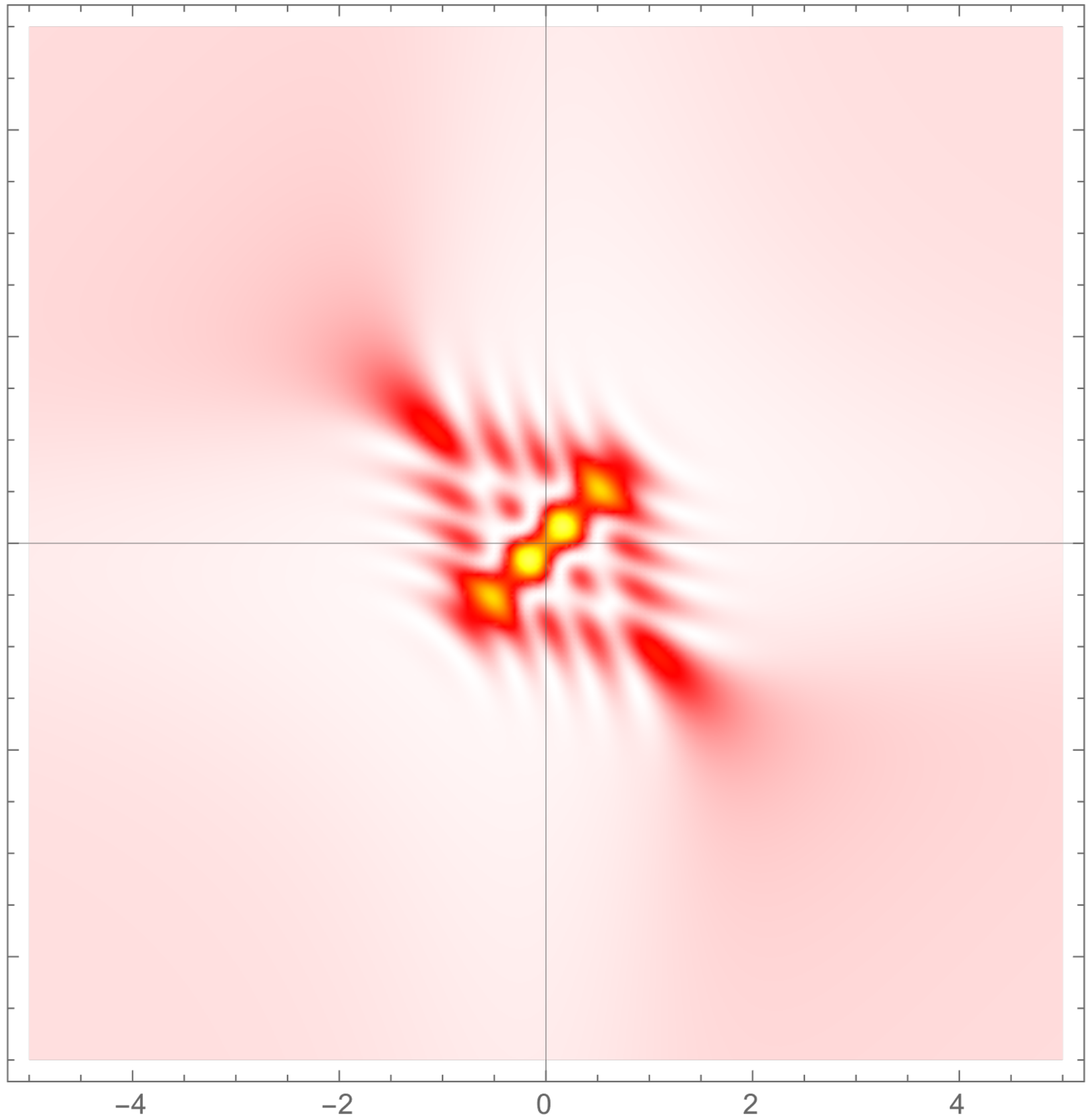
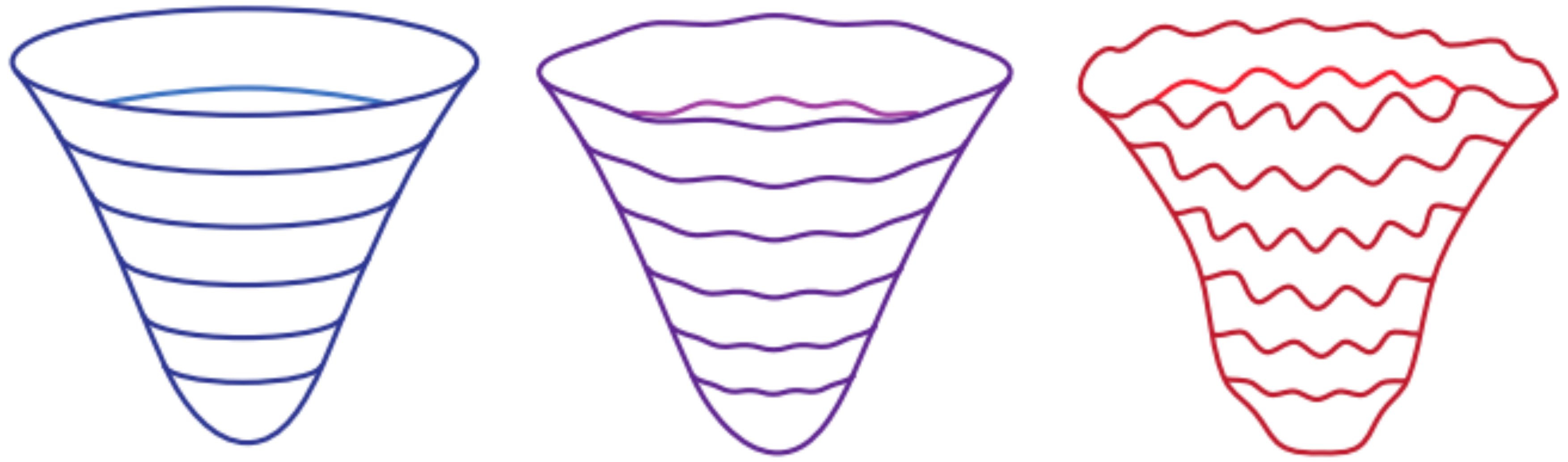
Complex saddle points in gravitational path integrals





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Work in collaboration with
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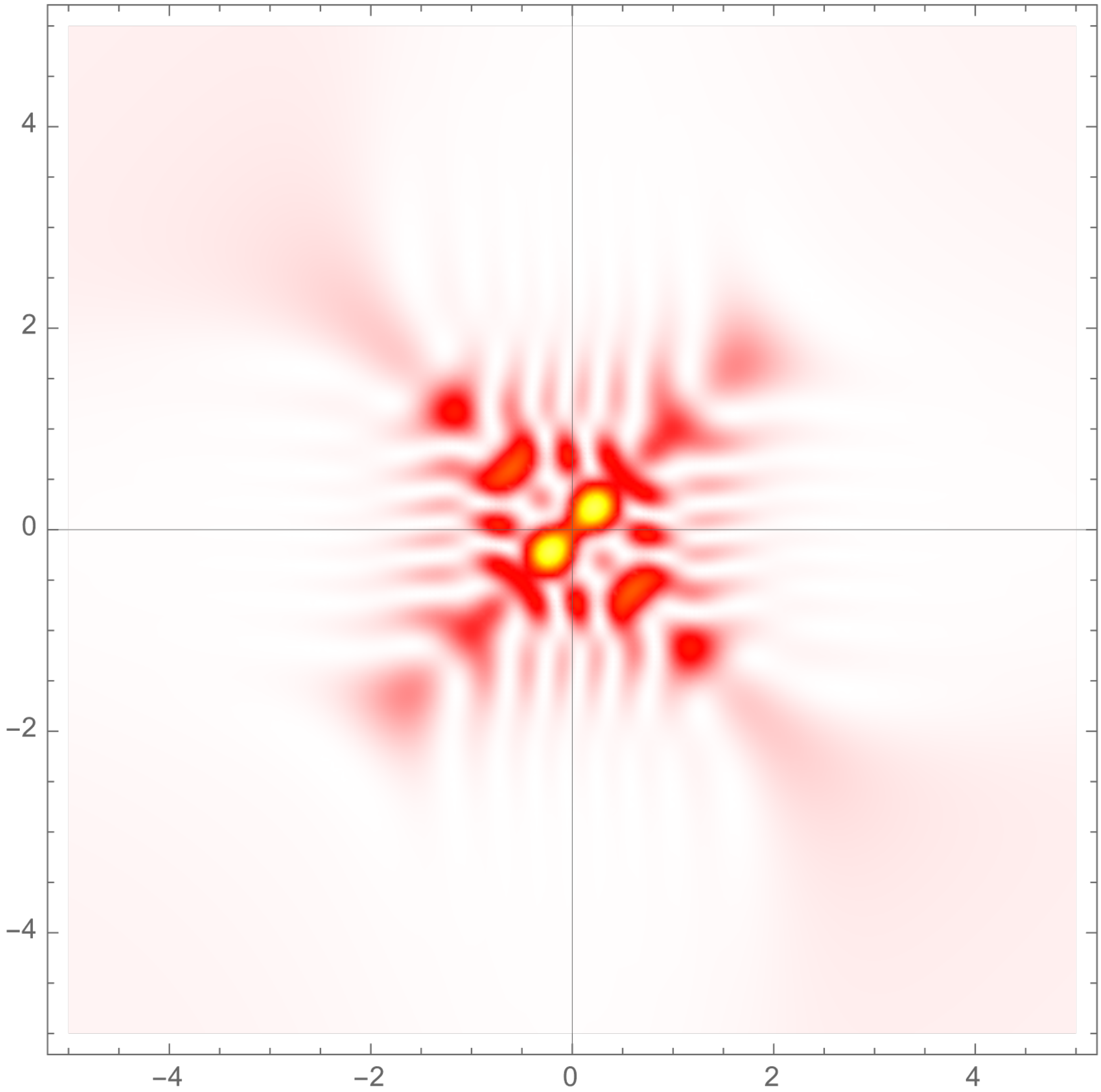
arXiv: 2207.12798, 2309.12420 and 2309.12427



Complex saddle points?

There are roughly two paths to quantum gravity

Canonical Quantization

Wheeler-DeWitt equation

$$\hat{\mathscr{H}}_0 \Psi[\mathscr{G}^{(3)}] = 0 \qquad \hat{\mathscr{H}}_i \Psi[\mathscr{G}^{(3)}] = 0$$

S_i

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Trans-series in perturbation theory (resurgence)

$$\Psi = \sum_i e^{iS_i/\hbar} \sum_{n=0}^{\infty} c_n^{(i)} \hbar^n$$

Resurgence theory: The perturbative solution of a differential equation assumes the form of a trans-series, where S_i is the Einstein-Hilbert action of a classical spacetime.

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Path Integral Quantization

Path integral over spacetimes

$$K[\mathscr{G}_1^{(3)}, \mathscr{G}_0^{(3)}] = \int_0^\infty \int_{\mathscr{G}_0^{(3)}}^{\mathscr{G}_1^{(3)}} e^{iS_{EH}[\mathscr{G};N]/\hbar} \mathscr{D}\mathscr{G} \mathrm{d}N$$

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Picard-Lefschetz theory: The highly oscillatory integral is dominated by constructed interference at saddle points of the Einstein-Hilbert action, solving the boundary value problem

$$\frac{\delta S_{EH}}{\delta \mathscr{G}} = 0 \mapsto G_{\mu\nu} = \kappa T_{\mu\nu}$$

Complex saddle points?

