Quantum deformation of 4d spin foam models

Winston J. Fairbairn

Department Mathematik
Universität Hamburg

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Introduction

- Spin foam models : discretised functional integrals for TFTs of BF type and quantum gravity
- Models for theories with zero 'cosmological constant' are based on the representation theory of Lie groups
- Problem : generally these models diverge
- Natural regularisation : consider models based on the representation theory of quantum groups
- Our work : application of this procedure to the EPRL model

Gravity in $2+1$ dimensions

- First order gravity on a 3-dimensional manifold M:
  - dual co-frame $e : TM \to \mathbb{R}^3$
  - $\mathrm{SO}(\eta)$ connection A ($\eta = (\pm, +, +)$ flat metric)

  Let $F_A = dA + \frac{1}{2}[A \wedge A]$ be the curvature of A, and $\langle , \rangle$ the Killing form on $\mathfrak{so}(\eta) \cong \mathbb{R}^3$
- Action for $2+1$ gravity with cosmological constant Λ

$$S_\Lambda = \int_M \langle e \wedge F_A \rangle - \frac{\Lambda}{3} \langle e \wedge [e \wedge e] \rangle \quad (1)$$

- Goal : make sense of the formal quantity

$$\mathcal{Z}(M) = \int D[e] D[A] \exp iS_\Lambda[e, A] \quad (2)$$

- Ponzano-Regge (PR) model : regularisation of $\mathcal{Z}(M)$ for Euclidean gravity with $\Lambda = 0$

Ponzano-Regge model [Ponzano, Regge '68; Freidel, Louapre '04; Barrett, Naish-Guzman '08]

- Let M be a closed, oriented triangulated 3-manifold
- PR model based on $\mathrm{Rep}(\mathrm{SU}(2))$. Data consists of :
  - Assignments $I : e \mapsto I(e) \in \mathrm{Irrep}(\mathrm{SU}(2))$ to the edges e of M
  - A state space for each triangle Δ of M

$$H_\Delta = \mathrm{Hom}(\bigotimes_{e \in \partial\Delta} I(e), \mathbb{C}) \ni \alpha,$$

  - An amplitude $A_t : \bigotimes_{\Delta \in \partial t} H_\Delta \to \mathbb{C}$ for each tetrahedron t

$$A_t(\alpha \otimes \beta \otimes \gamma \otimes \delta) = \{6j\}_t$$

- PR model :

$$PR(M) = \sum_I \prod_e \dim I(e) \prod_t \{6j\}_t \qquad (3)$$

- Problem : the above infinite sum generally diverges

[Bonzom, Smerlak '10]

The Turaev-Viro model [Turaev, Viro '92]

- Goal : regularise the divergencies of the PR model
- Idea : the quantum group $U_q(\mathfrak{su}(2))$ with $q^r = 1$ admits only a finite number of irreducible representations [Arnaudon, Roche '89]

  → Natural regularisation of the PR model : replace $\mathrm{SU}(2)$ by its quantum deformation $U_q(\mathfrak{su}(2))$ at root of unity
- Irreducible representations of $U_q(\mathfrak{su}(2))$ (after purification):

$$I \in \left\{0, \frac{1}{2}, 1, ..., \frac{r-2}{2}\right\}, \qquad q = \exp 2i\pi/r$$

- Constructing a model based on $\mathrm{Rep}(U_q(\mathfrak{su}(2)))$ leads to a finite model: the Turaev-Viro invariant

$$TV_q(M) = K \sum_I \prod_e [\dim I(e)]_q \prod_t \{6j_q\}_t \tag{4}$$

Relation to the cosmological constant Λ

- Using the equivalence [Witten '88] between 2 + 1 gravity and Chern-Simons theory one can show that :

$$TV_q(M) \propto \int D[e]D[A] \exp iS_\Lambda[e, A], \quad \text{if } \; q = \exp il_p/l_c \tag{5}$$

- Asymptotics of the quantum $6j$ symbol when $l(e) \to \infty$:

