Matter
in
Quantum BF Theory

"International LQG Seminars"
November 2006 - Perimeter Institute

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Outline

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- Introduction
- Classical theory
- Discretization
- Fermionic loop expansion
- Generating functional
- Fermions and quantum gravity

Part *II*: Quantisation of string-like sources coupled to BF theory: Physical scalar product and spinfoam models

- Classical theory

- Kinematical Hilbert space

- Generalized projection operator

→ Formal definition

→ Regularization

- Topological invariance

Part I: Fermions in 3d Spinfoam Quantum Gravity

gr-qc/0609040 to appear in GRG

• Spinfoam models :

- Canonical : Spacetime history interpolating between spin network states ≡ physical scalar product in LQG (Rovelli,Reisenberger - 97; Perez - 06)
- Covariant : Discretized path integrals of a large class of BF-like field theories (Baez - 98/00; Freidel, Krasnov - 99)

• Spinfoams & Matter :

- Matter "allready there" (Crane - 00/01; Mikovic - 02; Livine, Oeckl - 04)
- Matter added by coupling

→ Point particles (Freidel, Louapre - 04; Noui, Perez - 04; Freidel, Kowalski-Glikman, Starodubtsev -06)

→ Feynman diagrams (Barrett - 05; Freidel, Livine - 05; Baratin, Freidel - 06)

→ Strings & branes (Baez, Perez - 06; WF, Perez -to appear)

→ YM fields (Oriti, Pfeiffer - 02; Mikovic - 03)

- Here : Fermionic fields in $3d$ spinfoam Riemannian QG:

$$\Rightarrow \text{Compute } \mathcal{Z} = \int \mathcal{D}e\mathcal{D}\omega\mathcal{D}\overline{\psi}\mathcal{D}\psi \, e^{iS[e,\omega,\overline{\psi},\psi]} \tag{1}$$

- Strategy:

\- Integrate out the fermions $\Rightarrow$ functional determinant

$$\mathcal{Z} = \int \mathcal{D}e\mathcal{D}\omega \; det \; (e,\omega) e^{iS[e,\omega]} \tag{2}$$

\- Expand the determinant $\Rightarrow$ sum over fermionic loops Γ

$$det \; (e,\omega) = \sum_{\Gamma} \Gamma(e,\omega) \;\; \Rightarrow \;\; \mathcal{Z} = \sum_{\Gamma} \mathcal{Z}(\Gamma) \tag{3}$$

\- Compute the integrals

$$\mathcal{Z}(\Gamma) = \int \mathcal{D}e\mathcal{D}\omega \; \Gamma(e,\omega) e^{iS[e,\omega]} \tag{4}$$

- Framework : Einstein-Cartan geometry (M, ω) + spin structure
- Action : $S = S_{GR} + S_D$ with

$$S_{GR}[e,\omega] = \frac{1}{2\kappa} \int_M < e \wedge R[\omega] >, \qquad (5)$$

and

$$S_D[e,\hat{\omega},\overline{\psi},\psi] = \frac{1}{2} \int_M \epsilon_{IJK} e^I \wedge e^J \wedge \wedge \left(\frac{i}{2} (\overline{\psi} \sigma(e^K) \nabla \psi - \nabla \overline{\psi} \sigma(e^K) \psi) - \frac{1}{3} m e^K \overline{\psi} \psi \right), \qquad (6)$$

- e soldering form

- ω metric connection ($\mathfrak{so}(3)$-valued), $\hat{\omega}$ spin connection ($\mathfrak{spin}(3)$-valued)

- $\nabla \equiv$ covariant derivative w.r.t. $\hat{\omega}$

- $\sigma : \mathcal{C}(\mathbb{R}^3) \to M_2(\mathbb{C})$, $\psi \in C^\infty(M) \otimes \mathbb{C}^2$.

