD arxiv: 1701.02037 [hep-th]]

ILQGS, April 18 2017

Part I: Motivation and Main Results

Recent developments

[BD, Steinhaus 2013: From TQFT to quantum geometry]

[BD, Geiller 2016]

We constructed a (2+1)D quantum geometry based on Turaev-Viro TQFT:

Vacuum stated peaked on homogeneously curved geometries.

Curvature excitations described by defects.

How to generalize this construction to (3+1) D?

Key problem: braiding relations are central for the (2+1)D theory.

[Delcamp, BD 2016], relations to [Haggard, Han, Kaminski, Riello 14-15], [Baerenz, Barrett 2016]

canonical quantization

canonical formulation, including defects

We developed a strategy:

Lift (2+1)D TQFT to (3+1)D theory with line defects.

[BD arxiv: 1701.02037 [hep-th]]

Applied this strategy to Turev-Viro TQFT.

Results

- Rigorous implementation of quantum group structure into (3+1)D LQG.
  Strong evidence that this facilitates implementation of positive cosmological constant.

[Smolin, Major, Noui, Perez, Pranzetti, Dupuis, Girelli, Bonzom, Livine, Haggard, Han, Kaminski, Riello, Rovelli, Vidotto, …]

quantum group structure

$\mathrm{SU}(2)_{\mathrm{k}}$ where $$\mathrm{k} = \frac{6\pi}{\ell_p^2 \, \Lambda}$$

[Smolin, Major]

- A new family of (3+1)D quantum geometry realizations
  based on vacuum peaked on homogeneously curved geometry: Crane-Yetter TQFT.

- Finiteness properties:
    - Hilbert spaces (associated to fixed triangulations/ graphs) are finite dimensional.
    - Important for (numerical) coarse graining efforts.
    - All (graph preserving) geometric operators have discrete and bounded spectra.

Results

This quantum geometry features a very interesting self-duality. (Born reciprocity.)

- Spectra of curvature operator and (exponentiated) area operators coincide.

  Both operators are implemented as Wilson loop operators. [also Haggard, Han, Kaminski, Riello 2014-2015]

  Diagonalized by the spin network bases and curvature bases respectively.

Spectrum for normalized Wilson loop operator

$$\frac{\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1)(2k+1)\right)\sin\left(\frac{\pi}{\mathrm{k}+2}\right)}{\sin\left(\frac{\pi}{\mathrm{k}+2}(2k+1))\right)\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1))\right)} \quad \stackrel{\mathrm{k}\to\infty}{\longrightarrow} \quad 1-\frac{8}{3}\,j(j+1)\,k(k+1)\,\left(\frac{\pi}{\mathrm{k}+2}\right)^2$$

- Two bases dual to each other:
  - $\mathrm{SU}(2)_\mathrm{k}$ spin network basis based on dual graph
  - curvature bases based on one-skeleton of triangulation - also labelled by $\mathrm{SU}(2)_\mathrm{k}$ spins

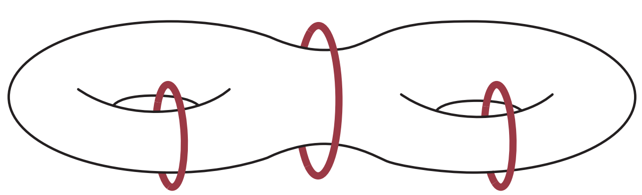
(Many more bases, including bases adjusted to coarse graining schemes.)

[Delcamp, BD, Riello JHP 2016, Delcamp, BD to appear]

Strategy: from (2+1)D TQFT to a (3+1)D theory with line defects

[Delcamp, BD: JMP 2017]

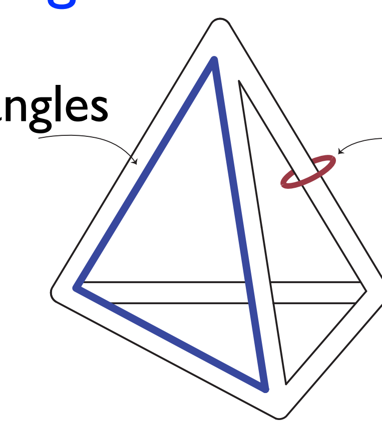
(2+1)D TQFT



assigns degrees of freedom
to non-contractible curves
on a surface

(3+1)D TQFT: 3-sphere with
one-skeleton of (tetrahedral)
triangulation removed

curves around triangles
are contractible in
3-sphere



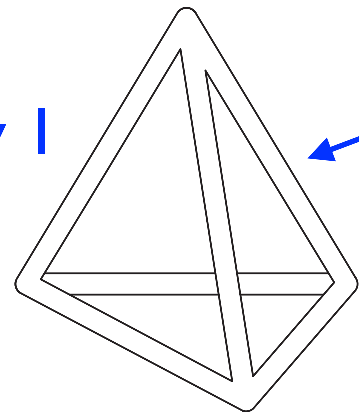
curves around the
edges of the
triangulation are
not contractible

want to assign degrees of
freedom
to curves around edges of
triangulation

Use (2+1) D theory to assign state space to a 3D triangulation.
But impose (contractibility/ flatness) constraints associated to curves
around triangles.

Heegaard splitting and diagrams

Handlebody 1

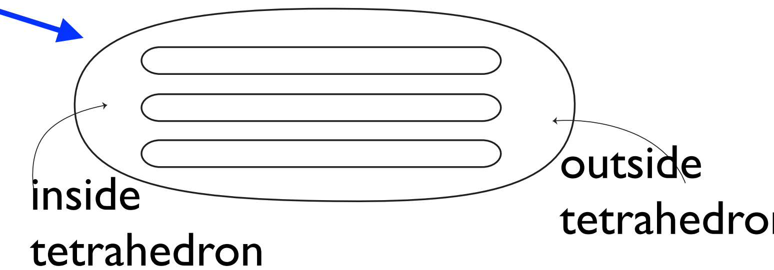


Compact 3D manifold
(e.g. 3-sphere)

$\equiv$

Handlebody 1 $\cup$ Handlebody 2

Handlebody 2

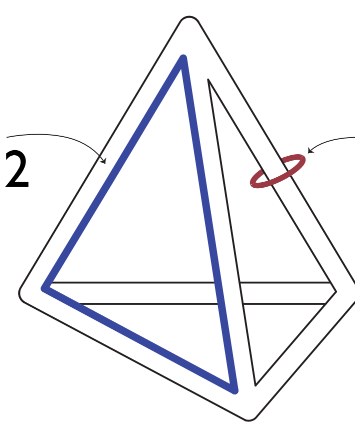


Handlebody 1 $\cap$ Handlebody 2

$\equiv$

Heegaard surface

two-handle cycle:
contractible in handlebody 2



one-handle cycle:
contractible in handlebody 1

A Heegaard diagram is a Heegaard surface decorated with generating basis of one-handle cycles and two-handle cycles.