[Mizoguchi,Tada '92]

$$\{6j_q\}_t \sim \frac{1}{\sqrt{12\pi V_t}} \cos\left(S_t + \frac{\pi}{4}\right), \quad V_t = \text{volume}(t), \tag{6}$$

where S_t is the Regge action with cosmological constant Λ for the tetrahedron t

$$S_t = \sum_{e \in \partial t} \theta_e l(e) - \frac{1}{l_c^2} V_t \tag{7}$$

4d BF theory [Horowitz '89; Cattaneo, Cotta-Ramusino, Fröhlich, Martellini '95; Baez '95]

- BF theory with semi-simple Lie group G on a 4-manifold M:
  - B field $B \in \Omega^2(M) \otimes \mathfrak{g}$
  - connection A on a principal G-bundle over M

  Let $F_A = dA + \frac{1}{2}[A \wedge A]$ be the curvature of A, and $\langle , \rangle$ the Killing form on $\mathfrak{g}$
- Action of 4d BF theory with cosmological constant Λ

$$S_\Lambda = \int_M \langle B \wedge F_A \rangle - \frac{\Lambda}{12} \langle B \wedge B \rangle \qquad (8)$$

- Goal : make sense of the formal quantity

$$\mathcal{Z}(M) = \int D[B] D[A] \exp i S_\Lambda [B, A] \qquad (9)$$

The Ooguri model [Ooguri '92]

- Ooguri model : regularisation of $\mathcal{Z}(M)$ in the case of $G = \mathrm{SU}(2)$ and $\Lambda = 0$
- Model based on $\mathrm{Rep}(\mathrm{SU}(2))$
  - The assignments are now to the triangles Δ and tetrahedra t of a triangulation of M

$$I : \Delta \mapsto I(\Delta) \in \mathrm{Irrep}(\mathrm{SU}(2)), \qquad t \mapsto \alpha_t \in H_t = \mathrm{Hom}(\bigotimes_{\Delta \in \partial t} I(\Delta), \mathbb{C})$$

  - The amplitude for the 4-simplexes σ are given by $15j$-symbols
- Partition function :

$$O(M) = \sum_{I,J} \prod_{\Delta} \dim I(\Delta) \prod_{t} \dim J(t)^{-1} \prod_{\sigma} \{15j\}_{\sigma} \qquad (10)$$

The Crane-Yetter model [Crane, Yetter, Kauffman '93]

- The Ooguri model diverges in general
- It can be regularised by considering a model based on $\mathrm{Rep}(U_q(\mathfrak{su}(2)))$ with $q^r = 1$

$$CY_q(M) = K \sum_{I,J} \prod_{\Delta} [\dim I(\Delta)]_q \prod_t [\dim J(t)]_q^{-1} \prod_{\sigma} \{15 j_q\}_{\sigma} \qquad (11)$$

- The Crane-Yetter model is finite and provides an invariant of topological 4-manifolds
- It is related to $\mathrm{SU}(2)$ BF theory on M with cosmological constant Λ via Chern-Simons theory on ∂M [Roberts '93; Baez '95]

$$CY_q(M) \propto \int D[B] D[A] \exp i S_{\Lambda}[B, A], \quad \text{if} \;\; q = \exp i l_p^2 / l_c^2 \qquad (12)$$

Gravity as a constrained BF theory [Plebanski '77; Freidel, De Pietri '98]

- First order gravity on a 4-dimensional manifold M:
  - dual co-frame $e : TM \to \mathbb{R}^4$
  - $\mathrm{SO}(\eta)$ connection A ($\eta = (\epsilon, +, +, +)$, $\epsilon = \pm 1$, flat metric)
- Action for $3+1$ gravity with cosmological constant Λ

$$S_\Lambda = \int_M \langle *(e \wedge e) \wedge F_A \rangle - \frac{\Lambda}{12} \langle *(e \wedge e) \wedge e \wedge e \rangle \qquad (13)$$

- Suggests that GR could be cast as a BF theory :

$$S_\Lambda = \int_M \langle B[e] \wedge F_A \rangle - \frac{\epsilon\Lambda}{12} \langle B[e] \wedge * B[e] \rangle, \quad \text{with} \quad B[e] = *(e \wedge e)$$