• Pick a triangulation $\mathcal{T}$ of the spacetime manifold M :
$M = \cup_{k=1}^{n} \Delta_k^3$

• Consider the dual two-skeleton $\{v, e, f\} \subset \mathcal{T}^*$ of $\mathcal{T}$

• Field discretization :

$$\begin{aligned} e &\rightarrow e_f \in \mathbb{R}^3 \\ \hat{\omega} &\rightarrow U_e \equiv g_e \in Spin(3) \simeq SU(2) \\ \omega &\rightarrow \Lambda_e \equiv \Lambda_e(g_e) \in \overset{1}{\pi}(Spin(3)) \simeq SO(3) \\ R &\rightarrow \Lambda_f = \prod_{e \in \partial f} \Lambda_e \\ \psi^\alpha &\rightarrow \psi_v^\alpha \in \mathcal{G}_{\mathcal{T}}, \end{aligned} \tag{7}$$

Notation :

- $\mathcal{G}_{\mathcal{T}} \equiv$ Grassmann algebra $\bigwedge \left(E \bigoplus \bar{E}^*\right)$, $E = \oplus_v \mathbb{C}_v^2$
- $(\overset{j}{\pi}, \overset{j}{\mathbb{V}})$ spin j unitary, irreducible representation of $Spin(3)$

- Discretized action : $S_{\mathcal{T}} = S_{GR,\mathcal{T}} + S_{D,\mathcal{T}}$ with

$$S_{GR,\mathcal{T}}[e_f, g_e] = \sum_f Tr\left(e_f \Lambda_f\right), \tag{8}$$

and

$$S_{D,\mathcal{T}}[e_f, g_e, \overline{\psi}_v, \psi_v] = \frac{1}{2}\Big(\frac{i}{4}\sum_{uv} A_{I\,uv} \times \tag{9}$$

$$\times \left(\overline{\psi}_u\, \sigma^I U_{uv}\, \psi_v - \overline{\psi}_v\, U^{\dagger}_{uv}\, \sigma^I\, \psi_u\right) - \frac{1}{3}\sum_v m V_v \overline{\psi}_v \psi_v\Big),$$

where

$$A_{I\,uv} \equiv \sum_{f,f'\supset uv} \epsilon_{IJK} e^J_f e^K_{f'} sgn(f,f') \overset{n\to\infty}{\longrightarrow} \int_{\Delta^2=uv^*} (*e \wedge e)_I \tag{10}$$

$$V_v \equiv \sum_{f,f',f''\supset v} \epsilon_{IJK} e^I_f e^J_{f'} e^K_{f''} sgn(f,f',f'') \overset{n\to\infty}{\longrightarrow} \int_{\Delta^3=v*} e^{\wedge^3}$$

- The fermionic action can be compactly re expressed as

$$S_{D,\mathcal{T}}[e_f, g_e, \overline{\psi}_v, \psi_v] = (\psi, W\psi) \tag{11}$$
$$= \sum_{uv} \overline{\psi}_u W_{uv}(e_f, g_e)\psi_v \in \bigwedge{}^2(E \oplus \overline{E}^*).$$

- The (discretized) path integral yields

$$\mathcal{Z}(\mathcal{T}) = \prod_f \int_{\mathbb{R}^3} de_f \prod_e \int_{Spin(3)} dg_e \int_{\mathcal{G}_\mathcal{T}} \prod_v d\bar{\psi}_v d\psi_v \; e^{iS_\mathcal{T}} \tag{12}$$

- Integrating out the fermions, we obtain

$$\mathcal{Z}(\mathcal{T}) = \prod_f \int_{\mathbb{R}^3} de_f \prod_e \int_{Spin(3)} dg_e \,(det\; W(e_f, g_e))\, e^{iS_{GR,\mathcal{T}}} \tag{13}$$
$$= < det\; W >_\mathcal{T} \;.$$

- How to calculate the fermionic determinant ?