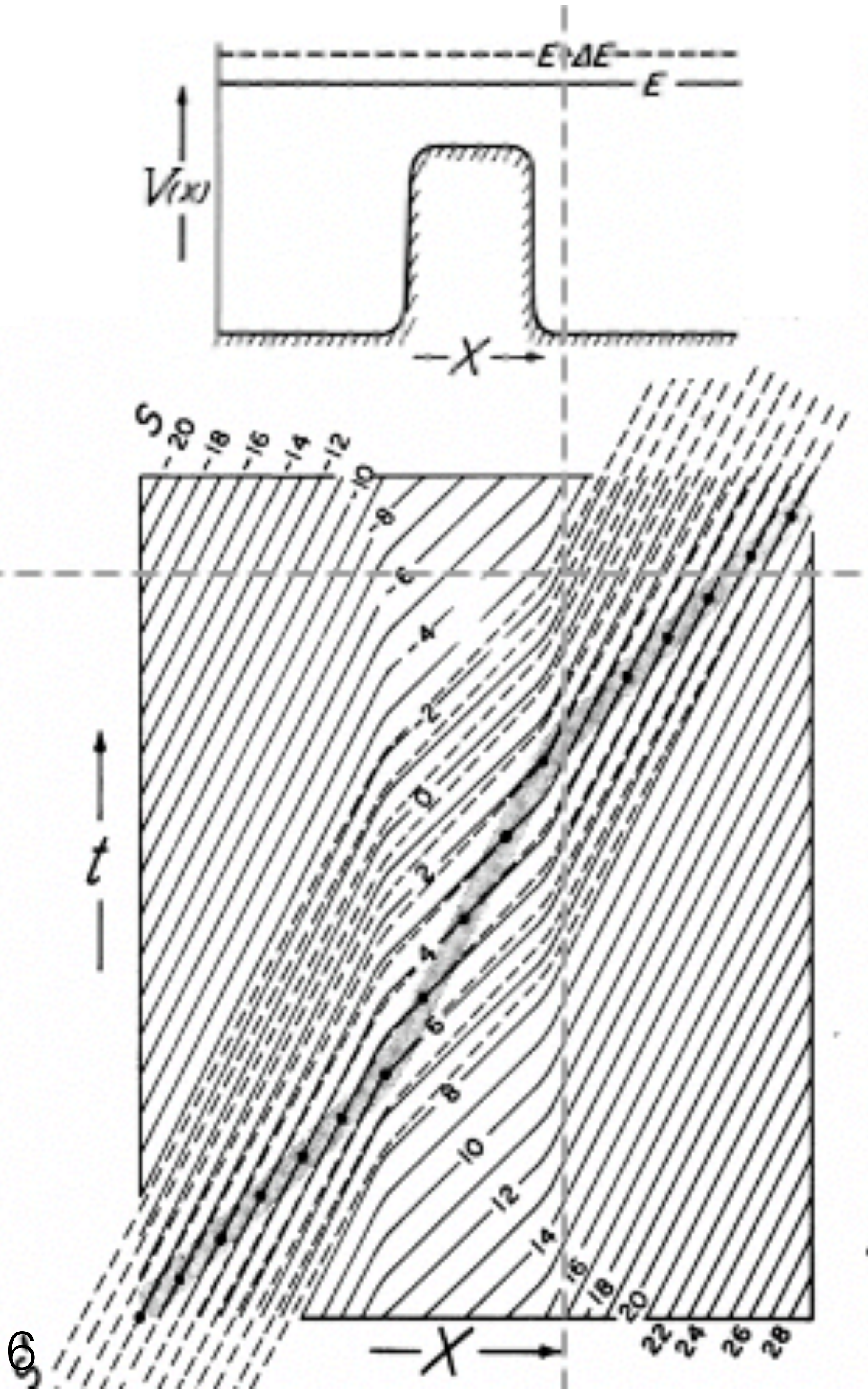
In **quantum gravity**, the **path integral for gravity** has influenced many explorations

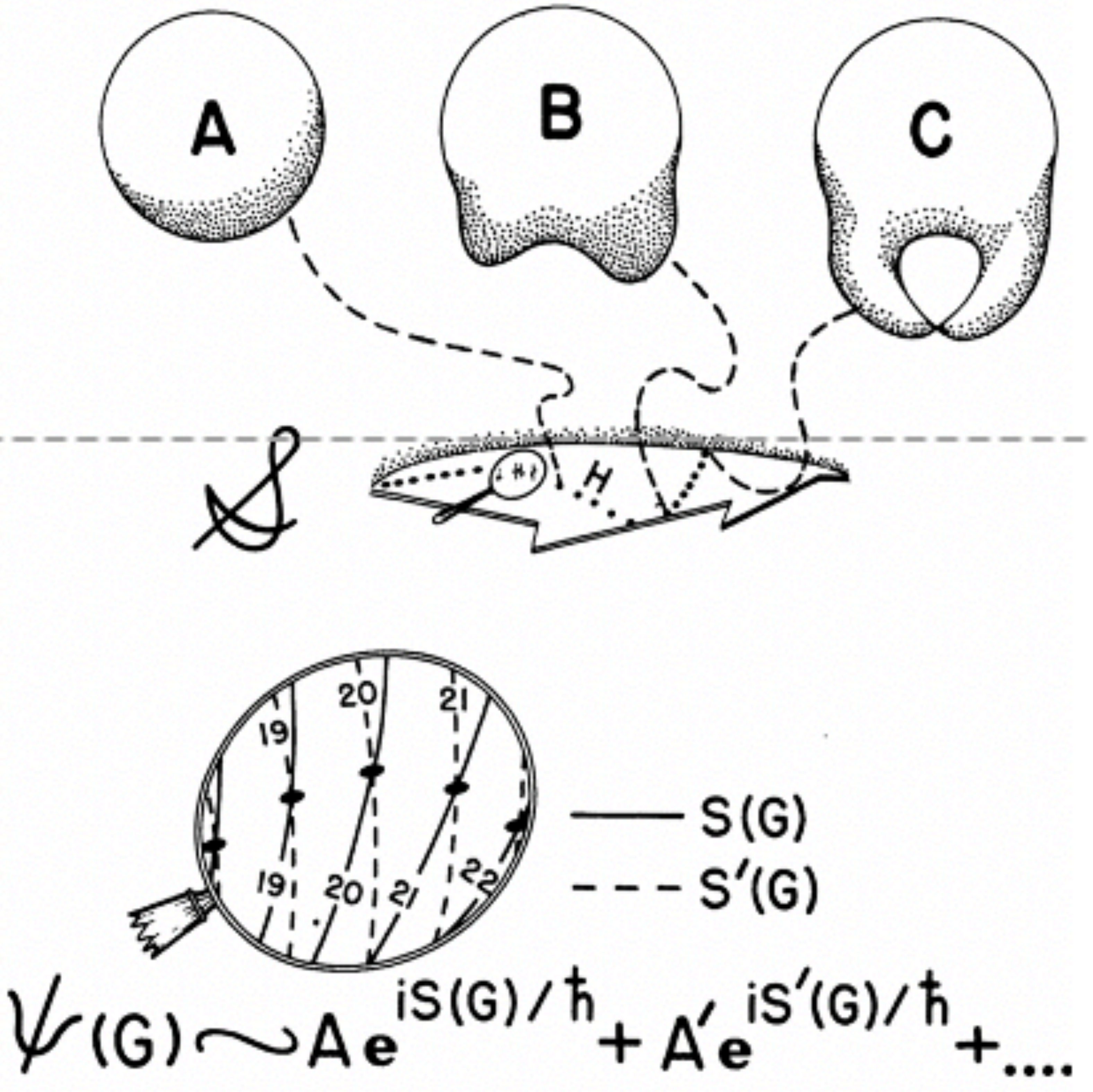
Wheeler: A classical trajectory emerges as an interference phenomena in quantum mechanics. Classical spacetime spacetime may emerge as an interference effect in superspace

- The path integral over spacetimes

$$K[\mathcal{G}_1^{(3)}, \mathcal{G}_0^{(3)}] = \int_0^\infty \int_{\mathcal{G}_0^{(3)}}^{\mathcal{G}_1^{(3)}} e^{iS_{EH}[\mathcal{G};N]/\hbar}\, \mathcal{D}\mathcal{G}\,\mathrm{d}N$$

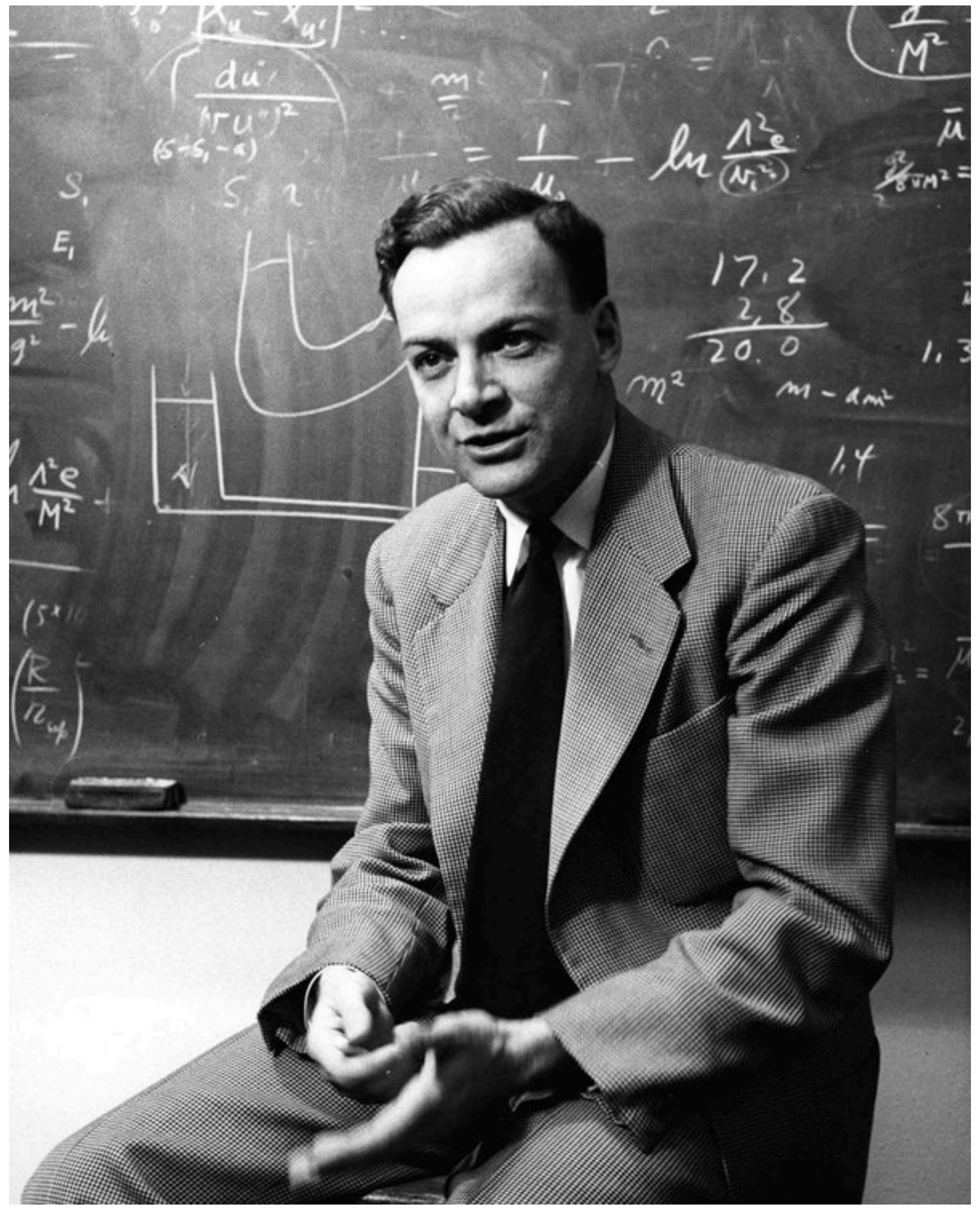
Classical spacetime emerges from **real saddle points**. **Quantum transitions** follow from **relevant complex saddle points.**