- Gravity $\equiv$ BF theory + constraints on the B field

$$S_\Lambda^{\text{Plebanski}} = \int_M \langle B \wedge F_A \rangle - \frac{\epsilon\Lambda}{12} \langle B \wedge * B \rangle + \mathcal{C}(B) \qquad (14)$$

The EPRL/FK models [Engle, Pereira, Rovelli, Livine '08; Freidel, Krasnov '08]

- EPRL/FK models : spin foam models for Plebanski's theory with $\Lambda = 0$ and Immirzi parameter γ
- Idea : implement the Plebanski constraints as a choice of measure in the path integral for BF theory
- Concretely :
  - Consider a generalised Ooguri model for $\text{Rep}(\mathrm{SO}(\eta))$
  - Impose constraints on the data of the model, i.e., at the level of representation labels $I(\Delta)$ and state spaces H_t
- Questions :
  - Are the EPRL/FK models finite ? [Perini, Rovelli, Speziale '09]
  - How to introduce a cosmological constant ?

    [Bianchi, Krajewski, Rovelli, Vidotto '11]
- Possible answer : consider the quantum deformation of the models [Smolin '95; Major, Smolin '96; Smolin '02; Rovelli '10]

Quantum deformation of the EPRL model

- Our work : construction and analysis of a q-deformation of the Euclidean and Lorentzian versions of the EPRL model
  - The Euclidean model is based on

$$U_q(\mathfrak{spin}(4)) = U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2)), \quad \text{with } q \text{ root of unity}$$

  - The Lorentzian model on

$$U_q(\mathfrak{sl}(2,\mathbb{C})) = \mathcal{D}U_q(\mathfrak{su}(2)) = U_q(\mathfrak{su}(2)) \hat{\otimes} F_q(\mathrm{SU}(2))^{op}, \quad \text{with } q \text{ real}$$

- Basic ingredients for the construction of the model :
  1. A given subset of representations $\mathcal{I} \subset \mathrm{Rep}(U_q(\mathfrak{so}(\eta)))$ (q-analogue of EPRL representations)
  2. A specific class of intertwiners ι between the elements of $\mathcal{I}$ (q-analogue of EPRL intertwiners)
- From here on, we will focus on the Lorentzian model

q-Lorentzian EPRL representations

- Unitary, irreducible representations (of the principal series) of $U_q(\mathfrak{sl}(2,\mathbb{C}))$, $q = e^{-\kappa}$ [Pusz '93, Buffenoir, Roche '99] :

$$\text{Irrep}(U_q(\mathfrak{sl}(2,\mathbb{C}))) = \{(n,p), \quad n \in \mathbb{Z}/2, p \in [0, 4\pi/\kappa[\,\} \quad (15)$$

- Let $\gamma \in \mathbb{R}$ be a parameter. An EPRL representation is a map

$$\phi_\gamma : \text{Irrep}(U_q(\mathfrak{su}(2)) \to \text{Irrep}(U_q(\mathfrak{sl}(2,\mathbb{C}))),$$

defined by

$$K \mapsto (n(K), p(K)) = (K, \gamma K) \quad (16)$$

- Remark: the pre-image of the map ϕ_γ is restricted to

$$\mathcal{L} = \{K \in \mathbb{N}/2 \mid K < 4\pi/\gamma\kappa\} \subset \text{Irrep}(U_q(\mathfrak{su}(2)))$$

q-Lorentzian EPRL interwiner : Definition I.