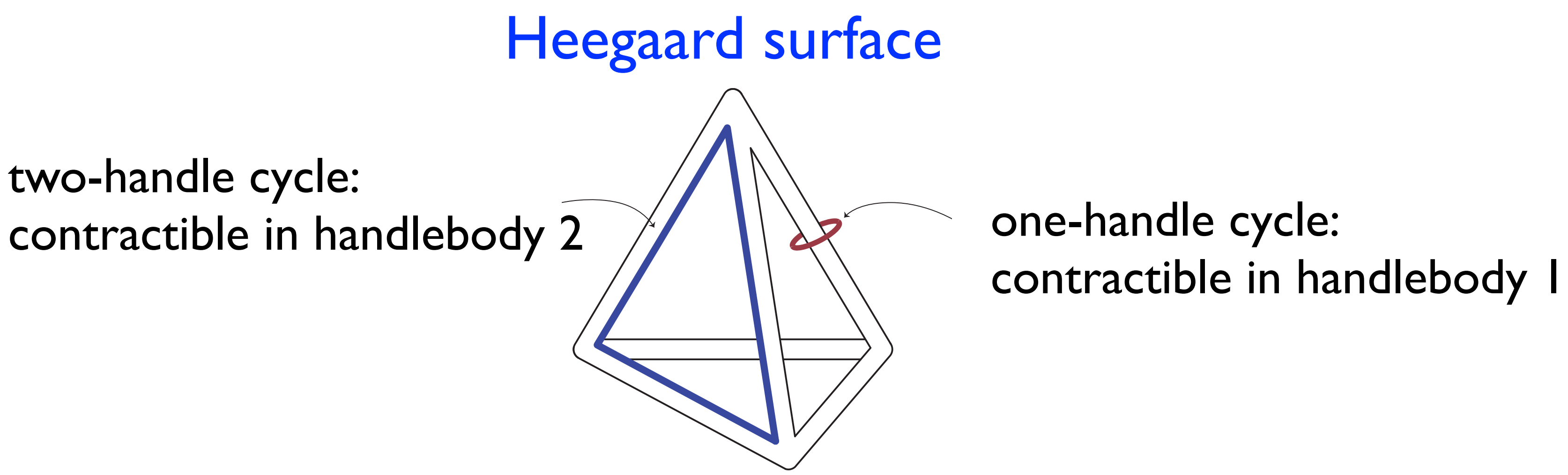
Heegaard diagrams encode uniquely topology of 3D manifold.

Heegaard diagrams

Heegaard diagrams can be constructed from a triangulation of the 3D manifold.

Set of cycles around triangles generates (over-completely) all curves that are contractible even if we do take out the one-skeleton of the triangulation.

Thus it is sufficient to impose flatness constraints for the cycles around the triangles.



Part II: Explicit construction

Strategy

1. Hilbert space, operators and bases for a closed surface.

2. Apply this to a Heegaard surface constructed via a triangulation.

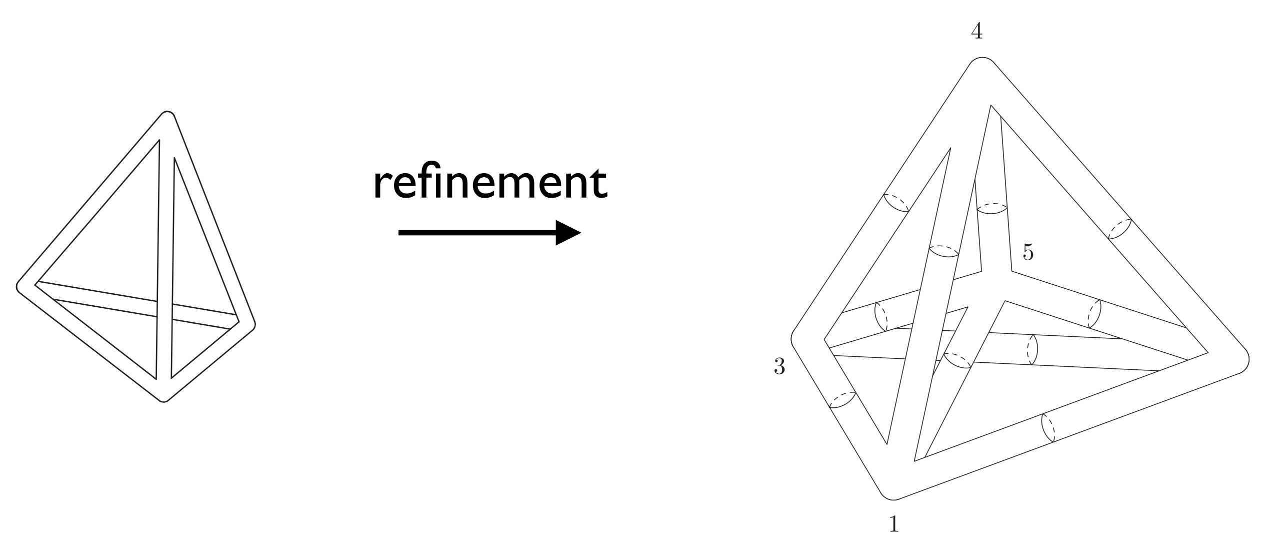
3. Impose constraints for 2-handle cycles and find operators and bases consistent with these constraints.

Remark: fixed triangulation

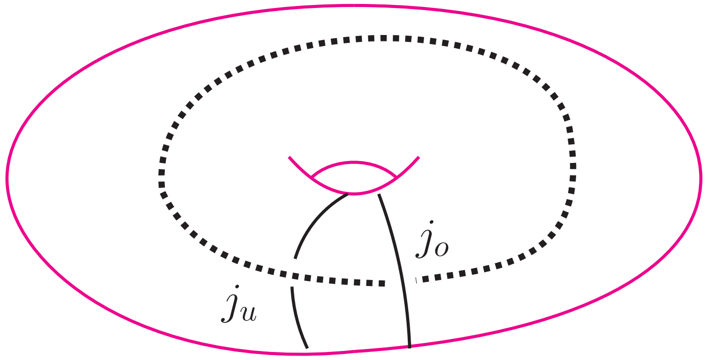
Remark:

This talk is mostly focussed on describing Hilbert space and operators for a fixed triangulation.
Refinements implementing a vacuum based on the Crane-Yetter TQFT can be defined. The operators that we will discuss here are consistent with respect to these refinements.

Open possibility: refinements implementing an Ashtekar-Lewandowski type vacuum and finding operators consistent with these refinements.