- Idea : write $W = V + \frac{1}{m}D$ and expand the determinant in $1/m$

$$
\begin{aligned}
det\ W &= \int_{\mathcal{G}_{\mathcal{T}}} \prod_v d\bar{\psi}_v d\psi_v\ e^{i(\psi, W\psi)} && (14)\\
&= \int_{\mathcal{G}_{\mathcal{T}}} \prod_v d\bar{\psi}_v d\psi_v\ e^{i(\psi, V\psi)} \sum_k \frac{1}{k!} (\frac{i}{m})^k (\psi, D\ \psi)^k \\
&:= \sum_\Gamma \Gamma(e_f, g_e)
\end{aligned}
$$

$\Rightarrow$ Finite expansion

- Accordingly, $\mathcal{Z}(\mathcal{T}) = \sum_\Gamma \mathcal{Z}(\Gamma, \mathcal{T})$ where

$$
\mathcal{Z}(\Gamma, \mathcal{T}) = \prod_f \int_{\mathbb{R}^3} de_f \prod_e \int_{Spin(3)} dg_e\ (\Gamma(e_f, g_e))\ e^{i \sum_f Tr(e_f \Lambda_f)} \qquad (15)
$$

• What is the structure of the contributions $\Gamma(e_f, g_e)$?
→ Typically, fermionic loops γ + decorations (= polynomial functions of the triad)

$$\begin{aligned} \Gamma(e_f, g_e) &:= \Gamma_\gamma(e_f, g_e) \\ &= \prod_{v \notin \gamma} V_v^2 \left[\prod_{v \in \gamma} V_v \prod_{e \in \gamma} A_{I_e, e} \, Tr_D \left(\prod_{e \in \gamma} \sigma^{I_e} U_e \right) \right] \end{aligned} \tag{16}$$

• How to compute the integrals over the triads in the path integral ?
→ Generating functional techniques (Freidel, Krasnov -99)

- Idea :

$$e_f = -i \left(\frac{\delta}{\delta J_f} \right) e^{i \left(\sum_f Tr(e_f J_f) \right)} \, |_{J=0} \, . \tag{17}$$

$\Rightarrow$ degree n monomial function of the triad $\mathcal{O}(e_f) \rightarrow$ differential operator $\hat{\mathcal{O}}(\frac{\delta}{\delta J_f})$ of order n:

$$< \mathcal{O}(e_f) > := (-i)^n \hat{\mathcal{O}}(\frac{\delta}{\delta J_f}) \, \left[\mathcal{Z}^{(I)}(J, \gamma, \mathcal{T}) \right]_{J=0} , \tag{18}$$

where

$$\mathcal{Z}^{(I)}(J, \gamma, \mathcal{T}) = \prod_f \int_{\mathbb{R}^3} de_f \prod_e \int_{Spin(3)} dg_e \times \tag{19}$$

$$\left[Tr_D \left(\prod_{e \in \gamma} \sigma^{I_e} U_e \right) \right] e^{i S_{\mathcal{T},J}[e_f, g_e, e^{J_f}]} ,$$

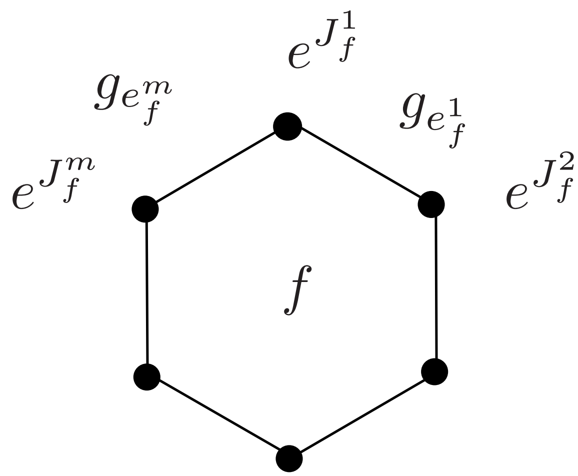
and the source action is given by :

$S_{\mathcal{T},J} = \sum_f Tr(e_f \Lambda_f e^{J_f}) \overset{n \rightarrow \infty}{\rightarrow} \int_M (< e \wedge R > + < e \wedge J >)$

- Integration over simplicial triads :

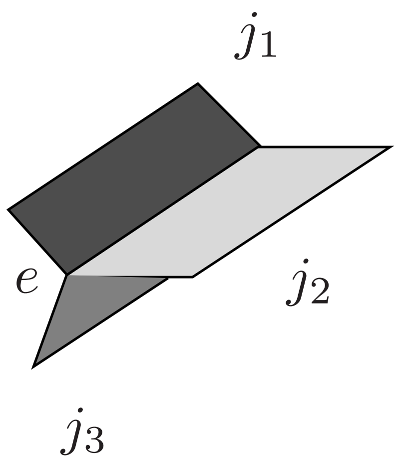
→ Idea : $\int de\, e^{iTr(eg)} = \delta(g) = \sum_j dim(j)\chi_{j_f}(g)$