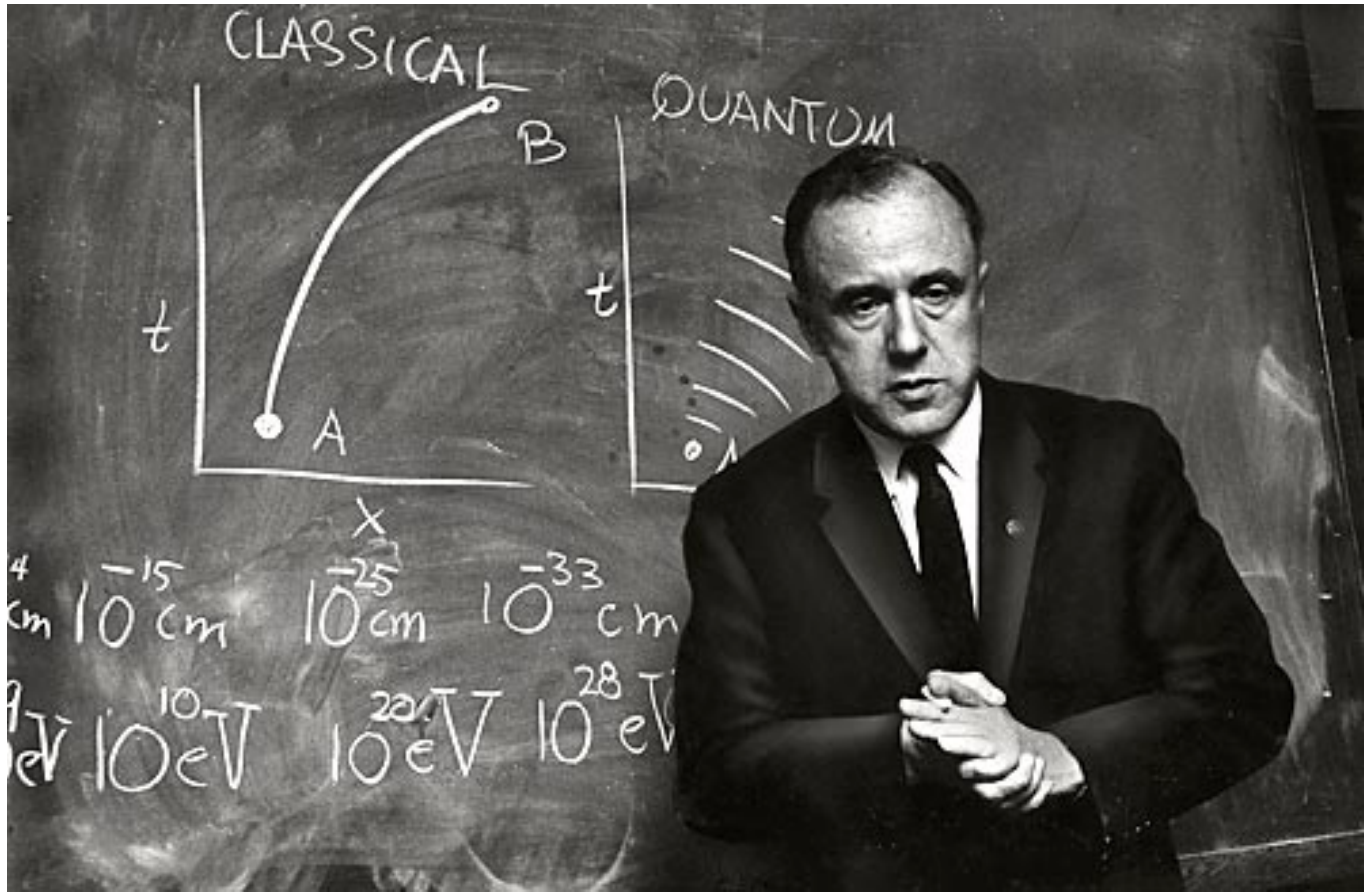




Overview

1. Oscillatory integrals and Picard-Lefschetz theory
2. Feynman sums over histories and the Wiener measure
3. Caustics, Stoke’s transitions and quantum tunnelling





Interference

Interference

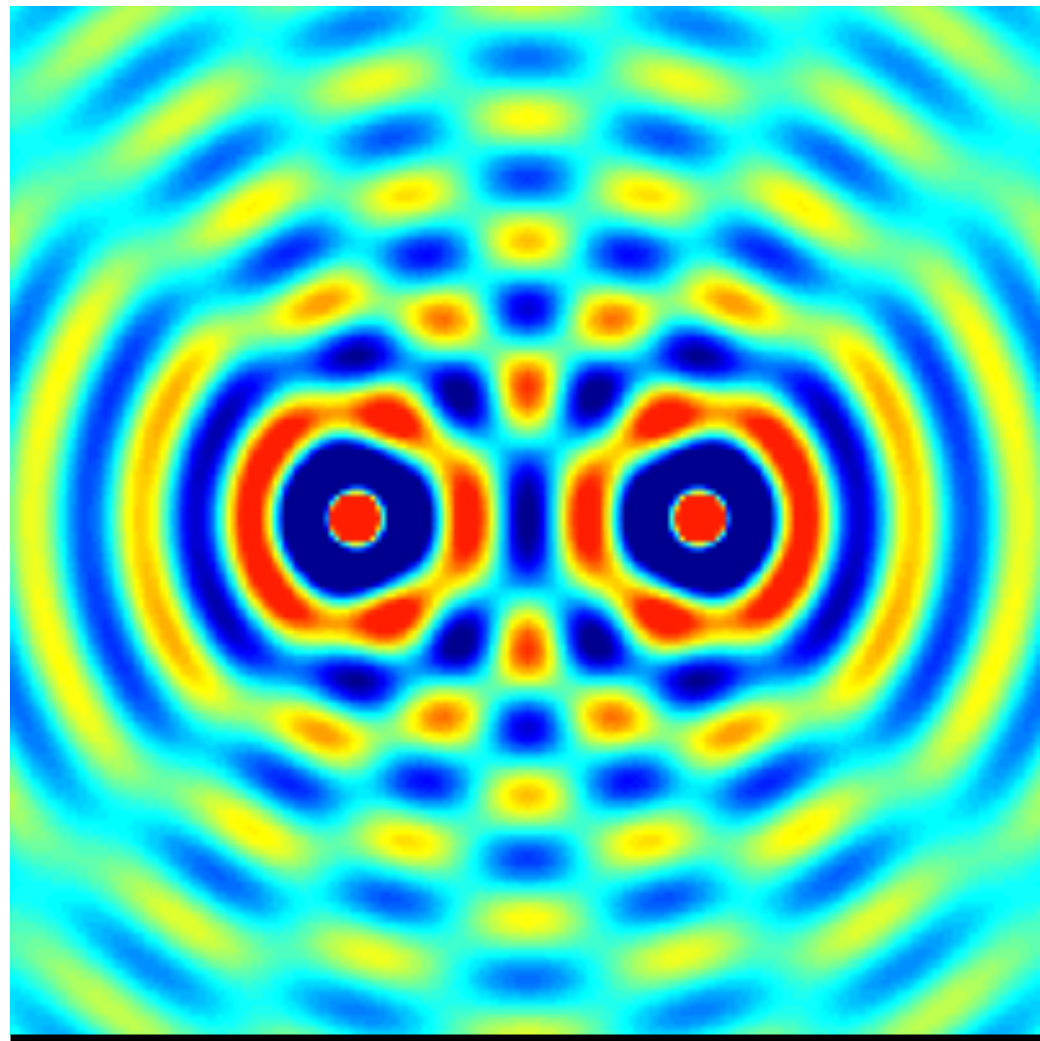
- Interference is among the **most universal phenomena in physics**

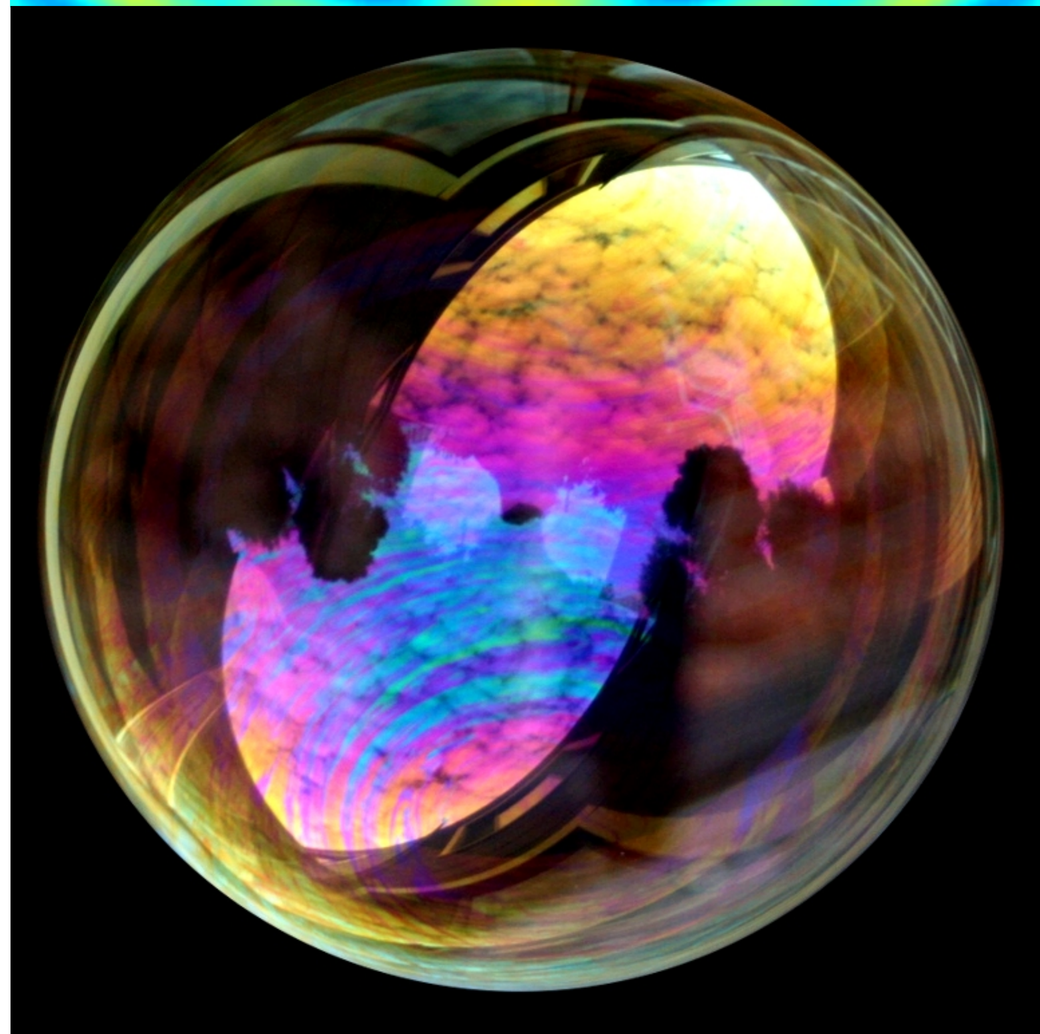
- Unfortunately, the associated multi-dimensional highly oscillatory integrals are generally **delicate to define and expensive to evaluate**

$$\Psi(y) = \int_{-\infty}^{\infty} e^{if(x,y)} \mathrm{d}x$$

- Conditionally convergent integrals need to be regulated with care

- **Picard-Lefschetz theory** formalizes saddle point methods and solves these problems





