A perturbative approach to DIrac observables

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ILQGS, Nov 29 2006

B.D., Johannes Tambornino: gr-qc/0610060, to appear in CQG
B.D., Johannes Tambornino: w.i.p.

Introduction

- What are the observables in a theory of quantum gravity?
- How can we construct these observables?
- tremendously difficult: Dirac observables have to include an infinite number of spatial derivatives (Torre 93)
- resort: approximation? (Gambini,Pullin 00)
- can also be used to address conceptual issues:
- Can we approximate the (local) observables of standard field theory?
  Will these observables have a local Poisson algebra?
- suggest an expansion around a fixed background/phase space point
- deviations from standard field observables could result in fundamental uncertainties for quantum gravity observables (Giddings, Marolf, Hartle 05).
- How does the choice of time/clocks influence the interpretation?

Overview

1. Approximate Dirac observables
2. Application to General Relativity
3. Interpretation in terms of propagators and interaction processes
4. Scalar field coupled to gravity
5. Commutator algebra of fields: How does the choice of clock variables matter?
6. Outlook and summary

Fluctuation variables

- $\mathbf{X}_0$ "background" phase space point on constraint hypersurface
- introduce fluctuations $x = X - \mathbf{X}_0$ around background; consider x^a as first order quantities

- notation: for a phase space function g

$^{(m)}g$ all terms of order m in g
$^{[m]}g$ all terms of order $\leq m$ in g
$^{(m+)}g$ all terms of order $> m$ in g

- expand (first class) constraints $C_j = \sum_{m=1} {}^{(m)}C_j$, start with first order

Approximate Dirac observables

F Dirac observable $\Rightarrow \quad \{F, C_j\} \simeq 0$ for all constraints C_j

truncate F to terms up to the m-th order

$$\Rightarrow \quad \{^{[m]}F\,,\,C_j\} = \{F, C_j\} - \{^{(m+)}F, C_j\} \simeq O(m)$$

Definition

An approximate Dirac observable of m-th order commutes with the constraints up to terms of order m.

- concept is generalizable to perturbation in parameters or perturbations around (solvable) sectors of the theory
- first order Dirac observables coincide with observables of linearized theory

How to compute approximate Dirac observables?

Problem: higher order constraints ${}^{[m]}C_j$ do not form an algebra; cannot find invariants under ${}^{[m]}C_j$

Here:

- use formalism of partial and complete observables (Rovelli 02, BD 04)
- in particular the series expansion for complete observables (BD 04)
- truncate this series to terms of order $\leq m$
- series restricts to finitely many terms and is a Dirac observable of m-th order
- moreover complete observables have a (dynamical) interpretation

Complete Observables

Complete observables can be understood as gauge invariant extensions:

Choose a (parameter dependend) gauge $T^K = \tau^K$.

Complete Observable $\mathbf{F}_{[f;\,T]}(\tau)$ associated to a phase space function f:

- restricts to f on the gauge surface: $\mathbf{F}_{[f;\,T]}(\tau)|_{\{T^M = \tau^M\}} \simeq f$
- is constant along the gauge orbits: $\{\,\mathbf{F}_{[f;\,T]}(\tau)\,,\,C_j\,\} \simeq 0$
- interpretation: gives the value of f at the moment at which $T^M = \tau^M$. The T^M are called clock variables.

Power Series for Complete Observables

To compute complete observables introduce a new basis of the constraints:

$$\tilde{C}_K = C_j \, (A^{-1})^j_K \quad \text{with} \quad A^K_j = \{T^K, C_j\}$$

$\Rightarrow$ $\tilde{C}_M$ acts as derivatives in T^K–diretion: $\{T^K, \tilde{C}_M\} \simeq \{T^K, C_j\}(A^{-1})^j_M = \delta^K_M$

$\Rightarrow$ $\{\tilde{C}_K\}$ 'weakly Abelian' i.e. $\{\tilde{C}_K, \tilde{C}_M\} = O(C^2)$

Taylor expansion of the complete observable away from gauge surface:

$$\mathbf{F}_{[f;T]}(\tau) \simeq \sum_{r=0}^{\infty} \frac{1}{r!} \{\cdots\{f, \tilde{C}_{K_1}\}, \cdots\}\tilde{C}_{K_r}\} \, (\tau^{K_1} - T^{K_1}) \cdots (\tau^{K_r} - T^{K_r})$$

satisfies

$$\mathbf{F}_{[f;T]}(\tau)|_{\{T^M = \tau^M\}} \simeq f \qquad \text{and} \qquad \{\mathbf{F}_{[f;T]}(\tau), \tilde{C}_K\} \simeq 0$$