Hilbert space for (2+1)D Turaev-Viro TQFT



Hilbert space for (2+1)D Turaev-Viro TQFT

here: for surfaces without punctures

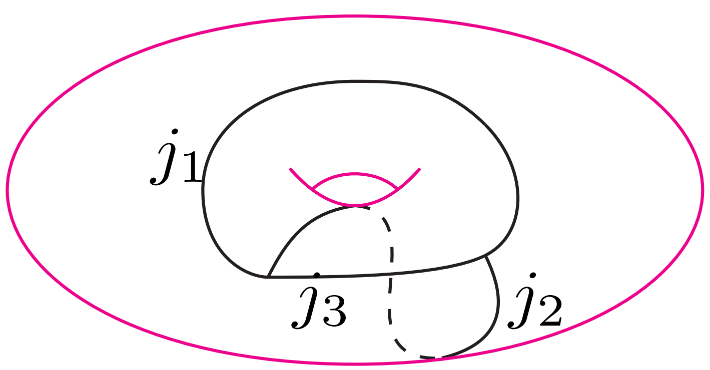
[Levin, Wen; Koenig, Kuperberg, Reichardt; Kirillov; BD, Geiller]

Kinematical (but gauge invariant) Hilbert space:

States based on spin-labelled three-valent graphs with $\mathrm{SU}(2)_{\mathrm{k}}$ coupling rules imposed on the nodes.

Admissible spins: $j = 0, \frac{1}{2}, 1, \ldots, \frac{\mathrm{k}}{2}$ labelling undirected edges of the graph.

Coupling rules: $i \leq j + k, \quad j \leq i + k, \quad k \leq i + j, \quad i + j + k \in \mathbb{N}, \quad i + j + k \leq \mathrm{k}$

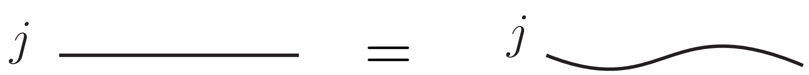


Hilbert space for (2+1)D Turaev-Viro TQFT

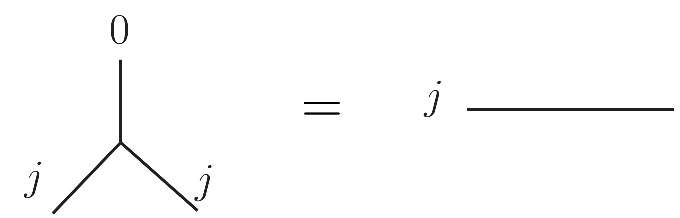
Physical Hilbert space - impose ‘flatness’ constraints:

Flatness constraint are imposed as equivalence relations between graph states:

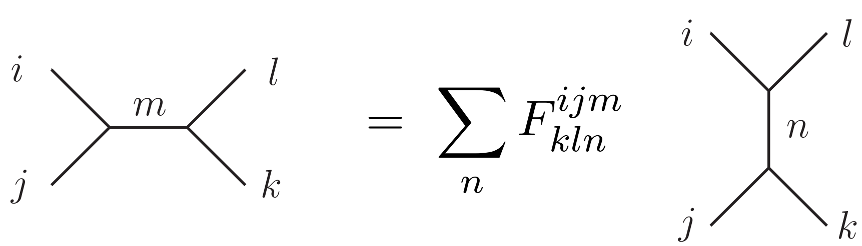
Strands can be (isotopically) deformed.



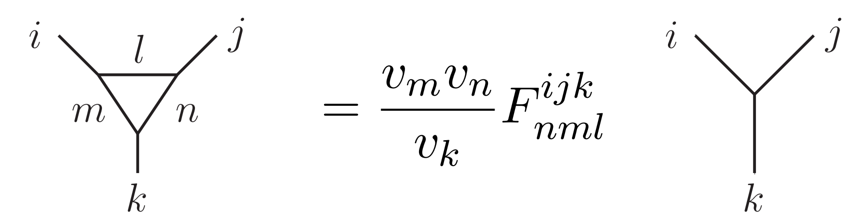
Strands with trivial spin can be omitted.



2-2 Pachner move. Involving the F-symbol.



3-1 Pachner move. Involving the F-symbol.



$$v_j \;=\; (-1)^j \sqrt{d_j}$$

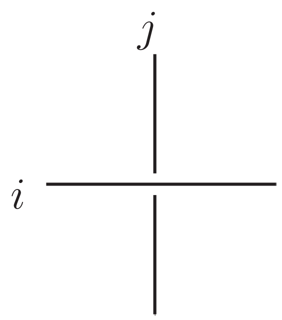
Rather involved now:

Finding a basis of independent states and operators consistent with equivalence relations.

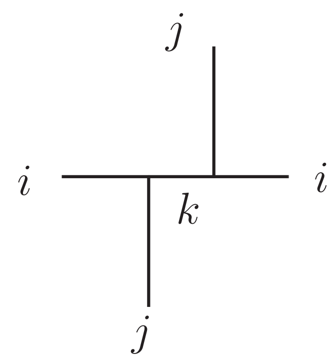
We need a) braiding and b) vacuum strands to define these.

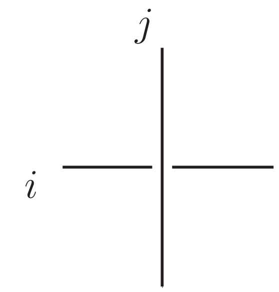
a) Braiding

Strands can cross each other. Such crossings can be resolved using the R-matrix of $\mathrm{SU}(2)_\mathrm{k}$.

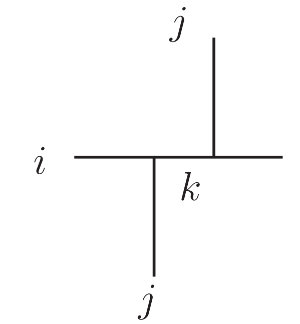


$$= \sum_k \frac{v_k}{v_i v_j} R_k^{ij}$$





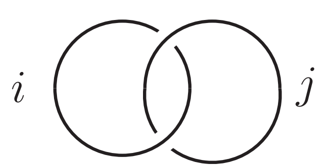
$$= \sum_k \frac{v_k}{v_i v_j} \left(R_k^{ij}\right)^*$$



We can thus define the so-called s-matrix as the evaluation of the Hopf link.

(Planar graphs are equivalent to a number times the empty graph. This number is called the evaluation of the planar graph.)

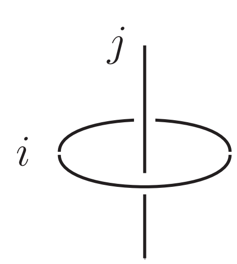
$s_{ij} :=$



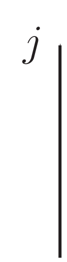
gives

$$s_{jk} = (-1)^{2k+2j} \frac{\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1)(2k+1)\right)}{\sin\left(\frac{\pi}{\mathrm{k}+2}\right)}$$

An important identity:



$$= \frac{s_{ij}}{s_{0j}}$$



b) Vacuum strands

Vacuum strands are defined as weighted sum over strands labelled by admissible spins:

$$:= \frac{1}{\mathcal{D}} \sum_k v_k^2$$

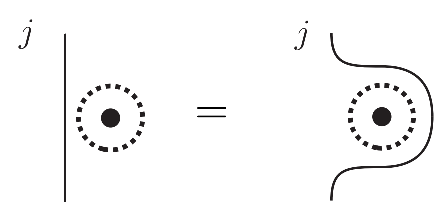
$$v_j = (-1)^j \sqrt{d_j}$$

$$\mathcal{D} := \sqrt{\sum_j v_j^4}$$

total quantum dimension

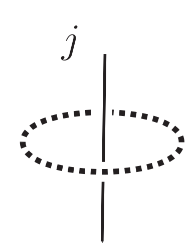
A vacuum loop is similar to a $\delta(g)$ function. Wilson lines (strands) can be deformed across a region enclosed by a vacuum loop.