Conditional convergence

Alternating sums occur in many places, ranging from classical systems, and wave optics, to quantum physics

- Absolutely v.s. conditionally convergent sums

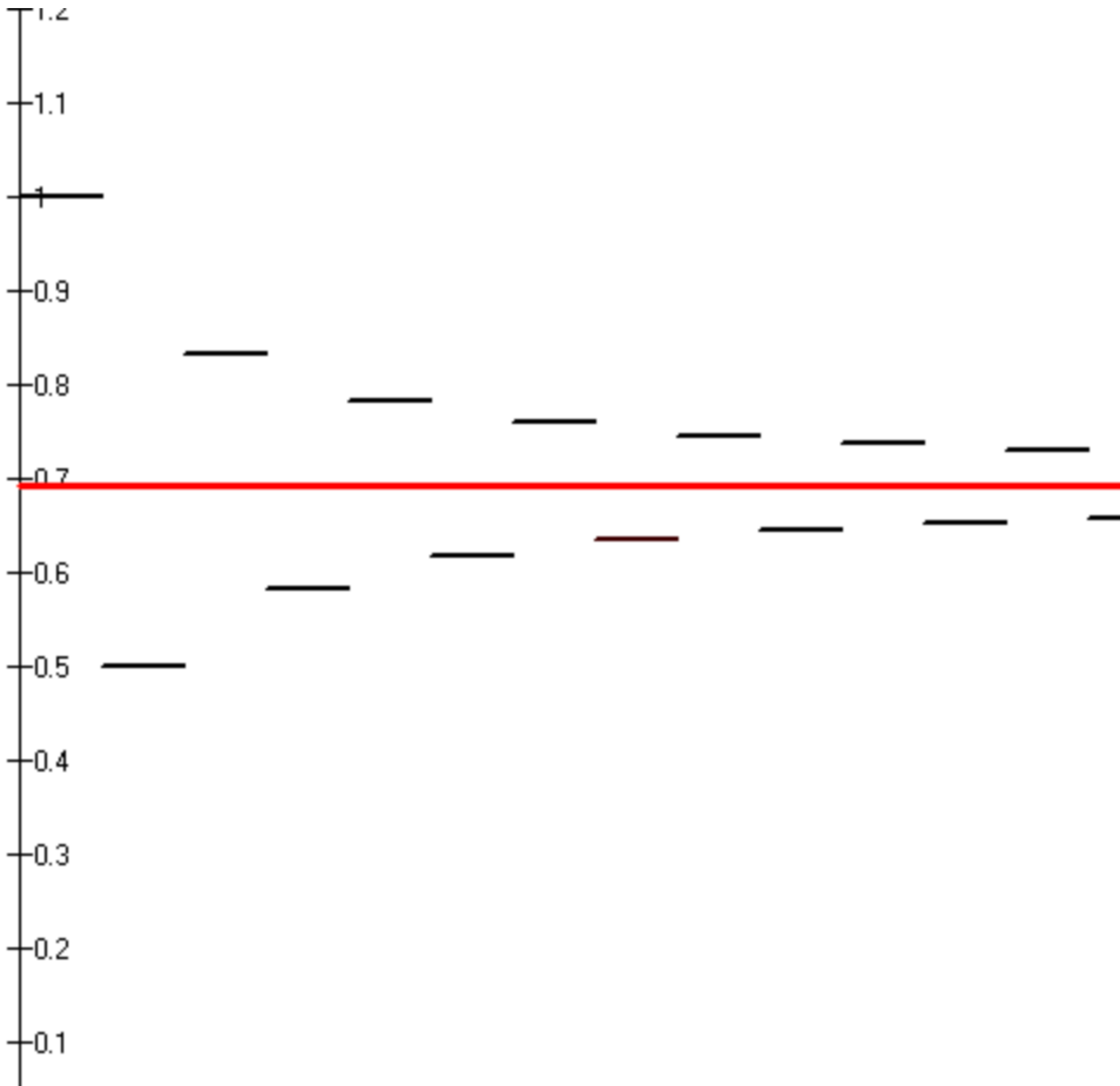
$$S = \sum_{i=1}^{\infty} a_i \qquad\qquad \sum_{i=1}^{\infty} \ a_i \ < \infty$$

- Conditional series depend on the ordering

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2$$

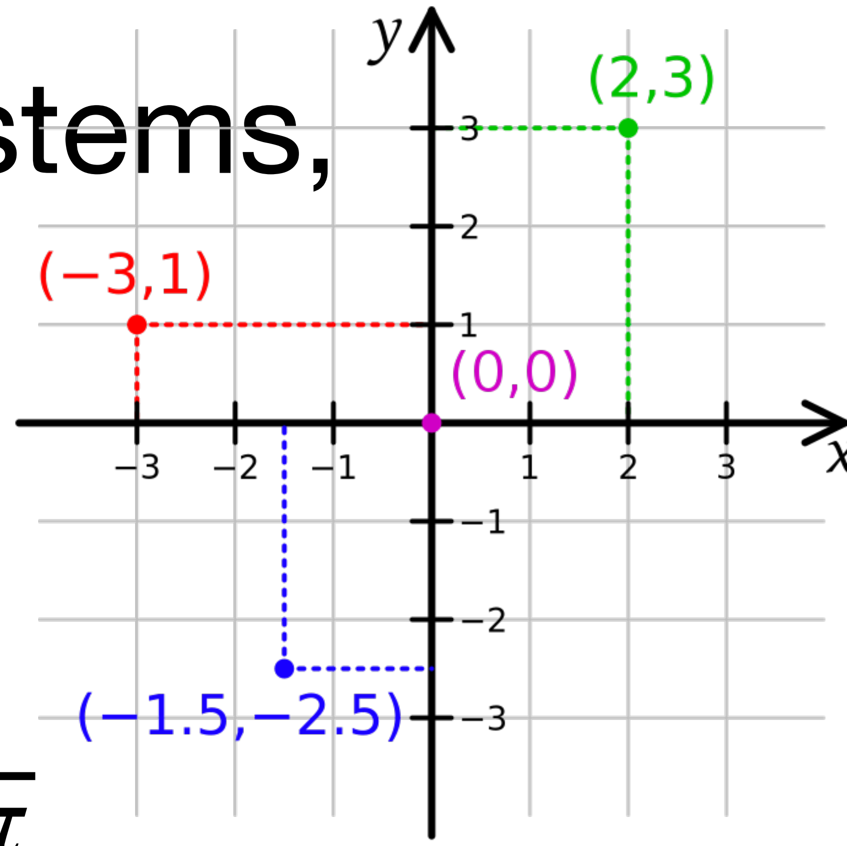
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots\right) = \frac{1}{2}\ln 2$$



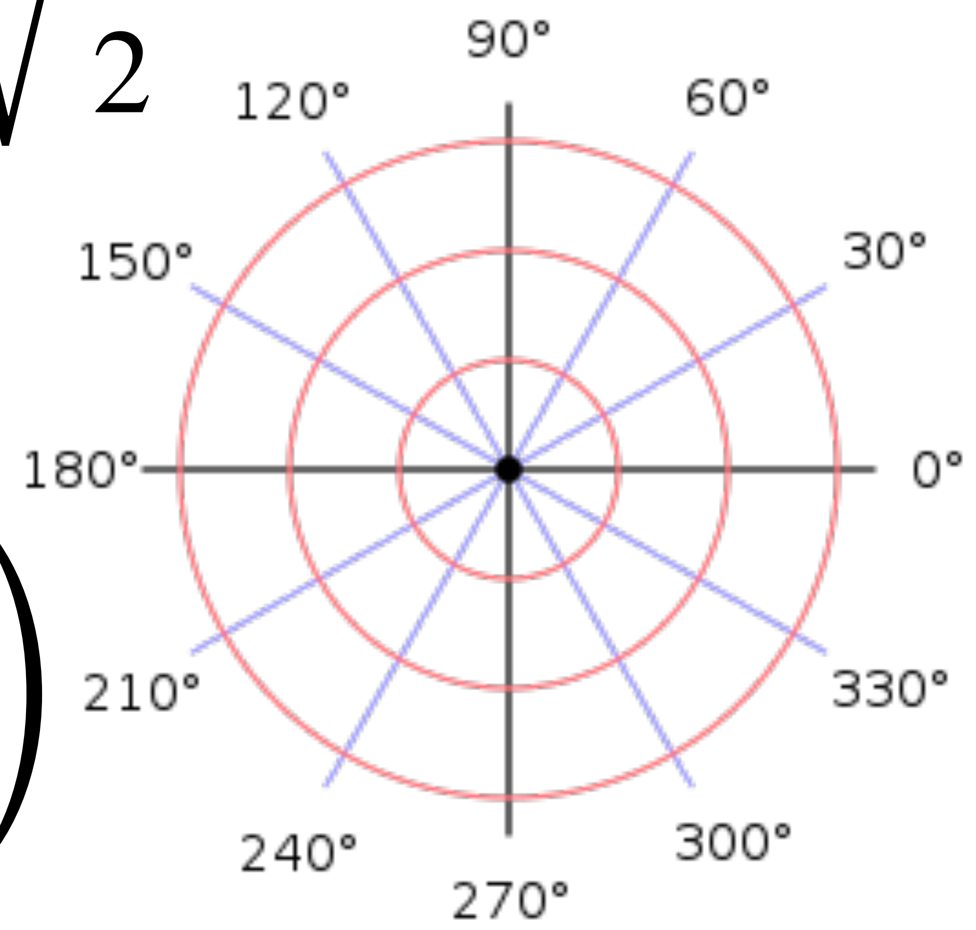
Oscillatory integrals

Oscillatory integrals occur in many places, ranging from classical systems, wave optics, to quantum physics



- Fresnel integral can be defined with the typical regularization

$$\int_{-\infty}^{\infty} e^{ix^2}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} e^{ix^2}\mathrm{d}x = -(1+i)\sqrt{\frac{\pi}{2}}\lim_{R\to\infty}\mathbf{erf}\left(\frac{i-1}{\sqrt{2}}R\right) = (1+i)\sqrt{\frac{\pi}{2}}$$



- Higher dimensional generalizations run into problems

$$\int\prod_{l=1}^{N} e^{iy_l^2}\mathrm{d}y_l = \lim_{R\to\infty}\frac{2\pi^{N/2}}{\Gamma(N/2)}\int_0^R e^{ir^2} r^{N-1}\mathrm{d}r = \lim_{R\to\infty}(i\pi)^{N/2}\left(1-\frac{\Gamma(N/2,-iR^2)}{\Gamma(N/2)}\right)$$

oscillates around the box cutoff regulator for N = 2 and diverges for N > 2.