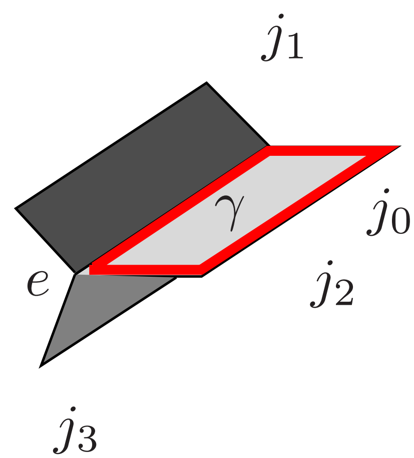
$$\mathcal{Z}^{(I)}(J,\gamma,\mathcal{T}) = \prod_e \int_{Spin(3)} dg_e \left[Tr_D\, (\prod_{e\in\gamma} \sigma^{I_e} U_e) \right] \times \prod_f \sum_{j_f} dim(j_f)\chi_{j_f}(e^{J_f^1} g_{e_f^1} ... e^{J_f^m} g_{e_f^m}), \quad (20)$$



• Integration over discretized connections :

→ Two cases : $e \notin \gamma$ and $e \in \gamma$



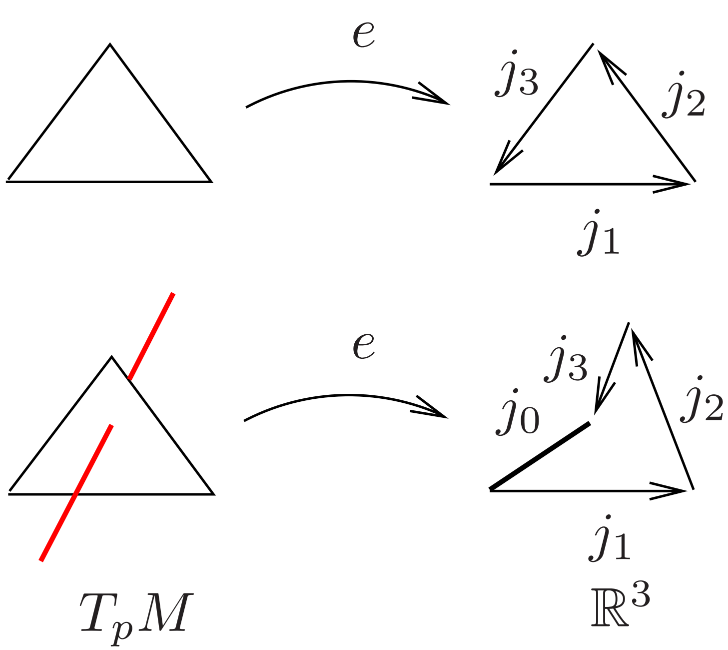


-If $e \notin \gamma$: $\int_{Spin(3)} dg \ \overset{j_i}{\pi}(g)^{\otimes_{i=1}^{3}} = \iota^{\dagger}\iota$,

-If $e \in \gamma$: $\int_{Spin(3)} dg \ \overset{j_i}{\pi}(g)^{\otimes_{i=0}^{3}} = \sum_s \iota_s^{\dagger}\iota_s$,

where $\iota \in Hom(\overset{j_i}{\mathbb{V}}^{\otimes_{i=0,1}^{3}}, \mathbb{C})$, $j_0 = 1/2$ and $\overset{j_0}{\pi}(g) = U$

- This is the effect of quantum mechanical torsion :



$$\equiv \quad \iota \in Hom(\overset{j_i}{\mathbb{V}}{}^{\otimes_{i=1}^{3}}, \mathbb{C})$$

$$\equiv \quad \iota \in Hom(\overset{j_i}{\mathbb{V}}{}^{\otimes_{i=0}^{3}}, \mathbb{C})$$