Power series in $(\tau^K - T^K)$ $\Rightarrow$ for $\tau^K = T^K(X_0) \Rightarrow$ power series in fluctuations

Approximate complete observables

Choose $\tau^K = T^K(X_0) \quad \Rightarrow \quad (\tau^K - T^K)$ at least first order

For f first order

$$^{[m]}\mathbf{F}_{[f;T]}(\tau) \simeq \sum_{r=0}^{m} \frac{1}{r!} \{\cdots\{f\,, {}^{[m]}\tilde{C}_{K_1}\},\cdots\}^{[m-r+1]}\tilde{C}_{K_r}\} \, (\tau^{K_1} - T^{K_1})\cdots(\tau^{K_r} - T^{K_r})$$

Only finitely many terms!

Convergence for $m \to \infty$ will depend on “quality” of clock variables.

If there exist a “perfect” set of clock variables, then:
There exists an exact Dirac observable that coincides with the approximate complete observable of order m (with respect to another set of clock variables) modulo terms of order $(m+1)$.

Application to General Relativity

(complex) connection variables: $A_a{}^j = \Gamma_a{}^j + \beta K_a{}^j, \qquad E^b{}_k$ where $\beta = i/2$

$$\{A_a{}^j(\sigma), E^b{}_k(\sigma')\} = \kappa \delta^a_b \delta^j_k \delta(\sigma, \sigma')$$

Minkowski background $\quad X_0 = (\mathbf{A}_a{}^j = 0,\, \mathbf{E}^b{}_k = \beta^{-1}\delta^b_k)$

fluctuation variables $\quad a_a{}^b := A_a{}^j \mathbf{E}^b{}_j \qquad e^c{}_d = (E^c{}_k - \mathbf{E}^c{}_k)\mathbf{E}^k{}_d$

first order of the constraints

$$\begin{aligned}
{}^{(1)}G_b &= \kappa^{-1}(\partial_a\, {}^{(LT+LL)}e^a{}_b + \beta\epsilon_{bcd}\, {}^{(LT+TL+AT)}a^{dc}) && \text{Gauß} \\
{}^{(1)}V_a &= \kappa^{-1}(\partial_a\, {}^{T}a_b{}^b - \partial_c\, {}^{TL}a_a{}^c) && \text{vector} \\
{}^{(1)}C &= \kappa^{-1}(2\beta\epsilon^{abd}\partial_a\, {}^{AT}a_{bd}) && \text{scalar}
\end{aligned}$$

LL left and right long. mode	LT left long. right transv. modes
T transv. trace part mode	TL left transv. right long. modes
AT antisym. transv. mode	STT symm. transv. trace–free modes

ADM clock variables

For every constraint $C_j(\sigma)$ we have to choose a clock variable $T^K(\sigma')$.

Convenient: $\quad {}^{(0)}A_j^K(\sigma,\sigma') := {}^{(0)}\{T^K(\sigma), C_j(\sigma')\} = \delta_j^K \delta(\sigma,\sigma')$

In ADM variables (ADM 62):

vector: $(LT + TL), LL$ mode of the 3–metric

scalar: T mode of the momentum

In connection variables:

$$
\begin{aligned}
{}^{G}T^a &= \beta^{-1}\epsilon^a{}_{bc}\ {}^{LT}e^{bc} - \Delta^{-1}(\partial^a\ {}^{LL}a^b{}_b + \tfrac{1}{2}\partial^a\ {}^{T}a^b{}_b) \\
{}^{V}T^a &= \Delta^{-1}(-\partial_b\ {}^{LT}e^{ba} - \partial_b\ {}^{TL}e^{ab} + \tfrac{1}{2}\partial^a\ {}^{T}e^b{}_b) - \tfrac{1}{2}\partial^a\ {}^{LL}e^b{}_b) \\
{}^{C}T &= (4\beta)^{-1}\Delta^{-1}(\epsilon_{cab}\partial^c\ {}^{AT}e^{ab} + \beta({}^{T}a^b{}_b + 2\ {}^{LL}a^b{}_b))
\end{aligned}
$$

satisfy $\quad \{T^K(\sigma)\,,\, {}^{(1)}C_j(\sigma')\} = \delta_j^K\, \delta(\sigma,\sigma')$

Interpratation: Metric is in a coordinate system which is as near to the Cartesian one as possible (Kuchař 1970). First order lapse and shift functions vanish on gauge and constraint surface.