Oscillatory integrals

Oscillatory integrals occur in many places, ranging from classical systems, wave optics, to quantum physics

- Absolutely v.s. conditionally convergent integrals

$$I = \int f(x)\mathrm{d}x \qquad \int f(x)\ \ \mathrm{d}x < \infty$$

$$\boxed{\int_{-\infty}^{\infty} e^{if(x)}\,\mathrm{d}x \quad \text{or} \quad \int_{x(0)=x_0}^{x(1)=x_1} e^{iS[x(t)]}\,\mathscr{D}x(t)}$$

- Fubini's theorem

$$\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int \left[\int f(x,y)\,\mathrm{d}x\right]\mathrm{d}y = \int \left[\int f(x,y)\,\mathrm{d}y\right]\mathrm{d}x$$

- Dominated convergence theorem

$$\lim_{n\to\infty}\left[\int f_n(x)\mathrm{d}x\right] = \int \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x \quad \text{when} \quad f_n(x)\ \ \le g(x) \quad \forall n \quad \text{with} \quad \int g(x)\ \ \mathrm{d}x < \infty$$

Saddle point methods

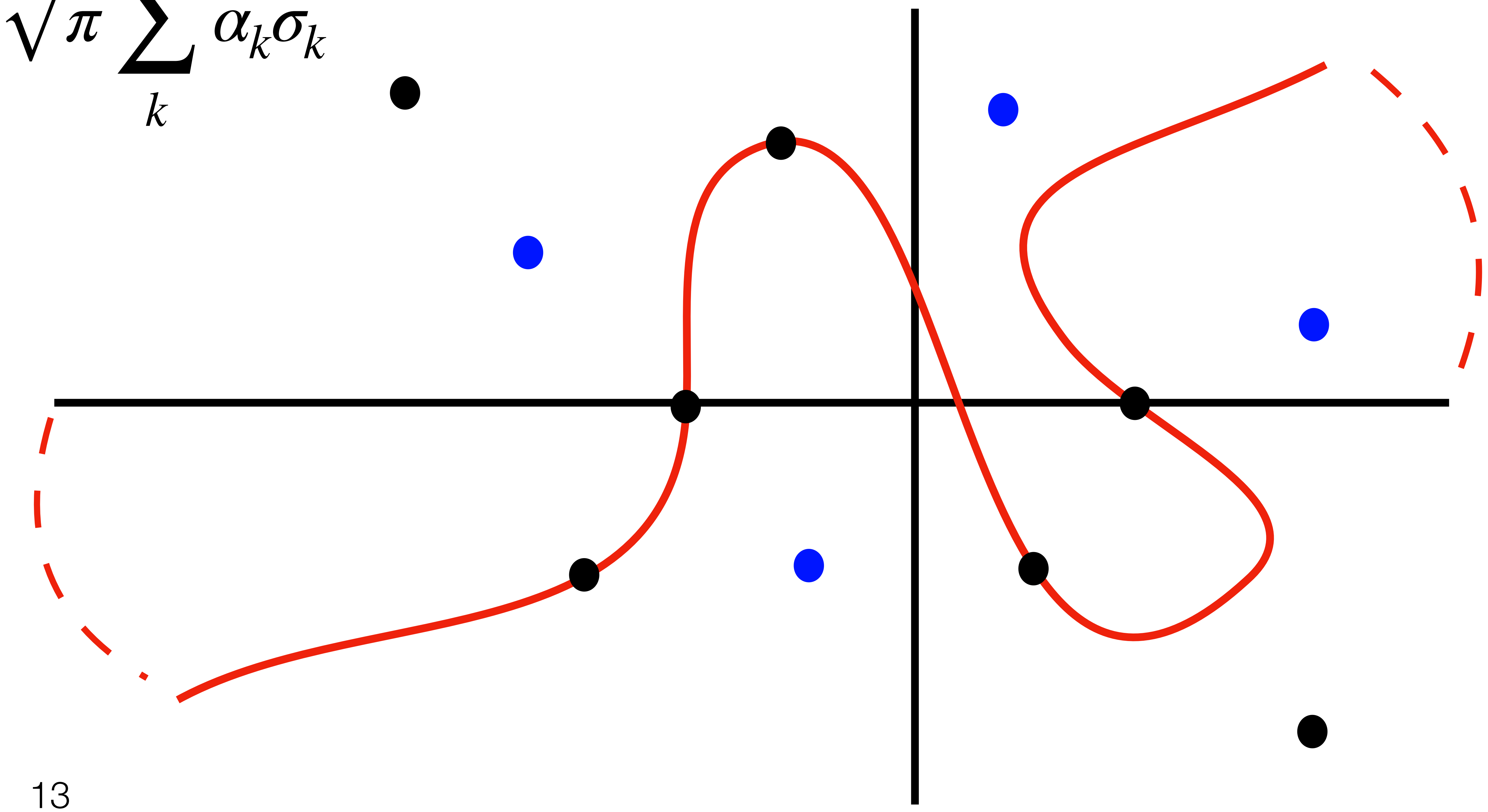
Using Cauchy's integral formula for integrals over analytic functions:

- Saddle point methods (WKB or Eikonal approximation)

$$\int_{-\infty}^{\infty} e^{if(x)}\,\mathrm{d}x \approx \sum_k \int \alpha_k \exp\left[-x^2/\sigma_k^2\right]\mathrm{d}x = \sqrt{\pi}\sum_k \alpha_k\sigma_k$$

- What is the optimal contour?

- Which saddle points to include?



Picard-Lefschetz theory

Picard-Lefschetz theory: For theories with **analytic actions:**

$$I = \int_{\mathbb{R}} e^{if(x)}\mathrm{d}x$$

- **Analytically continue** the integrand into the complex plane

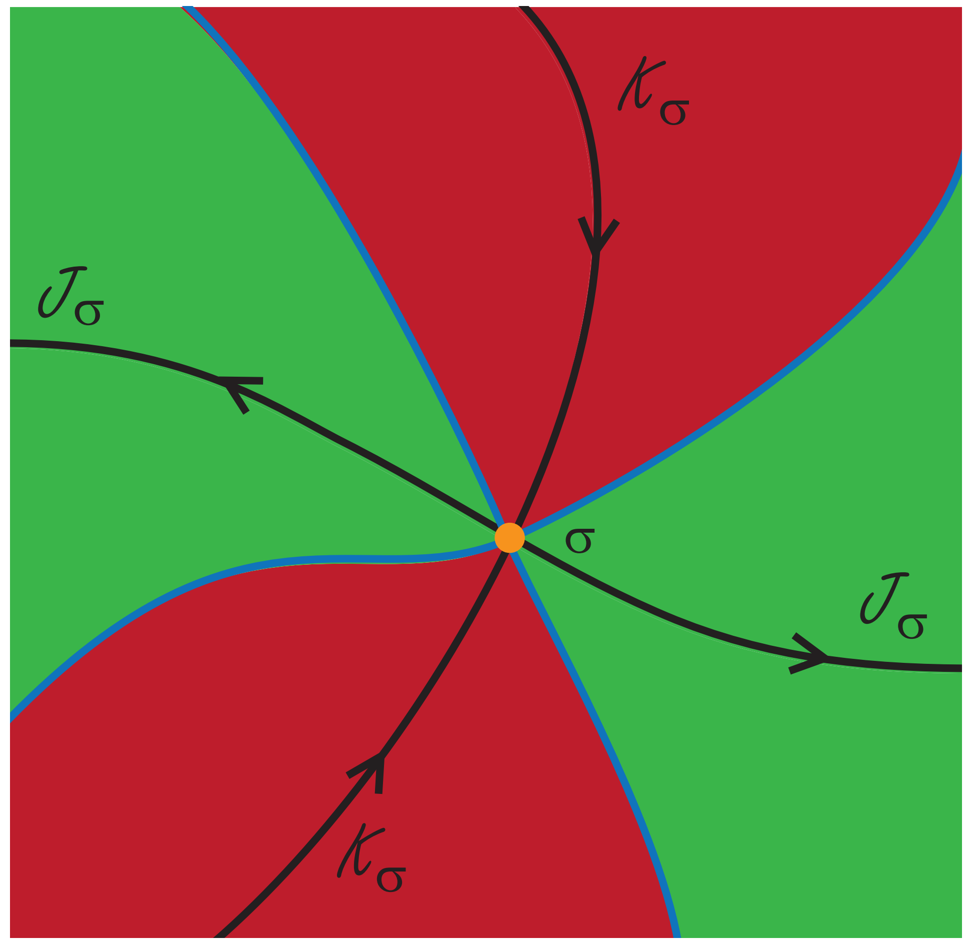
$$if(x) = h(x) + iH(x)$$

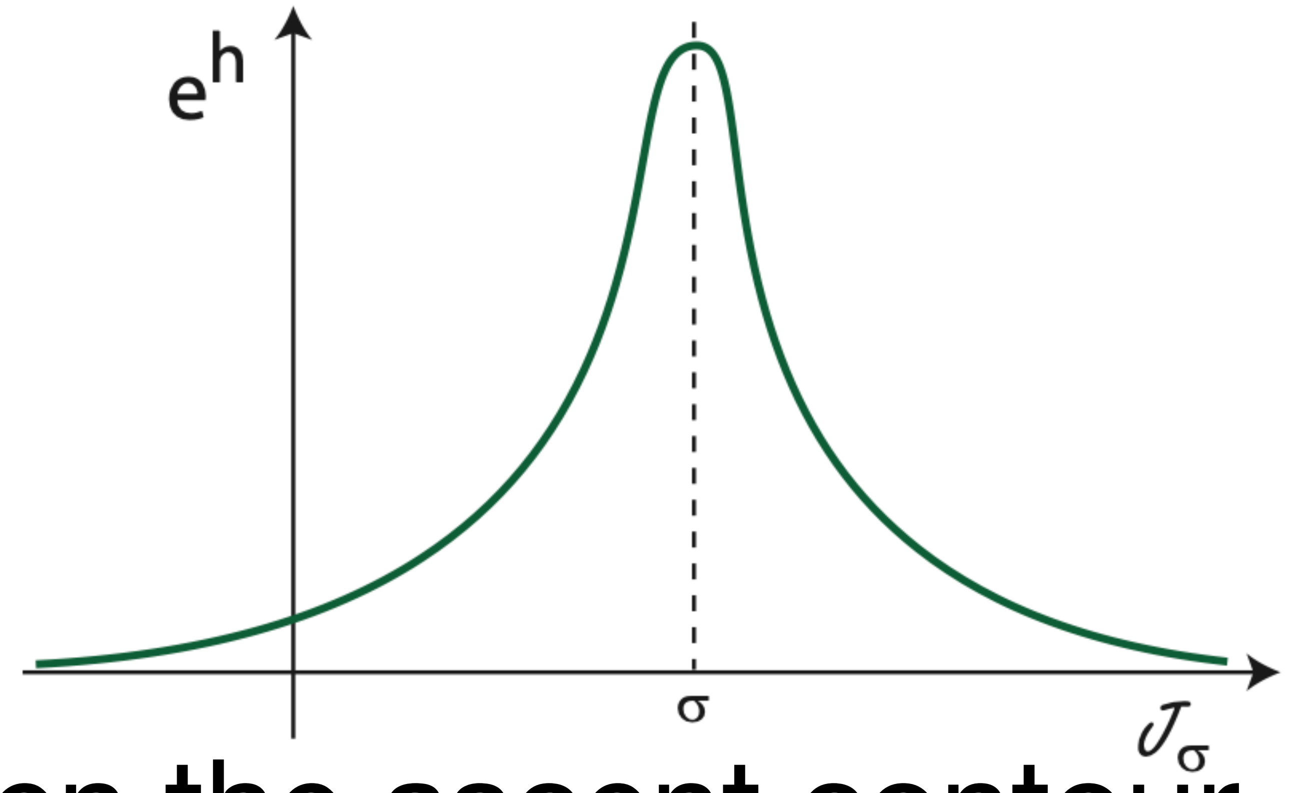
- Find all **saddle points**

$$I = \sum_i n_i e^{iH(x_i)} \int_{\mathcal{J}_i} e^{h(x)}\mathrm{d}x$$

Absolutely convergent

- Find the **steepest ascent** and **descent contours** associated with the real part of the exponent