⇒ Dislocation of quantum spacetime ⇔ deformation of the (vacuum) quantum triangles (Baez, Barrett -99)

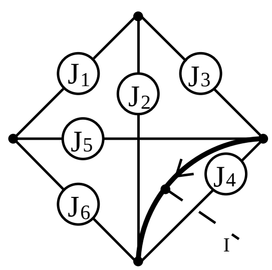
- In addition, each group element U_e comes attached with a sigma matrix σ^{I_e}

$\Rightarrow$ Rethink the Pauli matrices as elements of $Hom(\overset{1}{\mathbb{V}}, \overset{1/2}{\mathbb{V}} \otimes \overset{1/2}{\mathbb{V}}{}^{*})$

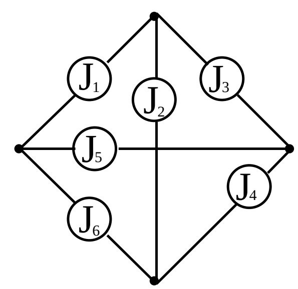
- Result :

$$\mathcal{Z}^{(I)}(J,\gamma,\mathcal{T}) = \prod_{f} \sum_{j_f} dim(j_f) \times \tag{21}$$

$$\left[\prod_{e\in\gamma}\sum_{j_e}\prod_{v\in\gamma}\left\{\sqrt{6}\right.\right.$$



$$\left.\left.\right\}_v\right]\prod_{v\notin\gamma}\left\{\right.$$



$$\left.\right\}_v$$

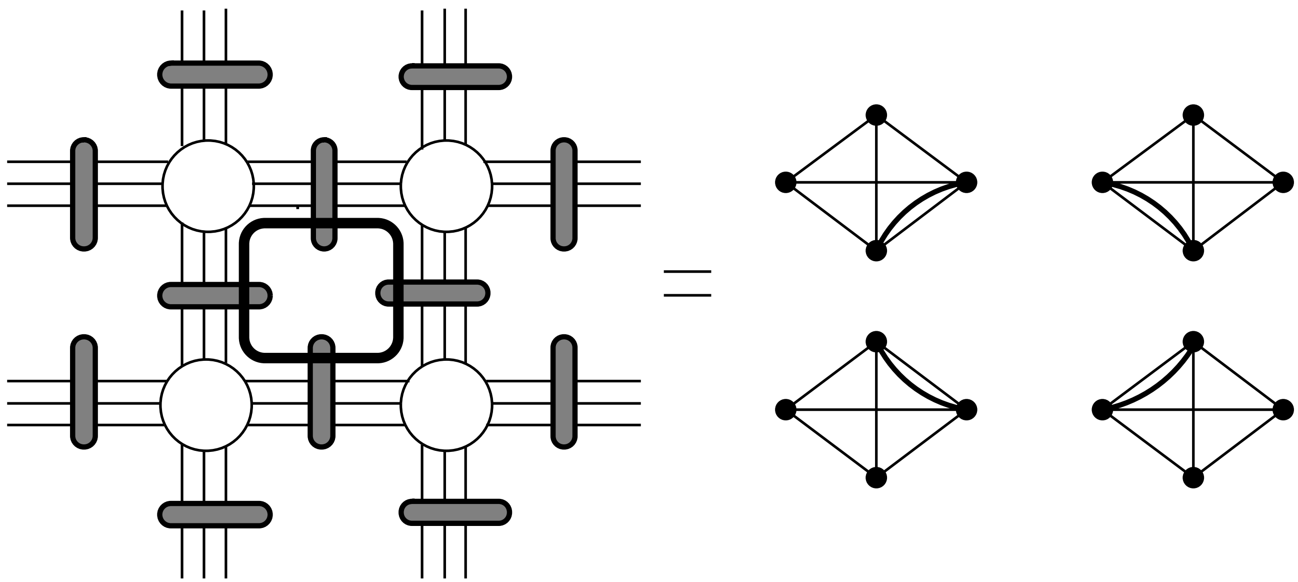


Figure 1: Graphical picture of the emergence of the fermionic seven-j symbol.