The new constraints

Can invert the matrix $A^K_j(\sigma,\sigma')$ perturbatively to obtain the constraints $\tilde{C}_K$, e.g.

$$^{(1)}(A^{-1})^j_K(\sigma,\sigma') = -\delta^j_L \, ^{(1)}A^L_k(\sigma,\sigma')\delta^k_K$$

Furthermore: can iteratively add to the $\tilde{C}_K$ terms which are at least $O(C^2)$, such that the resulting constraints $\check{C}_K$ are of the (deparametrized) form

$$\check{C}_K = {}^{(1)}C_j + {}^{(2+)}\check{H}_j(STT, T^L)$$

- Both sets of constraints can be used in the series expansion for the complete observables. The deparametrized form is easier to deal with.
- Only finitely many operations necessary to calculate complete observable of order m for parameter values $\tau^K(\sigma) \equiv 0$.
- Can we introduce dynamics? What happens if we change the parameter values τ^K?

Asymptotic conditions

In computing the complete observable terms of the following form appear:

$$\int_{\Sigma} \{\,\cdot\,, {}^{(1)}C_j(\sigma) + \ldots\}\, \delta^j_K \left(\tau^K(\sigma) - T^K(\sigma)\right) d\sigma$$

$\Rightarrow \tau^K(\sigma)$ and $T^K(\sigma)$ can be understood as smearing functions for the constraints.

Have to specify asymptotic conditions (Thiemann 95):

$$\begin{array}{c} a_{ab} \sim r^{-2} \text{ and odd parity in leading order} \\ e^{ab} \sim r^{-1} \text{ and even parity in leading order} \end{array}$$

The clock variables T^K have the correct asymptotic behaviour to be allowed as smearing functions for the constraints (without introducing boundary terms).

Choosing $\tau^0(\sigma) \equiv t$ corresponds to choosing constant lapse: have to add a boundary term to the scalar constraint. This boundary term is equal to the ADM energy and leads to the vanishing of the first order for the integrated scalar constraint.

$$\text{ADM energy} \simeq {}^{(2+)}C := \int_\Sigma {}^{(2+)}C\, d\sigma \simeq \int_\Sigma \check{H}_0(\sigma)\, d\sigma =: \check{H}_0$$

Dynamics

Set $\tau^0(\sigma) \equiv t$ and $\tau^A(\sigma) \equiv 0$ for $A \neq 0$ in $\mathbf{F}_{[f;\,T]}(\tau^K)$.

Can rewrite the series expansion in two ways:

(a) $\mathbf{F}_{[f;\,T]}(t) \simeq \mathbf{F}_{\left[\alpha^t_{\check{H}_0}(f)\,;\,T\right]}(\tau^K \equiv 0)$ with $\alpha^t_{\check{H}_0}(f) = \sum_{r=0} \frac{t^r}{r!}\{f, \check{H}_0[1]\}_r$

(b) $\mathbf{F}_{[f;\,T]}(t) \simeq \alpha^t_{\check{H}_0}(\mathbf{F}_{[f;\,T]}(\tau^K \equiv 0))$

(a) easier to calculate with:
first evolve f with $\check{H}_0$, then calculate complete observable (gauge invariant extension);
can restrict $\check{H}_0(STT, T^K)$ to STT modes!
$\Rightarrow$ interpretation in terms of scattering of gravitons

(b) first calculate complete observable, then evolve with the ADM–Hamiltonian $\check{H}_0 \simeq {}^{(2+)}C$
$\Rightarrow$ our time evolution is generated by the ADM–Hamiltonian and corresponds to time translations at infinity

The second order approximation

(1) evolve ${}^{(1)}f$ with $\check{H}_0(STT, T^K \equiv 0)$ up to second order

(2) calculate complete observable (gauge invariant extension) up to second order;

(1) Because ${}^{(1)}\check{H}_0 = 0$ we can define the "free" propagator

$$\alpha^t := \alpha^t_{(2)\check{H}_0} \quad .$$

The second order time evolution can be written as

$${}^{[2]}\alpha^t_{\check{H}}(f) = \alpha^t(f) + \int_0^t dt' \; \alpha^{t'}\left(\left\{\alpha^{(t-t')}(f), {}^{(3)}\check{H}\right\}\right) \quad .$$

$\Rightarrow$ field f propagates $\rightarrow$ interaction process $\rightarrow$ resulting fields propagate generalizes to higher order

(2) Calculate gauge invariant extension; for the second order term it is sufficient to calculate the first order complete observables of the two fields involved.