- Deform the integration domain to the **relevant descent thimbles**

Thimble is relevant when the ascent contour **intersects** the original integration contour

Picard-Lefschetz theory

- Fresnel integral

$$\int_{-\infty}^{\infty} e^{ix^2}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} e^{ix^2}\mathrm{d}x = (1+i)\sqrt{\frac{\pi}{2}}$$

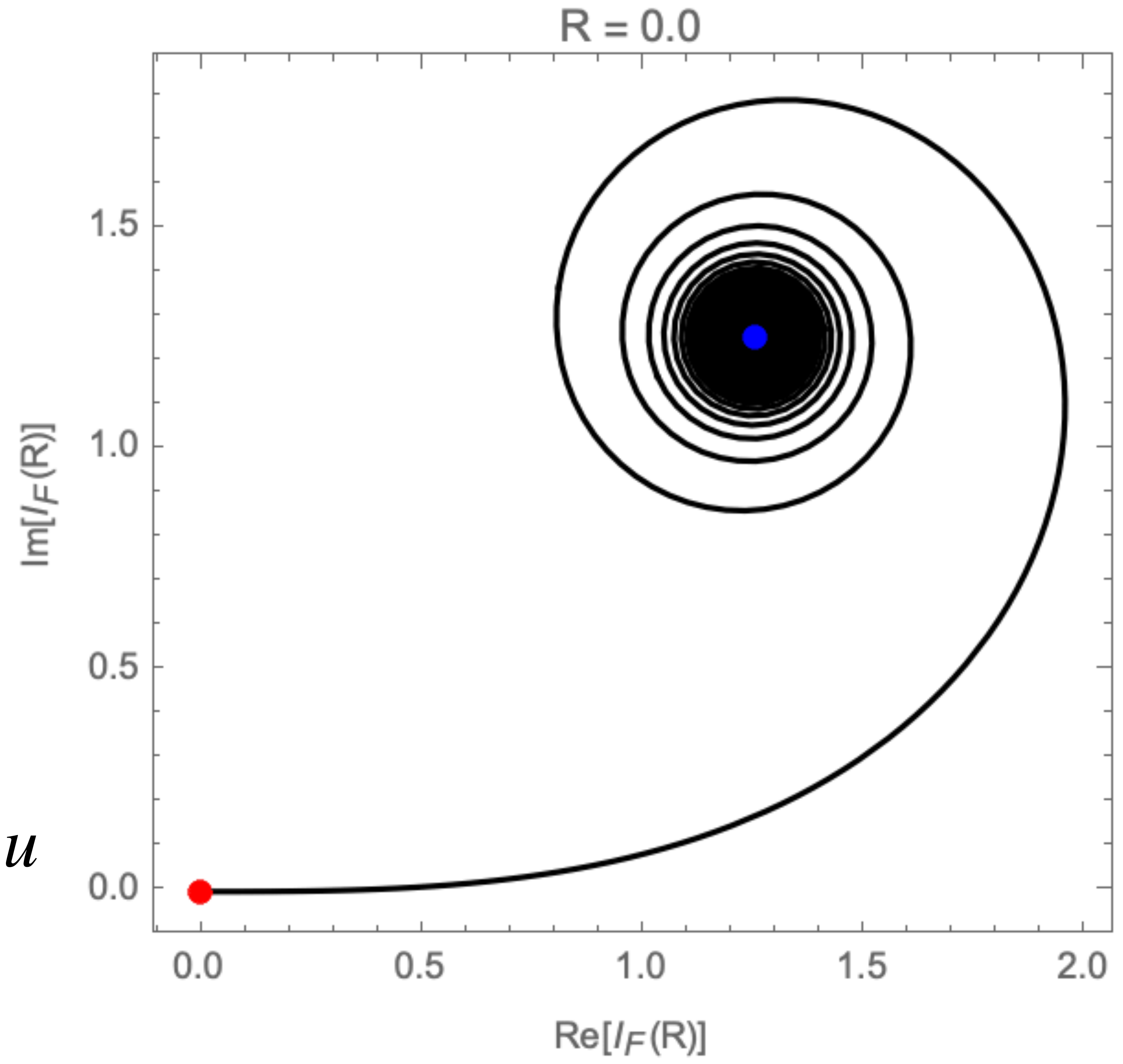
- Multi-dimensional extension

$$\iint_{\mathbb{R}^2} e^{i(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \lim_{R\to\infty} 2\pi\int_0^R r\, e^{ir^2}\mathrm{d}r = \lim_{R\to\infty}\left[i\pi - \pi e^{iR^2}\right]$$

- Complex analysis

$$\int_{\mathbb{R}} e^{ix^2}\,\mathrm{d}x = \frac{1+i}{\sqrt{2}}\int_{\mathbb{R}} e^{-u^2}\,\mathrm{d}u = (1+i)\sqrt{\frac{\pi}{2}} \qquad x = \frac{1+i}{\sqrt{2}}u$$

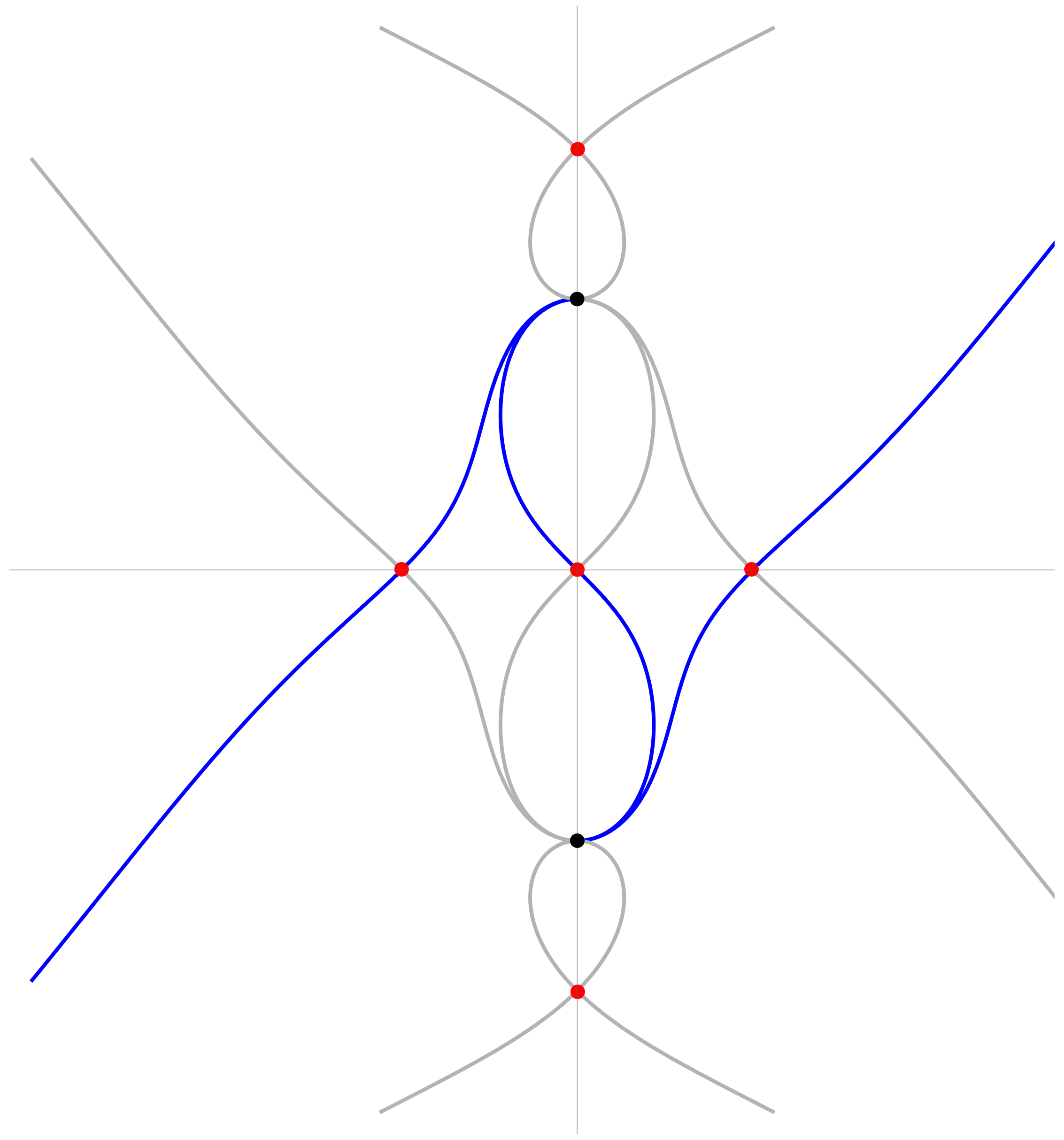
$$\iint_{\mathbb{R}^2} e^{i(x^2+y^2)}\,\mathrm{d}(x,y) = i\int_{\mathbb{R}^2} e^{-(u^2+v^2)}\,\mathrm{d}(u,v) = i\pi$$