• The weight associated to the contribution Γ in the total partition function $\mathcal{Z}(\mathcal{T})$ is given by

$$\mathcal{Z}(\Gamma,\mathcal{T}) = \left[\prod_{v\notin\gamma} \hat{V}_v^2 \prod_{v\in\gamma} \hat{V}_v \prod_{e\in\gamma} \hat{A}_{I_e,e} \left(\mathcal{Z}^{(I)}(J,\gamma,\mathcal{T})\right)\right]_{J=0} . \tag{22}$$

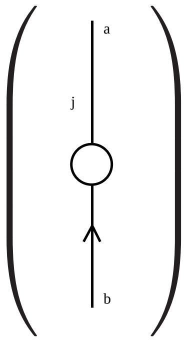
• How do the differential operators $\hat{\mathcal{O}} := \hat{\mathcal{O}}(\frac{\delta}{\delta J})$ act on the generating functional ?

$$\rightarrow \text{If } J = J^I T_I, \quad \frac{\delta}{\delta J_I} \overset{j}{\pi}(e^J)\mid_{J=0} = \overset{j}{\pi}_* (T^I), \tag{23}$$

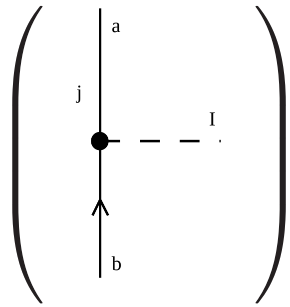
where $\overset{j}{\pi}_* \in Hom(\overset{1}{\mathbb{V}}, \overset{j}{\mathbb{V}} \otimes \overset{j}{\mathbb{V}}{}^*)$

- Graphically :

$$\frac{\delta}{\delta J_I}$$



$$\Big|_{J=0} = \Theta(j)$$



$$. \qquad (24)$$

- Introducing

$$\hat{V}_v = N' \sum_{f,f',f'' \supset v} \epsilon_{IJK} \frac{\delta}{\delta J_{fI}} \frac{\delta}{\delta J_{f'J}} \frac{\delta}{\delta J_{f''K}} sgn(f, f', f''), \qquad (25)$$

and

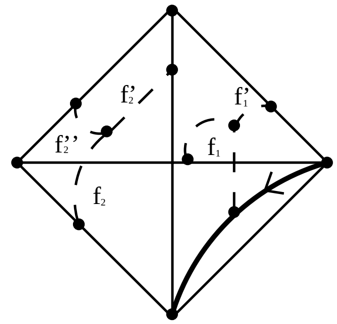
$$\hat{A}_{I\,uv} = N \sum_{f,f' \supset uv} \epsilon_{IJK} \frac{\delta}{\delta J_{Jf}} \frac{\delta}{\delta J_{Kf'}} sgn(f, f'), \qquad (26)$$

and using $\epsilon_{IJK} \in Hom(\overset{1}{\mathbb{V}} \otimes \overset{1}{\mathbb{V}} \otimes \overset{1}{\mathbb{V}}, \mathbb{C})$,

- We obtain $\mathcal{Z}(\mathcal{T}) = \sum_\Gamma \mathcal{Z}(\Gamma, \mathcal{T})$ with

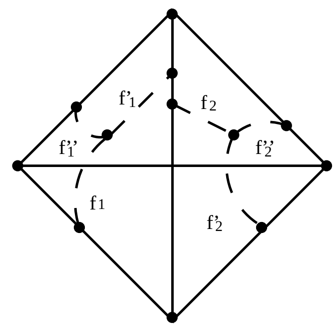
$$\mathcal{Z}(\Gamma, \mathcal{T}) = \prod_f \sum_{j_f} dim(j_f) \times \tag{27}$$

$$\left[\prod_{e \in \gamma} \sum_{j_e} \prod_{v \in \gamma} \sum_{grasp} K(v) \left\{ \vphantom{\prod} \right.\right.$$



$$\left.\left. \vphantom{\prod} \right\}_v \right] \times$$

$$\prod_{v \notin \gamma} \sum_{grasp} K'(v) \left\{ \vphantom{\prod} \right.$$



$$\left. \vphantom{\prod} \right\}_v .$$

Part II: Quantisation of string-like sources coupled to BF theory : Physical scalar product and spinfoam models