Sliding property:



Vacuum loops encircling a strand force the associated spin label to be trivial.

Killing property:

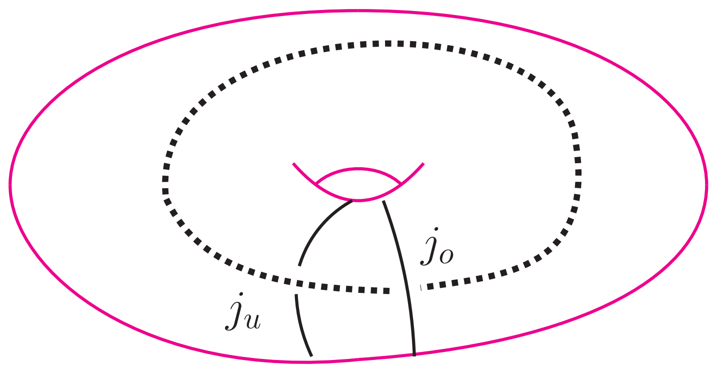


$$= \mathcal{D}\, \delta_{j0}$$

Hilbert space for (2+1)D: Bases

[Kohno 1992; Alagic et al 2010]

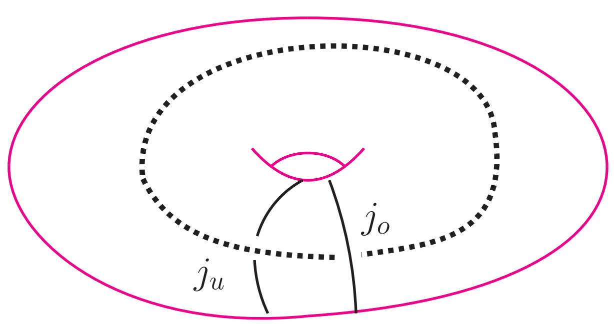
For the torus:



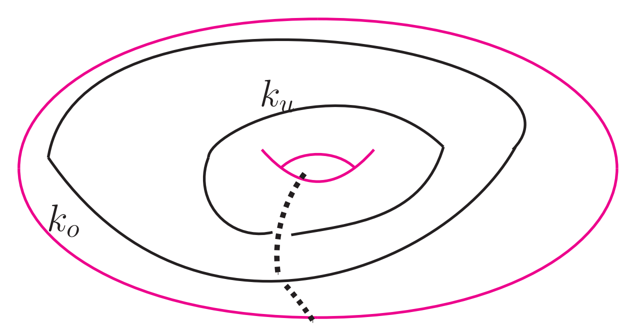
Basis states parametrized by two spins (j_u, j_o) labelling an under- and over-crossing strand.

We will see that this basis diagonalizes over- and under-crossing Wilsonloops parallel to the vacuum loop.

S-transformation (generalized Fourier transformation):



$$= \sum_{k_o, k_u} \mathbb{S}_{j_o j_u, k_o k_u}$$



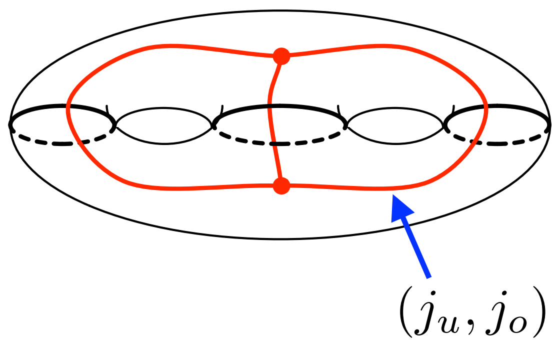
$$\mathbb{S}_{j_o j_u, k_o k_u} = \frac{1}{\mathcal{D}^2} s_{j_o k_o} s_{j_u k_u}$$

Hilbert space for (2+1)D: Bases

[Kohno 1992; Alagic et al 2010]

For g>1 surface:

To each pant decomposition of the surface we can associate a basis.



These bases states include a

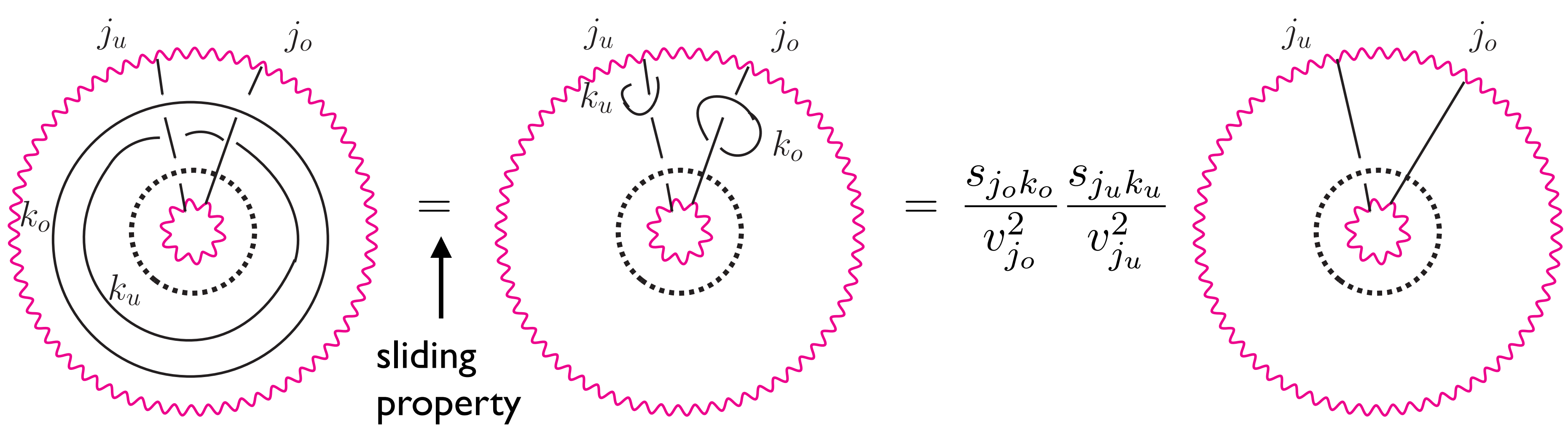
- set of vacuum loops
- over-crossing graph (dual to vacuum loops)
- under-crossing graph (dual to vacuum loops).