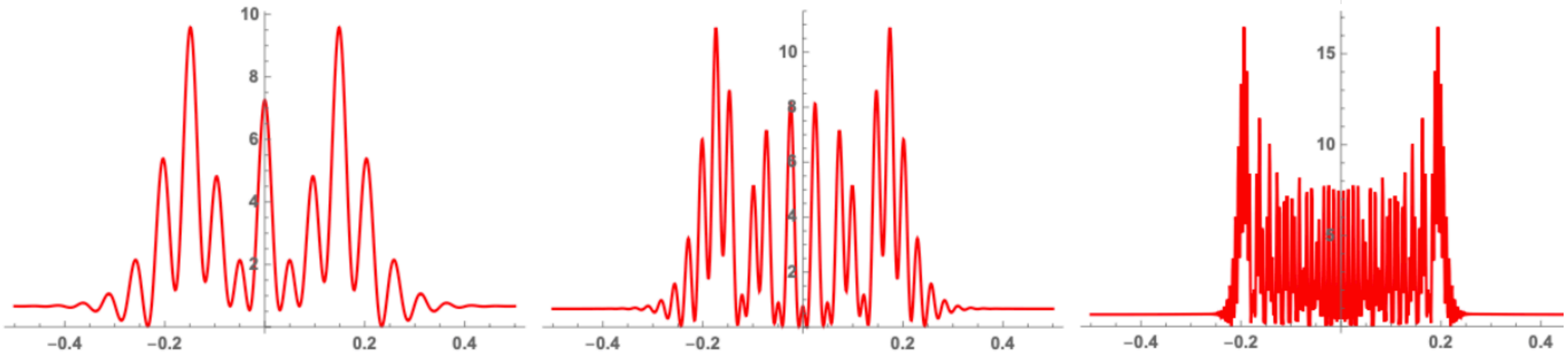


Picard-Lefschetz theory

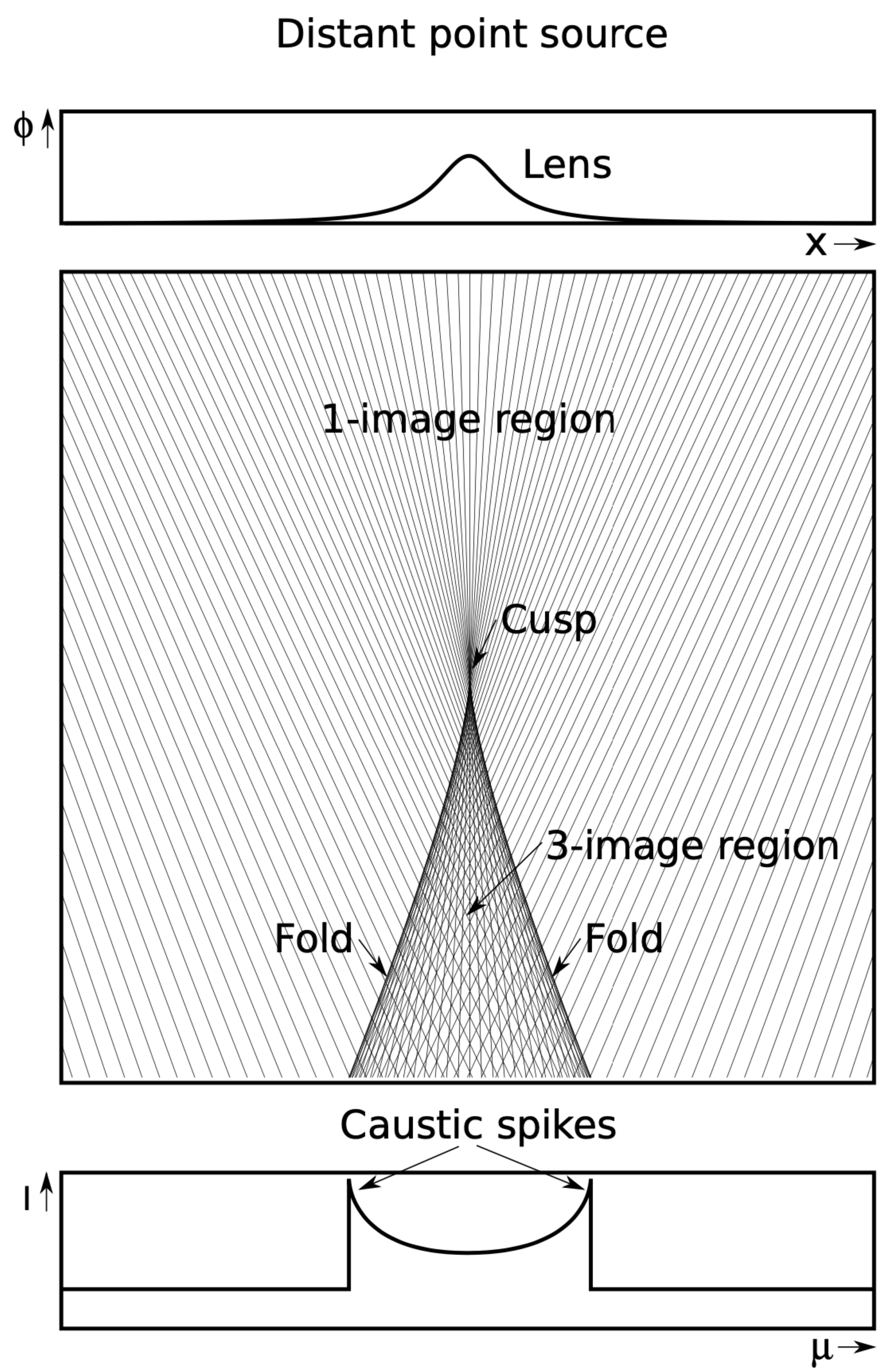
In wave optics, the Kirchhoff-Fresnel integral describes a thin lens. Picard-Lefschetz theory identifies real and complex rays and rewrites the integral into a sum of non-oscillatory integrals

$$\Psi(y) = \int_{-\infty}^{\infty} e^{i\nu\left(\frac{(x-y)^2}{2} + \frac{\alpha}{1+x^2}\right)} \mathrm{d}x$$





(a) $\alpha = 2, \nu = 50$ (b) $\alpha = 2, \nu = 100$ (c) $\alpha = 2, \nu = 500$



Picard-Lefschetz theory

Picard-Lefschetz theory yields the optimal deformation of analytic conditionally convergent integrals and yields the unique result:

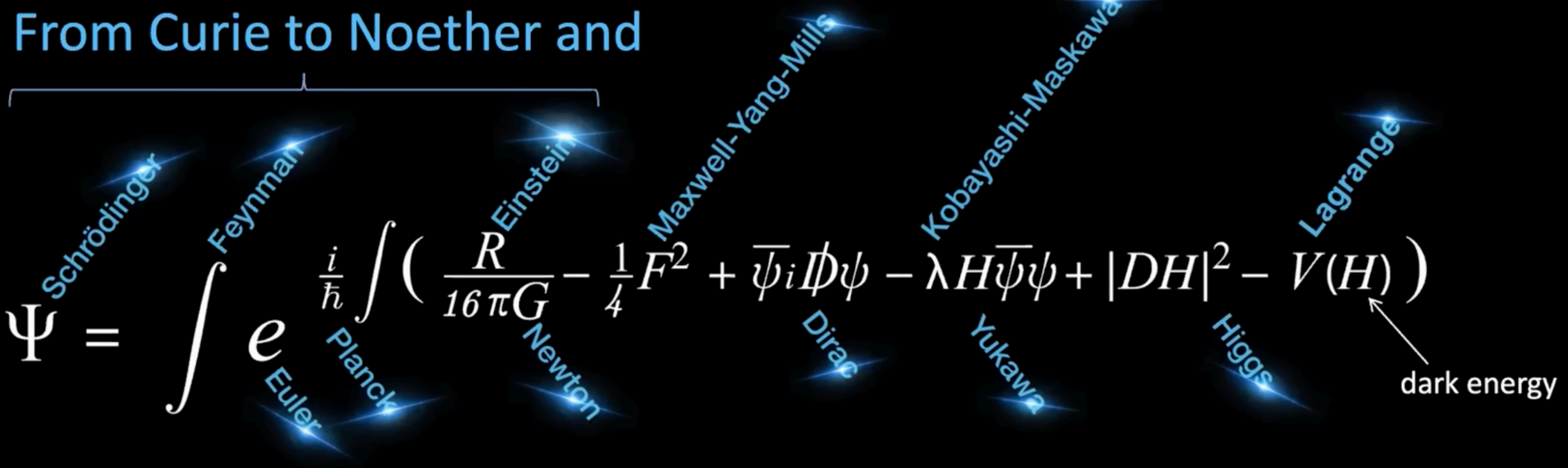
$$\int_{\mathbb{R}^n} e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} \equiv \lim_{R\to\infty} \int g_R(\mathbf{x}) e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} = \lim_{R\to\infty} \sum_i \int_{\mathcal{J}_i} g_R(\mathbf{x}) e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} = \sum_i \int_{\mathcal{J}_i} e^{if(\mathbf{x})}\mathrm{d}\mathbf{x}$$

For a regulator g, that converges to 1 as $R \to \infty$, is analytic in the complex plane, decays rapidly enough that no contributions from infinity are introduced. Extreme paths cancel out.

Feynman Path integral

Path integral

Interference gives our cleanest description of the Universe as formalized by the Feynman path integral. But how is this infinite-dimensional conditionally convergent oscillatory integral defined?



$$\Psi = \int e^{\frac{i}{\hbar}\int\left(\frac{R}{16\pi G} - \frac{1}{4}F^2 + \overline{\psi} i \not{D}\psi - \lambda H \overline{\psi}\psi + |DH|^2 - V(H)\right)}$$

$$\psi = (u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, e_L, \nu_L, e_R, \nu_R) \times 3$$

Path integral

General idea: the path integral has support for paths for saddle points of the action

$$G[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{iS[x]/\hbar}\mathscr{D}x$$

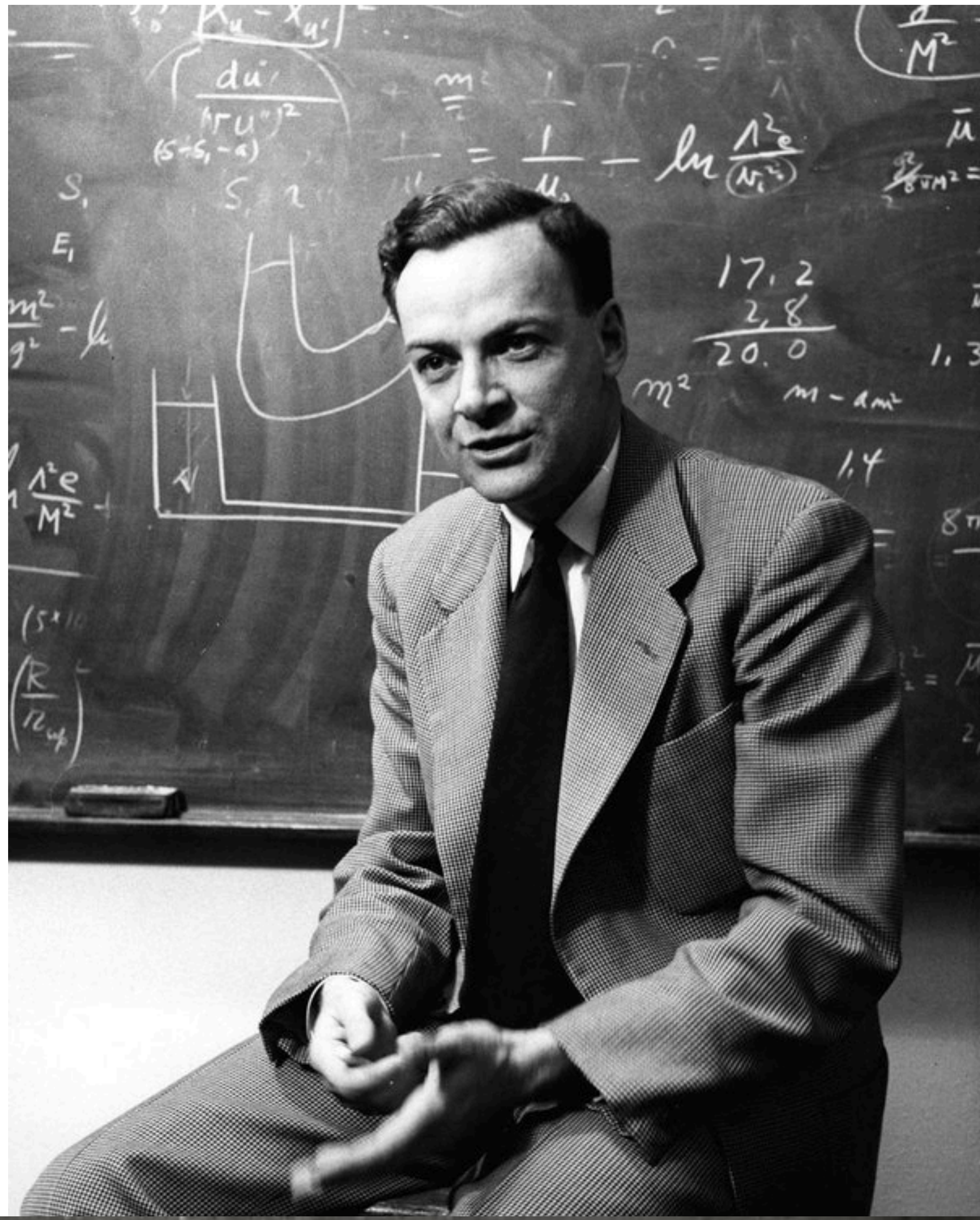
$$\frac{\delta S}{\delta x} = 0 \mapsto \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0 \mapsto m\ddot{x} = -\frac{\partial}{\partial x}V(x)$$