Work in progress in collaboration with Alejandro Perez (CPT Marseille)

• Idea : generalize the coupling of point particles to $3d$ gravity to higher dimensions

$\Rightarrow$ Natural sources $\equiv$ strings & branes (Baez, Perez -06)

• Action : S $= S_{BF} + S_{int}$

$$S_{BF}[B,\omega] = \int_M < B \wedge F[\omega] > \qquad (28)$$

$$S_{int}[\omega, B, q, \lambda] = \tau \int_W < (B + Dq), \lambda v \lambda^{-1} > \qquad (29)$$

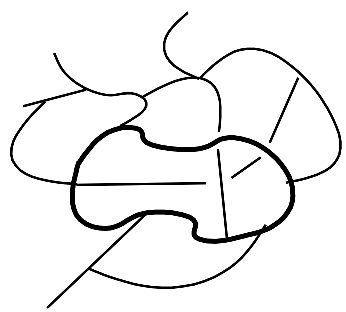
- B : $\mathfrak{g}$-valued two-form
- ω : $\mathfrak{g}$-valued connection one-form
- $D \equiv$ covariant derivative w.r.t. ω
- W : string worldsheet
- q : $\mathfrak{g}$-valued one-form
- $p := Ad_\lambda(v) \in C^\infty(W, \mathfrak{g})$, $\lambda \in C^\infty(W, G)$

• Canonical Quantization : $M = \Sigma \times \mathbb{R}$, $\mathcal{S} = \Sigma \cap W$

$\rightarrow$ Quantum kinematical constraints $\Rightarrow$ $\mathcal{H}_{kin}$ spanned by string spin network states (inspired by Thiemann - 97)

string spin network state $\equiv$ open graph with open edges ending on the string $\mathcal{S}$:

- edges and endpoints coloured by unitary, irreducible representations of G
- vertices (including the endpoints) coloured by intertwining operators :



$\rightarrow$ Hamiltonian constraint $\Rightarrow$ $\mathcal{H}_{phys}$?

- Problem : solve the dynamics
- Idea (Rovelli,Reisenberger - 97): introduce the rigging map

$$\eta_{\rm phys} : Cyl \rightarrow Cyl^* \tag{30}$$

whose range lies in the kernel of the Hamiltonian constraint :
$\forall \Psi \in \mathcal{H}_{kin}$, $\eta_{\rm phys}(\Psi) := \delta(\hat{H})\Psi \equiv \hat{P}\Psi$ with the Hamiltonian given by:

$$\hat{H}^i_a(x) = \epsilon^{ijk}\hat{F}_{ajk}(x) + \int_{\mathcal{S}} ds \dot{x}^i(s) < X_a, p > \; \delta^{(3)}(x - x_{\mathcal{S}}(s)), \tag{31}$$

where $\mathfrak{g} = \mathbb{R}\{X_a\}_a$, $F \equiv$ curvature of the spatial connection A, and the generalized projection operator $\hat{P}$ defined by

$$\hat{P} \equiv \delta(\hat{H}) = \prod_{x\in\Sigma} \delta(\hat{H}(x)) = \int_{\mathcal{N}} \mathcal{D}N \; exp\left(i \int_{\Sigma} < N \wedge \hat{H} > \right), \tag{32}$$

- Physical inner product :

$$< \eta_{\rm phys}(\Psi_1), \eta_{\rm phys}(\Psi_2) >_{\rm Phys} := < \hat{P}\, \Psi_1, \Psi_2 > := [\eta_{\rm phys}(\Psi_1)](\Psi_2), \tag{33}$$

for any two string spin network states Ψ_1, $\Psi_2 \in \mathcal{H}_{kin}$.