Hilbert space for (2+1)D: Operators

Operators consistent with equivalence relation:
Insertion of under- and over-crossing Wilson loops.

Ribbon operators: parallel under- and over-crossing loop, labelled by (j_u, j_o) .
For classical group: ribbon operators combine holonomy and (integrated) flux operators.

Wilson loops parallel to vacuum loops in basis states act diagonally:



Over- and under-crossing graphs and Wilson loops decouple.
Eigenvalues of Wilson loops determined by s-matrix.

From (2+1)D to (3+1)D

We discussed:

- choice of basis for (2+1)D Hilbert space
- consistent operators: under- and over-crossing Wilson loops.

For these constructions braiding relations play a very important role.
Using the encoding of a 3D manifold into a Heegaard surface we can export these braiding relations to the (3+1)D theory.

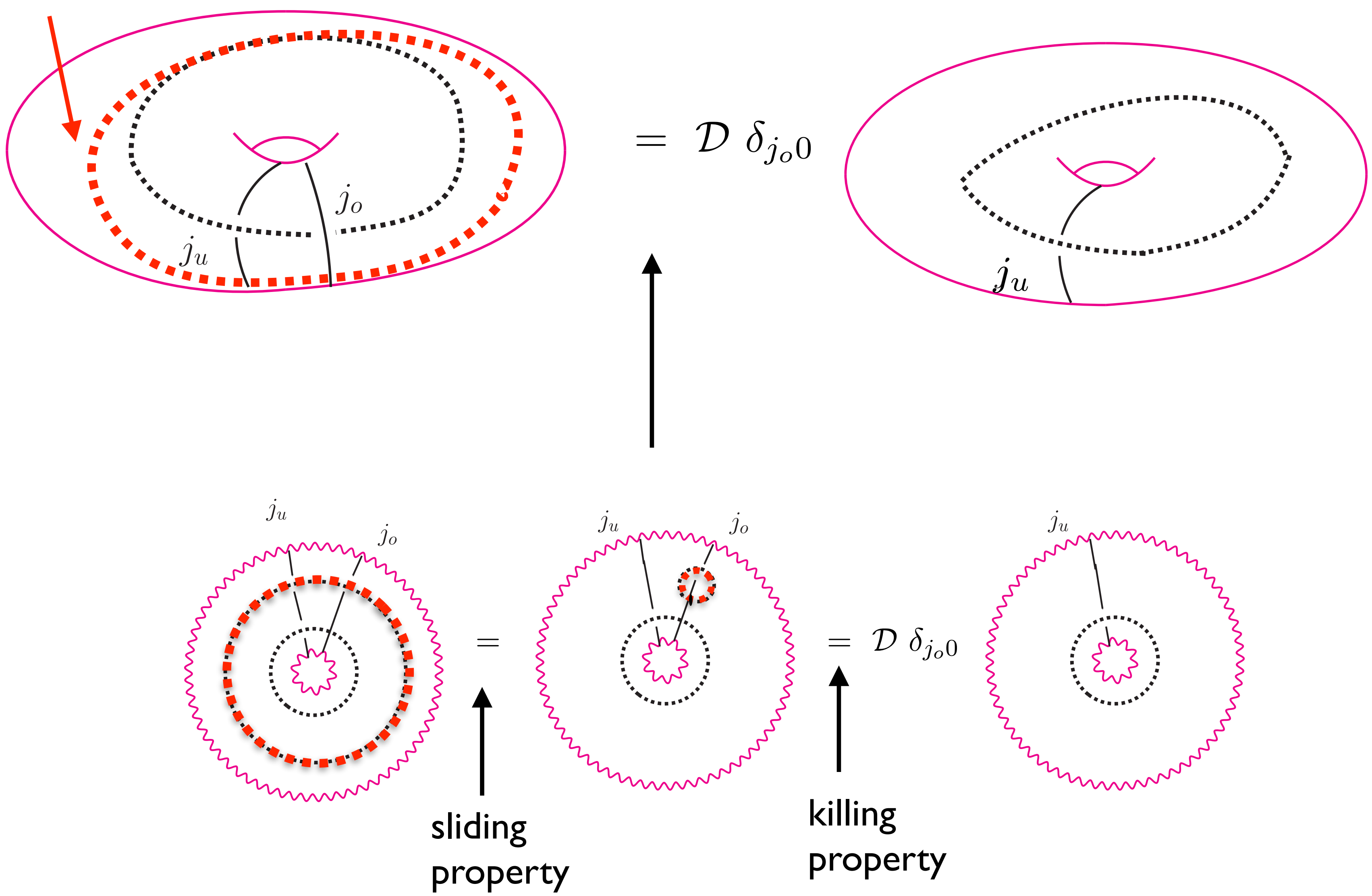
To proceed:

a) Construct bases for Heegaard surface.
b) Impose constraints.
c) Find operators preserving constraints.

Example: defect loop in 3-sphere

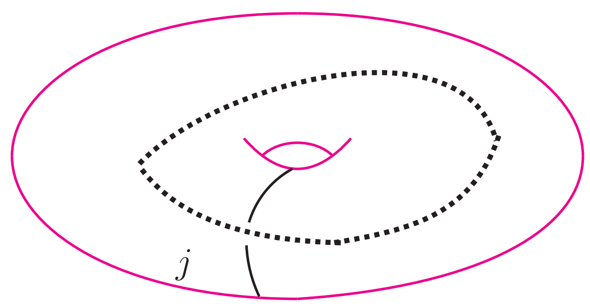
The corresponding Heegaard surface: a torus.
Flatness constraint along equator of this torus.

flatness constraint (over-crossing vacuum loop)
along equator

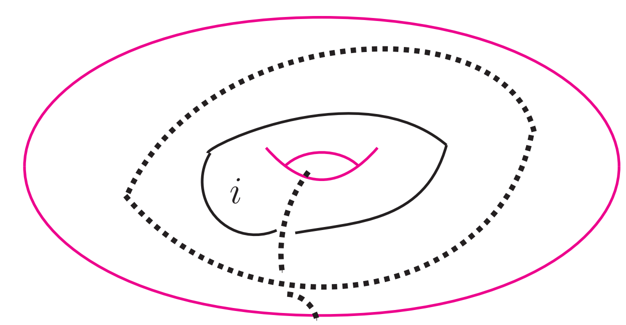


The flatness constraints surpress the over-crossing graph copy.

Example: defect loop in 3-sphere

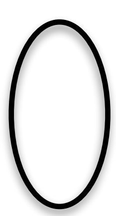


$$= \sum_i \frac{s_{ji}}{\mathcal{D}}$$



Here over-crossing graph copy given by vacuum loop.

j



Spin network basis: based on graph dual to curvature defects.

(quantum group) Fourier transform

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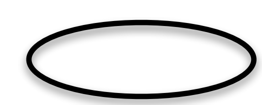


Curvature basis: based on graph along curvature defects

Diagonalizes (under-crossing) Wilson loop around equator.