- Notation : principal representations (π_α, V_α), $\alpha = (n, p)$
- The EPRL representation $\alpha(K)$ factorises as

$$V_{\alpha(K)} = \bigoplus_{I=K}^{\infty} V_I, \qquad V_I : U_q(\mathfrak{su}(2))\text{-module} \qquad (17)$$

- Let $f_\alpha^K : V_\alpha \to V_K$ be the projection on the lowest weight factor. The dual map induces an embedding

$$f^* : \mathrm{Hom}_{U_q(\mathfrak{su}(2))}(\otimes_{i=1}^n V_{K_i}, \mathbb{C}) \to \mathrm{Hom}_{U_q(\mathfrak{sl}(2,\mathbb{C}))}(\otimes_{i=1}^n V_{\alpha_i}, \mathbb{C})$$

- To all $U_q(\mathfrak{su}(2))$-intertwiner $\Lambda_K : \otimes_{i=1}^n V_{K_i} \to \mathbb{C}$, this map associates a quantum EPRL intertwiner $\iota_{\alpha(K)}$

$$\iota_{\alpha(K)} = f^*(\Lambda_K) = \Lambda_K \circ \bigotimes_{i=1}^{n} f_{\alpha_i}^{K_i} \circ T_{\alpha_1,\ldots,\alpha_n} \qquad (18)$$

q-Lorentzian EPRL interwiner : Definition II.

- $T_{\alpha_1,\ldots,\alpha_n}$ is the q-analogue of the classical expression

$$\int_{\mathrm{SL}(2,\mathbb{C})} dX \left(\bigotimes_{i=1}^{n} \pi_{\alpha_i} \right) (X) \tag{19}$$

- Let $h : F_q(\mathrm{SL}(2,\mathbb{C})) \to \mathbb{C}$ be a Haar measure on the Hopf algebra $F_q(\mathrm{SL}(2,\mathbb{C}))$ dual to $U_q(\mathfrak{sl}(2,\mathbb{C}))$ [Buffenoir, Roche '99]

$$\text{Ex. : } f \in F(\mathrm{SL}(2,\mathbb{C})), \quad h(f) = \int_{\mathrm{SL}(2,\mathbb{C})} dX\ f(X)$$

- $T_{\alpha_1,\ldots,\alpha_n}$ is defined as [Noui, Roche '02]

$$T_{\alpha_1,\ldots,\alpha_n} = \sum_A \left(\bigotimes_{i=1}^{n} \pi_{\alpha_i} \right) (\Delta^{(n)}(x^A))\, h(x_A), \tag{20}$$

where $\{x^A\}_A$ is a basis of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\{x_A\}_A$ is the dual basis of $F_q(\mathrm{SL}(2,\mathbb{C}))$

q-Lorentzian EPRL interwiner : Properties

- **Theorem.** *1. Let* $\{e^l_a(\alpha) \mid l \in \mathbb{N}, l \geq |n|, a = -l, ..., l\}$ *be a basis of* V_α*. The evaluation of the 4-valent quantum EPRL intertwiner* $\iota_\alpha(\bigotimes_{i=1}^4 e^{l_i}_{a_i}(\alpha_i))$ *is a multiple series which converges absolutely*
2. The linear map $\iota_\alpha : \otimes_{i=1}^4 V_{\alpha_i} \rightarrow \mathbb{C}$ *is a intertwiner for the quantum Lorentz group*

- **Proposition.** *Let* R *be the matrix of the quantum double* $\mathcal{D}U_q(\mathfrak{su}(2))$ *and* $c_{\alpha_2,\alpha_1} : V_{\alpha_2} \otimes V_{\alpha_1} \rightarrow V_{\alpha_1} \otimes V_{\alpha_2}$ *with*

$$c_{\alpha_2,\alpha_1} = \tau_{\alpha_2,\alpha_1} \circ (\pi_{\alpha_2} \otimes \pi_{\alpha_1})(R), \tag{21}$$

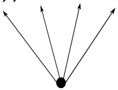
the associated braiding. The quantum EPRL intertwiner ι_α *is not invariant under the action of c :*

$$\iota_{\alpha_1\alpha_2\alpha_3\alpha_4} \circ c_{\alpha_2,\alpha_1} \neq \iota_{\alpha_2\alpha_1\alpha_3\alpha_4} \tag{22}$$

Graphical calculus : ingredients

- 4-simplex amplitude defined using graphical calculus
- Elements for the diagrammatic calculus :
  - EPRL tensors : elements of $\left[\bigotimes_{i=1}^{4} V_{\alpha_i}\right] \otimes F_q(\mathrm{SL}(2,\mathbb{C}))$