- Interesting duality :

$$\int_\Sigma < N \wedge H > \;=\; \int_\Sigma < N \wedge F > + \tau \int_{\mathcal{S}} < N, p > \tag{34}$$

$$= \; S^{3d}_{BF+part}[N, A; \tau], \tag{35}$$

If $G = SO(3)$ $\equiv$ action of $3d$ gravity coupled to a (spinless) point particle (Freidel, Louapre - 04) :

-$N \longrightarrow e$

-mass and worldline of the particle $\longrightarrow$ τ and $\mathcal{S}$

• Regularization of the physical inner product $< \hat{P}\,\Psi_1, \Psi_2 > \equiv$ express P in terms of basic observables of the theory

$\rightarrow$ Pick a triangulation $\mathcal{T}(\Gamma)$ adapted to the graph $\Gamma = \Gamma_1 \cup \Gamma_2$ and consider the dual two-skeleton $\{v, e, f\}$:

-$\mathcal{S} \in \{\Delta^1\} \subset \mathcal{T}(\Gamma)$

-$\Gamma \in \{v, e\} \subset \mathcal{T}^*(\Gamma)$

$\rightarrow$ discretize the fields à la Ponzano-Regge coupled to point particles (Freidel, Louapre - 04):

$$\begin{aligned} N &\rightarrow N_f \in \mathbb{R}^3 \\ F &\rightarrow h_f = \prod_{e \in \partial f} h_e \\ \tau p &\rightarrow p_f = \lambda(x_f) e^{\tau v} \lambda(x_f)^{-1},\ x_f = \mathcal{S} \cap f \end{aligned} \tag{36}$$

- Accordingly,

$$< \; P[\mathcal{T}(\Gamma)] \; \Psi_1, \Psi_2 > = < \left[\prod_{f \notin \mathcal{F}_{\mathcal{S}}} \delta(h_f) \prod_{f \in \mathcal{F}_{\mathcal{S}}} \delta(h_f \lambda_f e^{\tau v} \lambda_f^{-1}) \right] \Psi_1, \Psi_2 >, \tag{37}$$

where $\mathcal{F}_{\mathcal{S}}(\square) = \{f \in \partial\square \mid f \cap \mathcal{S} \neq \emptyset\}$

- $P[\mathcal{T}(\Gamma)]$ promoted to a well defined operator $\hat{P}[\mathcal{T}(\Gamma)]$:

-$\delta(g) = \sum_\rho dim(\rho)\chi_\rho(g)$

-$\chi_\rho(g) \rightarrow$ Wilson loop operator $\hat{\chi}_\rho(g)$

$\Rightarrow$ Prescription to compute background independent transition amplitudes of string-like currents coupled to BF theory

• Removal of the regulator : we show that
$\forall \Psi_1, \Psi_2 \in \mathcal{H}_{kin}$,

$$< P[\mathcal{T}(\Gamma)]\, \Psi_1, \Psi_2 >=< P[\mathcal{T}'(\Gamma)]\, \Psi_1, \Psi_2 >, \tag{38}$$

forall homeomorphically equivalent triangulations $\mathcal{T}(\Gamma)$ and $\mathcal{T}'(\Gamma)$ adapted to the graph Γ.

$\Rightarrow$ The transition amplitudes are topological invariants and accordingly independent of the triangulation

• Construction of a spinfoam model of massive fermionic fields coupled to $3d$ Riemannian QG $\Leftrightarrow$ computation of the path integral of the coupled system

$\rightarrow$ Discretization of the manifold and of the fields

$\rightarrow$ Integration over the fermions $\Rightarrow$ path integral for QG with insertion of an observable $\equiv$ functional determinant

$\rightarrow$ Finite expansion of the determinant in fermionic loops

$\rightarrow$ Computation of the integrals order by order

• Open issue: local degrees of freedom $\equiv$ dependence on the triangulation

$\Rightarrow$ GFT (Krasnov -05; Freidel, Oriti, Ryan -05; Oriti, Ryan -06)

• Perspectives :

$\rightarrow$ Asymptotics ($j \rightarrow \infty$) ?

$\rightarrow$ Generalization to four dimensions

$\rightarrow$ Computation of the fermionic propagator in 3d spinfoam QG

• Construction of the physical scalar product of BF theory coupled to string-like sources $\Leftrightarrow$ definition of the generalized projection operator
$\rightarrow$ Regularization
$\rightarrow$ Removal of the regulator

• Open issues and perspectives:
- Compute the associated spinfoam vertex amplitude
- Physical interpretation of the theory and its link to conventional string theory