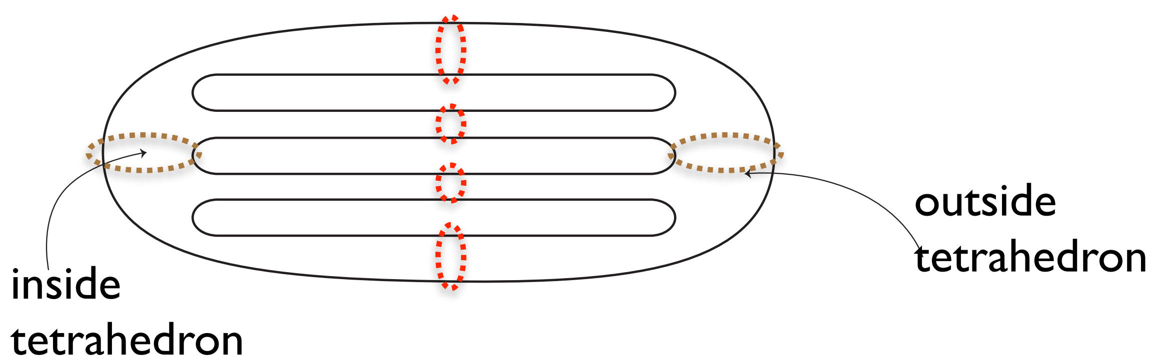
Measure area (of surface spanned by curvature defect).

Diagonalizes (under-crossing) Wilson loops around meridian.

Measures curvature (of curvature defect).

Spin network basis for general 3D triangulation

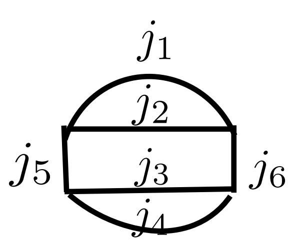
- Heegaard surface from thickening of one-skeleton of triangulation.
- Flatness constraints: (over-crossing) vacuum loops along triangle boundaries.



- Basis determined by pant-decomposition. Choose one adjusted to the dual graph.

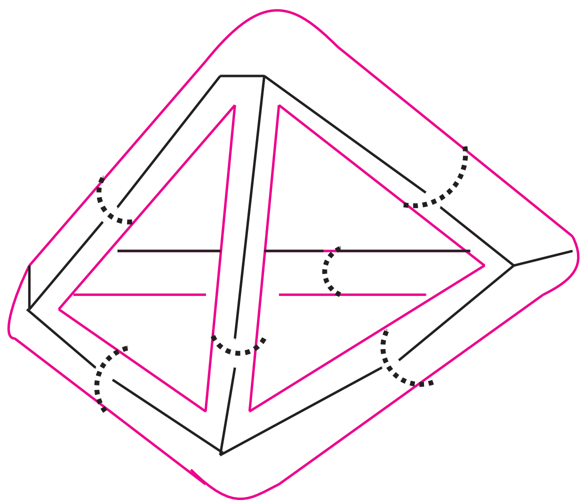
- Flatness constraints surpress over-crossing graph copy:

  Left with under-crossing graph dual to triangulation:
  (quantum deformed) spin network basis.



Curvature basis for general 3D triangulation

- Choose pant-decomposition adjusted to the one-skeleton of the triangulation
- After imposing flatness constraints: curvature basis.



Under-crossing graph along one-skeleton of triangulation which can be freely labelled by spins: labels of the curvature basis.
Over-crossing graph given by vacuum loops around triangles.

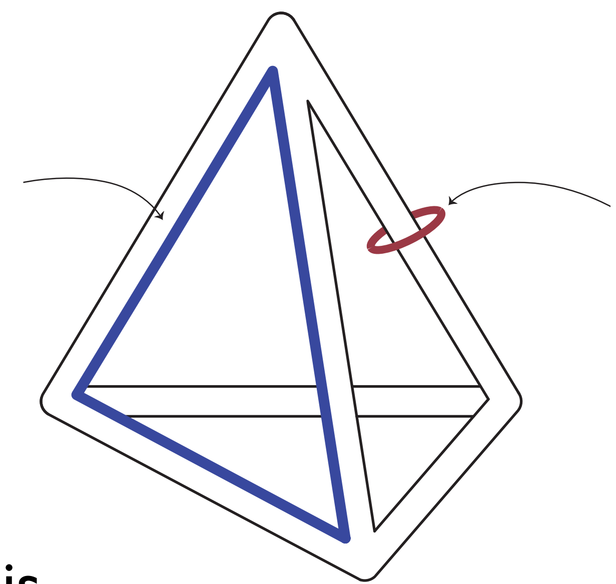
- (Curvature or Crane-Yetter) vacuum state:
  trivial spins associated to all edges of (triangulation) graph.

Non-degenerate vacuum state for all topologies.
Crane-Yetter invariant is 'trivial'.

Operators for the (3+1)D theory

Under-crossing Wilson loops preserve flatness constraints.

Wilson loops around triangles.



Wilson loops around edges.

- diagonalized by curvature basis
- measures curvature around edges

- diagonalized by spin network basis
- measure area of triangles:
    1. classical group case:
       ribbon operators preserving constraints
       map to integrated flux operators
       associated to triangles [Delcamp, BD JMP 2017]
    2. [HHKR]: Wilson loop around triangle
       measures homogeneous curvature
       which is proportional to area
    3. spectra match in classical limit

For normalized k-Wilson loop:

$$\frac{\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1)(2k+1)\right)\sin\left(\frac{\pi}{\mathrm{k}+2}\right)}{\sin\left(\frac{\pi}{\mathrm{k}+2}(2k+1))\right)\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1))\right)} \quad \xrightarrow{\mathrm{k}\to\infty} \quad 1-\frac{8}{3}\,j(j+1)\,k(k+1)\,\left(\frac{\pi}{\mathrm{k}+2}\right)^2$$

Operators for the (3+1)D theory

Under-crossing Wilson loops encode curvature and area operators.