Complete observable of order m associated to f and (t, σ) can be explicitely calculated.

The time generator

$$^{(2)}\check{H}_{0|STT} = {}^{(2)}C_{|STT}$$

$\Rightarrow$ free propagator coincides with propagator in linerized theory/ matter fields on Minkowski

$$^{(3)}\check{H}_{0|STT} = {}^{(3)}C_{|STT}$$

$\Rightarrow$ "scattering" of gravitons or matter fields at gravitons or matter

$$^{(4)}\check{H}_{0|STT} \neq {}^{(4)}C_{|STT}$$

will include terms with inverse derivative operator Δ^{-1}
$\Rightarrow$ non-local choice of time/ clock variables

$\check{H}_0$ is not of finite order anymore (as opposed to C)

Gravity coupled to a scalar field

- complete observables associated to matter field can be understood as expansions in $\kappa^{1/2}$
- first order complete observable ${}^{(1)}\mathbf{F}_{[\phi(\sigma);T^\kappa]}(t) = \phi(t,\sigma)$ coincides with observable of field theory on Minkowski space
- second order complete observable includes (one) scattering between matter field and gravitons: to lowest order propagation of matter field on graviton background
- justifies (up to second order) to work with an effective matter Hamiltonian, where gravitational variables are time dependent but non–dynamical
- Poisson brackets between second order matter fields reflect to lowest order the causality structure of the graviton background
- higher order terms include backreaction but also non–local expressions
- higher order Poisson brackets will be difficult to interprete because of non–local terms

The second order complete observable

$$\mathbf{F}_{[\phi(\sigma);T]}(t) \simeq$$

$$\int_\Sigma S(t,\sigma;0,\sigma')\pi(\sigma') + S'(t,\sigma;0,\sigma')\phi(\sigma')\,d\sigma' \;-$$

$$\int_\Sigma S(t,\sigma;0,\sigma')\partial_b(\pi\,{}^V T^b)(\sigma') + S'(t,\sigma;0,\sigma')\,(\partial_b\phi)(\sigma')\,{}^V T^b(\sigma')\,d\sigma' -$$

$$\int_\Sigma S(t,\sigma;0,\sigma')(\partial_a(\partial^a\phi\,{}^C T)(\sigma') - m^2\phi(\sigma')\,{}^C T(\sigma'))+$$

$$S'(t,\sigma;0,\sigma')\pi(\sigma')\,{}^C T(\sigma')\,d\sigma' \;+$$

$$\int_0^t dt' \int_\Sigma d\sigma'\, 2S(t-t',\sigma;0,\sigma')\times$$

$$\int_\Sigma G'^{ab}_{cd}(t',\sigma';0,\sigma'')\,{}^{STT}a^{cd}(\sigma'') + G''^{ab}_{cd}(t',\sigma';0,\sigma'')\,{}^{STT}e^{cd}(\sigma'')\,d\sigma'' \times$$

$$\partial_a^{\sigma'}\partial_b^{\sigma'} \int_\Sigma S(t',\sigma';0,\sigma''')\pi(\sigma''') + S'(t',\sigma';0,\sigma''')\phi(\sigma''')\,d\sigma'''$$

Influence of the clock variables

How does the choice of clock variables matter?
Consider influence on the "space–time algebra": commutators between fields at different space–time points.

(A) Choose scalar fields as clock variables:

- expect local behaviour of space–time commutators
- commutator has a correction term $\sim$ energy of the field observed/energy of the clock field
- energy of the clock field cannot be made arbitrary large because of back reaction/ black hole formation
- fundamental restriction on observables? (Giddings,Marolf,Hartle 05)

(B) Choose ADM clocks:

- correction term for the commutator scales in the same way as backreaction terms
- $\kappa \rightarrow 0$ gives the commutators of field theory on flat background
- however non–local behaviour for third order expected

Outlook and Summary

- approximate Dirac observables with a dynamical interpretation can be calculated explicitely to an arbitrary order
- precise understanding of linearized theory and (quantum) field theory on a fixed background as approximations to full general relativity
- formalism can be used to address construction and interpretation of Dirac observables
- can be generalized to expansion around symmetry reduced/cosmological sectors $\Rightarrow$ use knowledge on symmetry reduced sectors to construct approximate Dirac observables for full theory
- would also include closed universes $\Rightarrow$ address conceptual issues for QFT on curved space–time from a new perspective
- need a better understanding of the (quantum) interpretation of complete observables, in particular role of clock variables