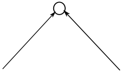
$$\Psi^{\alpha} = \left[\sum_{A} \left(\bigotimes_{i=1}^{4} \pi_{\alpha_{i(K_i)}} \left(\Delta^{(4)}(x^A)\right)\right) \circ \bigotimes_{i=1}^{4} f_{K_i}^{\alpha_i} \circ \Lambda^K\right] \otimes x_A \ :=$$



where $\Lambda^K \in \mathrm{Hom}_{U_q(\mathfrak{su}(2))}\left(\mathbb{C}, \bigotimes_{i=1}^{4} V_{K_i}\right)$ and $f_K^{\alpha} : V_K \to V_{\alpha}$

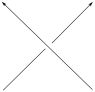
  - An invariant bilinear form $\beta : V_\alpha \times V_\alpha \to \mathbb{C}$

$$\beta :=$$



  - The braiding $c_{\alpha,\beta} : V_\alpha \otimes V_\beta \to V_\beta \otimes V_\alpha$

$$c_{\alpha,\beta} :=$$



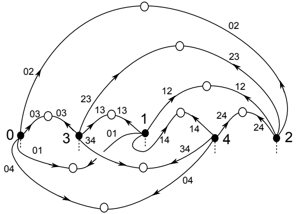
Graphical calculus : evaluation

- Closed diagram Γ_n with n vertices $\longrightarrow \phi(\Gamma_n) \in F_q(\mathrm{SL}(2,\mathbb{C}))^{\otimes n}$
- The evaluation of a closed diagram is defined as

$$ev(\Gamma_n) = (\epsilon \otimes h^{n-1})(\phi(\Gamma_n)), \quad \epsilon \text{ is the co-unit} \tag{23}$$

- The diagram for the 4-simplex amplitude is given by

$$\Gamma_5 = \tag{24}$$



- **Theorem.** *The multiple series* $ev(\Gamma_5)$ *converges absolutely*

The model

- Let M be a closed, oriented triangulated 4-manifold
- Model based on $\mathrm{Rep}(U_q(\mathfrak{sl}(2,\mathbb{C})))$. Data consists of :
  - An assignment $\alpha : \Delta \mapsto \alpha(\Delta) \in \mathrm{Irrep}(U_q(\mathfrak{sl}(2,\mathbb{C})))$ of an EPRL representation to each triangle Δ of M
  - A state space of EPRL tensors for each tetrahedron t of M

$$H_t = \left(\bigotimes_{\Delta \in \partial t} \alpha(\Delta) \right) \otimes F_q(\mathrm{SL}(2,\mathbb{C}))$$

  - An amplitude $A_\sigma : \bigotimes_{t \in \partial\sigma} H_t \to \mathbb{C}$ for each 4-simplex σ

$$A_\sigma(\Psi_1 \otimes \ldots \otimes \Psi_5) = ev(\Gamma_5) \ \in \mathbb{C}$$

- The resulting model is *finite*. It is given by

$$\mathcal{Z}_q(M) = \sum_{K \in \mathcal{L}} \sum_J \prod_\Delta [\dim K(\Delta)]_q \prod_\sigma A_\sigma(\{K(\Delta)\}, \{J(t)\}) \qquad (25)$$

Conclusion

- We have constructed and analysed a q-deformation of the EPRL model. We have :
  - Generalised the classical constructions to models based on $U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$ and $\mathcal{D}U_q(\mathfrak{su}(2))$
  - Defined the EPRL intertwiner, studied its convergence and properties under braiding
  - Constructed a convergent amplitude for the 4-simplexes
- Open question: relation to the cosmological constant Λ ?
  - If $q = e^{-l_p^2/l_c^2}$ (Lorentzian), bound on the area of the triangles

$$A(\Delta) < 32\pi^2 l_c^2 \qquad (l_p << l_c) \tag{26}$$

  - Need for an asymptotic formula [Ding, Han '11]