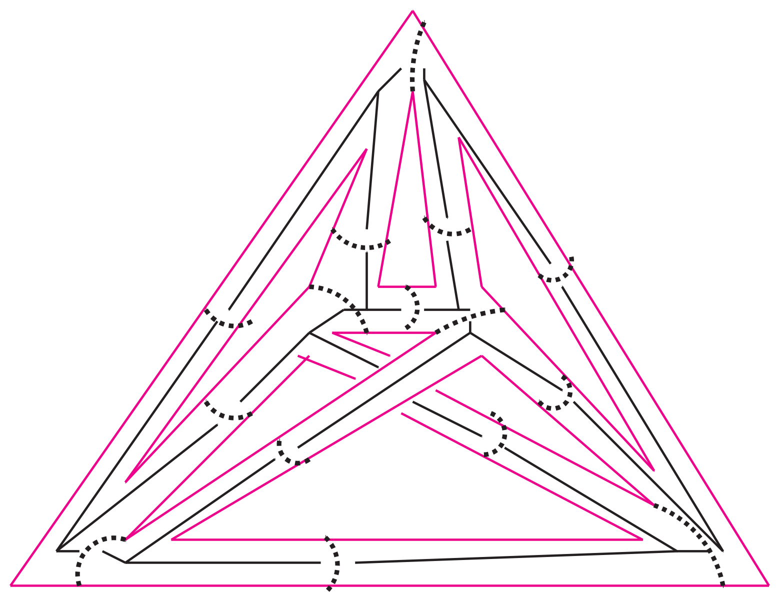
Spectra are discrete and bounded and coincide:

$$\frac{\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1)(2k+1)\right)\sin\left(\frac{\pi}{\mathrm{k}+2}\right)}{\sin\left(\frac{\pi}{\mathrm{k}+2}(2k+1))\right)\sin\left(\frac{\pi}{\mathrm{k}+2}(2j+1))\right)}$$

A self-dual quantum geometry.

Examples with even more self-duality

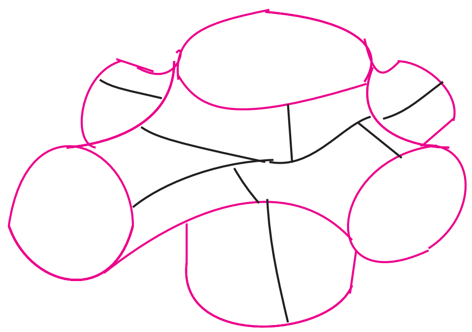
quantum-quantum 4-simplex



Curvature basis for 4-simplex.
(Over-crossing graph copy, which is
given by vacuum loops
around triangles, is suppressed.)

Spin network basis for 4-simplex.

quantum-quantum 3-torus



Curvature basis for 3 torus
with cubical lattice.
(Over-crossing graph copy and
vacuum loops are surpressed.)

Spin network basis for 3-torus.
(With Vacuum loops suppressed)

Conclusion

- enforcing a most important advantage of LQG/spin foams: relation to TQFT [Barrett, Crane, Smolin]
  - could be crucial for continuum limit (do we already have a geometric phase?)
  - exchange of elegant techniques between (now also canonical) quantum gravity and TQFT

- new vacua can serve as starting point of approximation scheme for dynamics [BD 2012-14]
  (Consistent Boundary Framework)

- this quantum geometry realization offers many advantages
  - spectra of intrinsic and extrinsic geometric operators are discrete and bounded
  - self-duality
  - finiteness properties important for (numerical) coarse graining schemes
  - new bases important for coarse graining

- new view on quantum geometries [BD, Steinhaus 2013: From TQFT to quantum geometry]
  - many new directions (next slide)
  - are there other quantum geometries (4D TQFTs) out there?
  - how do predictions depend on choice of representation?

Outlook

More quantum geometries:

- systematic way to construct 4D TQFTs with defects: [Delcamp, BD w.i.p.]
  lift other 3D TQFTs or string net models to 4D, e.g. group algebra models

- further generalizations ala [Baerenz, Barrett 2016]
  - weaken flatness constraints for triangles
  - allows for degenerate ground state (non-trivial 4D invariants)
  - introduces torsion degrees?

Analysis of current model:

- boundaries and torsion
  - compression bodies: Heegaard decomposition with boundary
  - expect surface anyons as excitations confined to boundary [Keyserlingk et al PRB 2013, …]
  - interpretation for lifted punctures with torsion defects?

- geometric interpretation of states and operators [Charles, Livine; Haggard, Han, Kaminski, Riello]
  - phase space
  - Barbero-Immirzi parameter

- refinements and coarse graining [Delcamp, BD w.i.p.]
  - fusion basis for (3+1)D

Thank you!

- B. Dittrich, (3+1)-dimensional topological phases and self-dual quantum geometries encoded on Heegaard surfaces, arXiv: 1701.02037

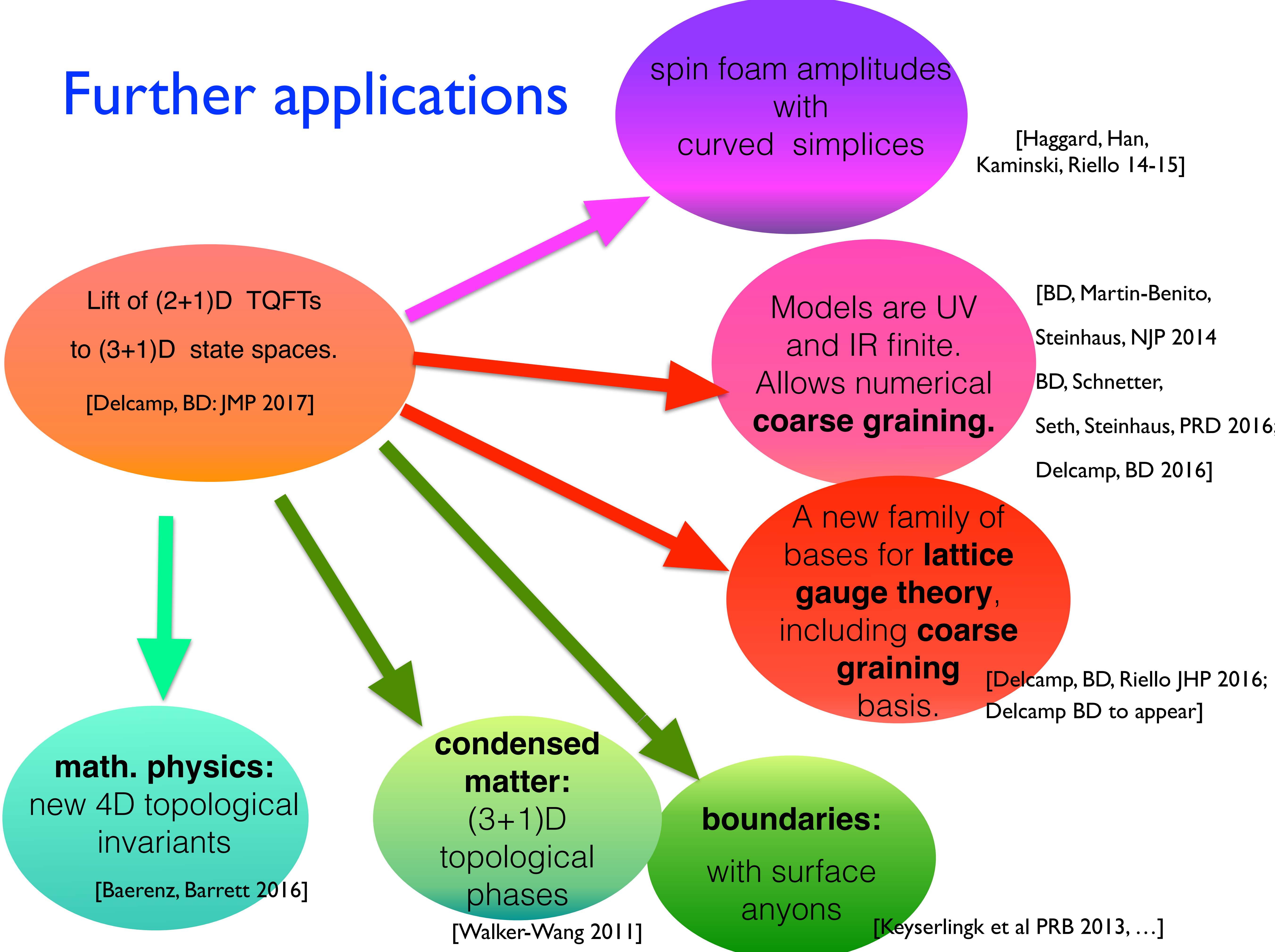
- C. Delcamp, B. Dittrich, From 3D TQFTs to 4D models with defects, to appear in JMP, arXiv: 1606.02384
- B. Dittrich, M. Geiller, Quantum gravity kinematics from extended TQFTs, NJP 2017, arXiv: 1606.02384

- M. Baerenz, J. Barrett, Dichromatic state sum models for four-manifolds from pivotal functors, arXiv: 1601.03580

- R. Koenig, G. Kuperberg and B.W. Reichardt, Quantum computation with Turaev-Viro codes, Annals of Physics 2010, arXiv: 1002.2816

- G. Alagic, S. P. Jordan, R. Koenig, B.W. Reichardt, Approximating Turaev-Viro 3-manifold invariants is universal for quantum computation, Phys Rev A 2010 , arXiv:1003.0923

Further applications

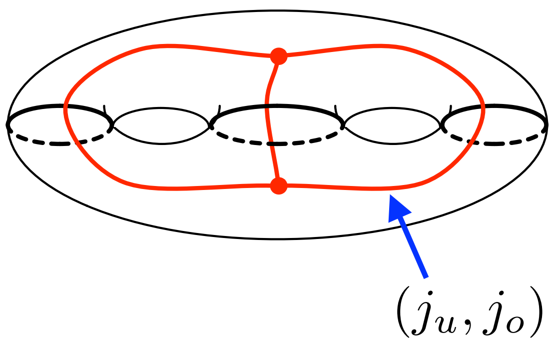


Hilbert space for (2+1)D: Bases

[Kohno 1992; Alagic et al 2010]

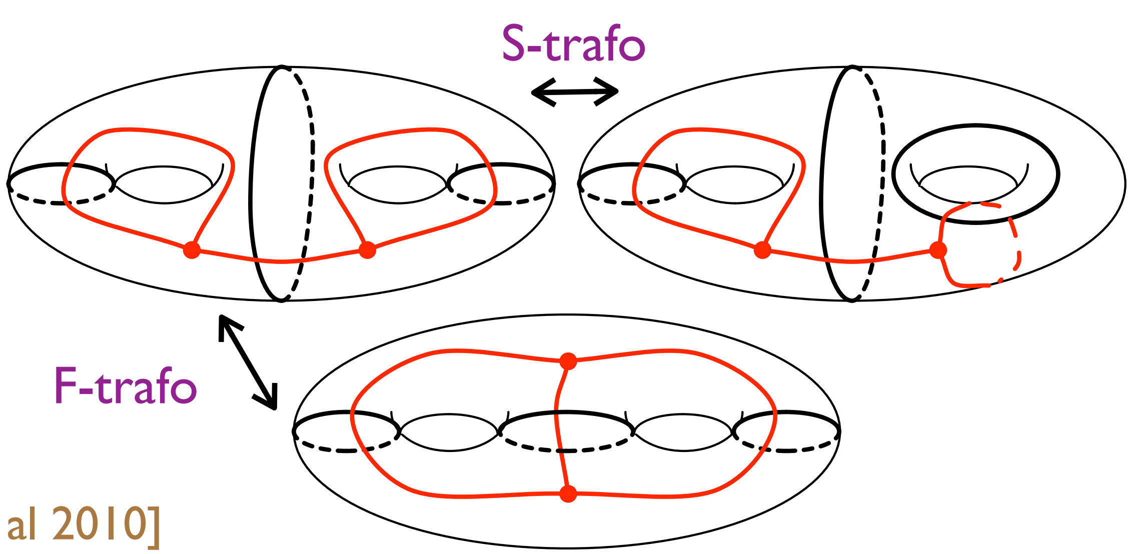
For g>1 surface:

Decompose surface into three-punctured spheres, aka ‘pants’:
By cutting surface along (3g-3) non-contractible curves.
This set of cutting curves defines the basis.



Construct the graph $\mathcal{F}$ dual to the cutting curves. Assign labels (j_u, j_o) to each edge of this dual graph.

Assign a vacuum loop to each cutting curve.
Double $\mathcal{F}$ to an under-crossing copy $\mathcal{F}_u$ labelled by j_u spins and an over-crossing copy $\mathcal{F}_o$ labelled by j_o spins.

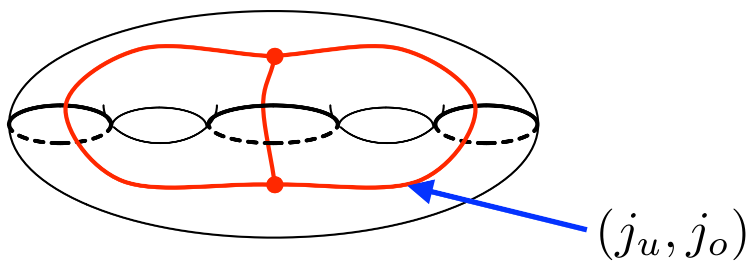


[Alagic et al 2010]

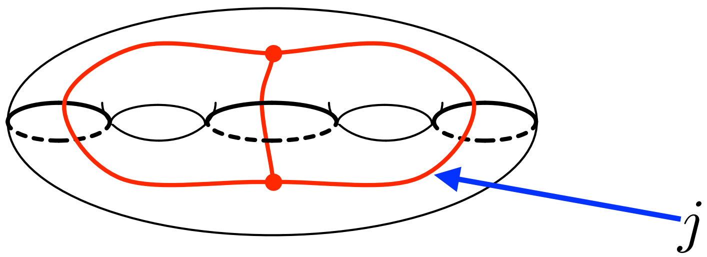
Transformations between bases can be generated by
- (generalized) S-transformations
- F-transformations (recoupling move)

Relation to Witten-Reshetikhin-Turaev TQFT

Basis for TV -TQFT



Basis for WRT -TQFT



Quantization of Chern-Simon theory.

$$Z_{TV} = |Z_{WTR}|^2$$

[Barrett et al JMP 2007] WRT partition function as boundary observable of Crane-Yetter model.