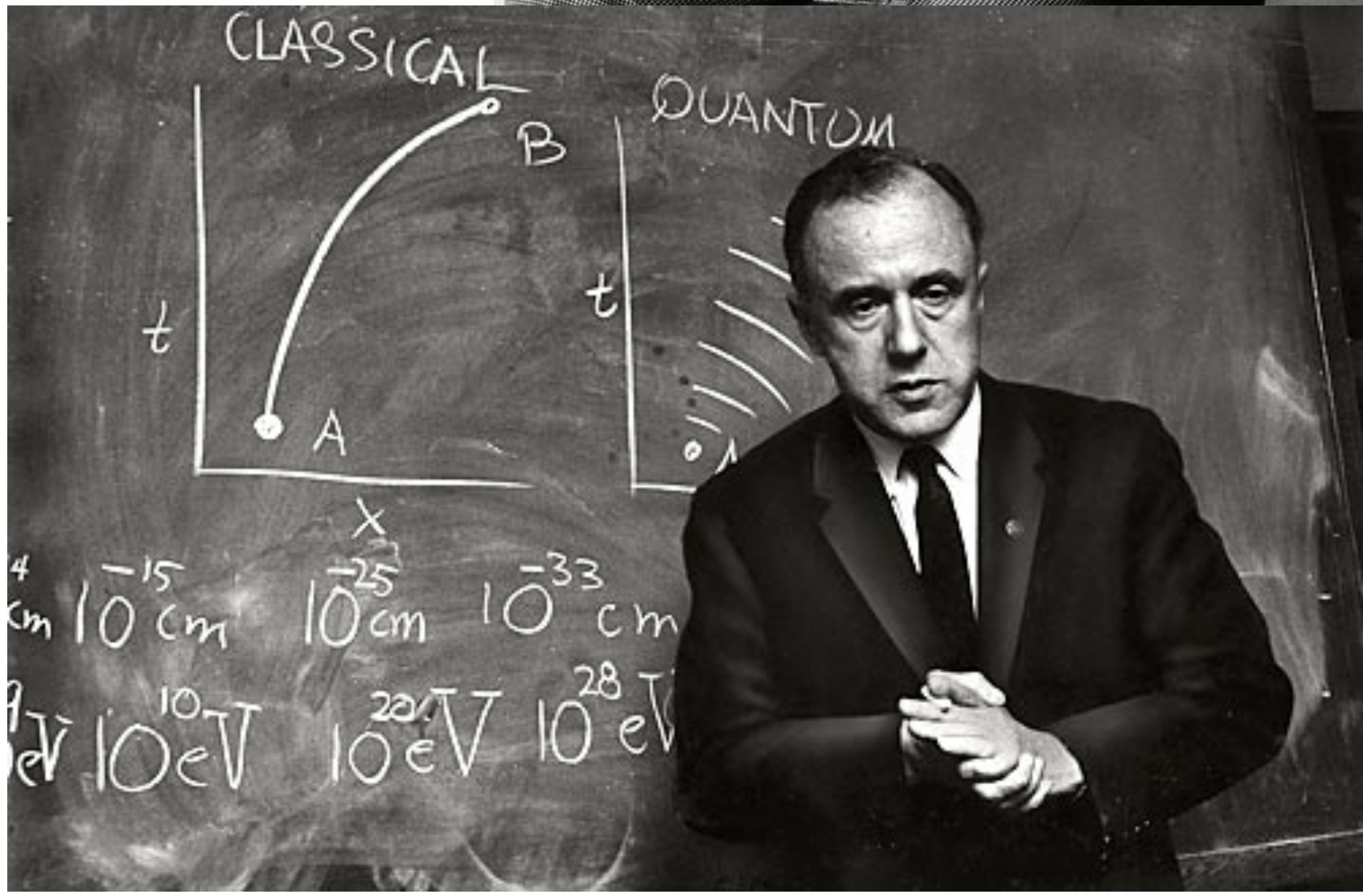
$$x(0) = x_0,\ x(T) = x_1$$

However, which saddles are relevant?

Can we make this idea rigorous?

Can everything be described in terms of classical paths?





Path integral

What is the problem? And why should we care?

- **Feynman and Hibbs**: *"...we feel that the possible awkwardness of the special definition of the sum over all paths may* ***eventually require new definitions to be formulated****. Nevertheless,* ***the concept of the sum over paths, like the concept of an ordinary integral, is independent of a special definition*** *and* ***valid in spite of the failure of such definitions****"*

- **Terence Tau**: *"****The point of rigour is not to destroy all intuition;*** *instead, it should be used to* ***destroy bad intuition while clarifying and evaluating good intuition****. It is* ***only with a combination of both rigorous formalism and good intuition*** *that one can* ***tackle complex mathematical problems.****"*

Infinite dimensional integrals

Integration theory is an application of measure theory

sigma-algebra $\mathscr{A}$ on the space Ω

1. $\Omega \in \mathscr{A}$
2. $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$
3. $A_n \in \mathscr{A}, n \in \mathbb{N} \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathscr{A}$

sigma-measure $\mu : \mathscr{A} \to [0,\infty]$

1. $\mu(\emptyset) = 0$
2. $A_n \in \mathscr{A}, n \in \mathbb{N}$, **pairwise disjoint**

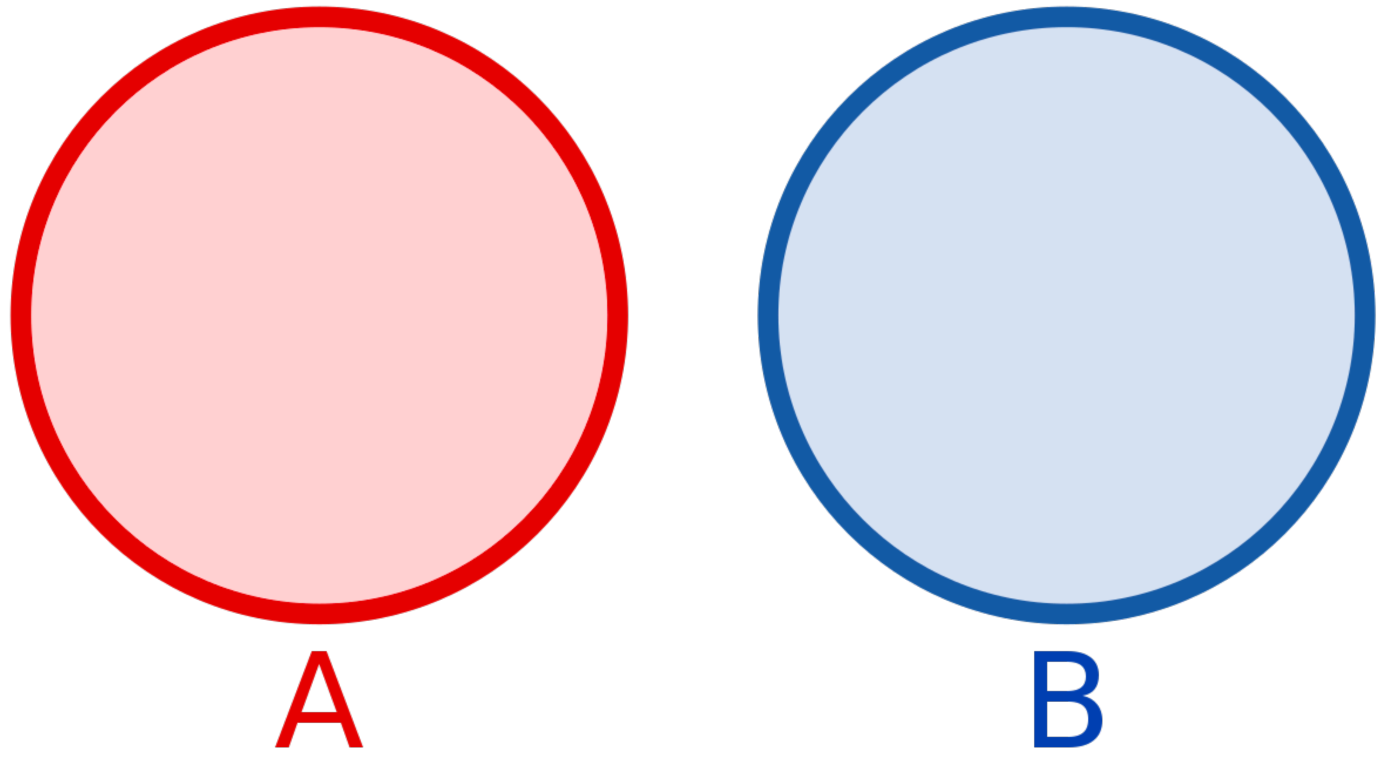
$$\Rightarrow \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

We define integrals of **positive simple functions** as a finite sum

$f = \sum_{i=1}^{r} \alpha_i 1_{A_i}$ **integrates to** $\int_\Omega f \, \mathrm{d}\mu = \sum_{i=1}^{r} \alpha_i \, \mu(A_i)$

leading to the general integral for **positive functions**

$\int_\Omega f \, \mathrm{d}\mu = \sup \left\{ \int_\Omega g \, \mathrm{d}\mu \,\middle|\, \textbf{where } g \textbf{ is simple and } 0 \leq g \leq f \right\}$



Infinite dimensional integrals

The infinite product of Lebesgue measures is not a sigma measure

- Lebesgue formalized the standard measure on geometric spaces
  $\mu([a,b]) = b - a$
- Unfortunately, the infinite product is **not a measure** due to **translation invariance** $\mathscr{D}x \stackrel{!}{=} \prod_{i=1}^{\infty} \mathrm{d}x_i$

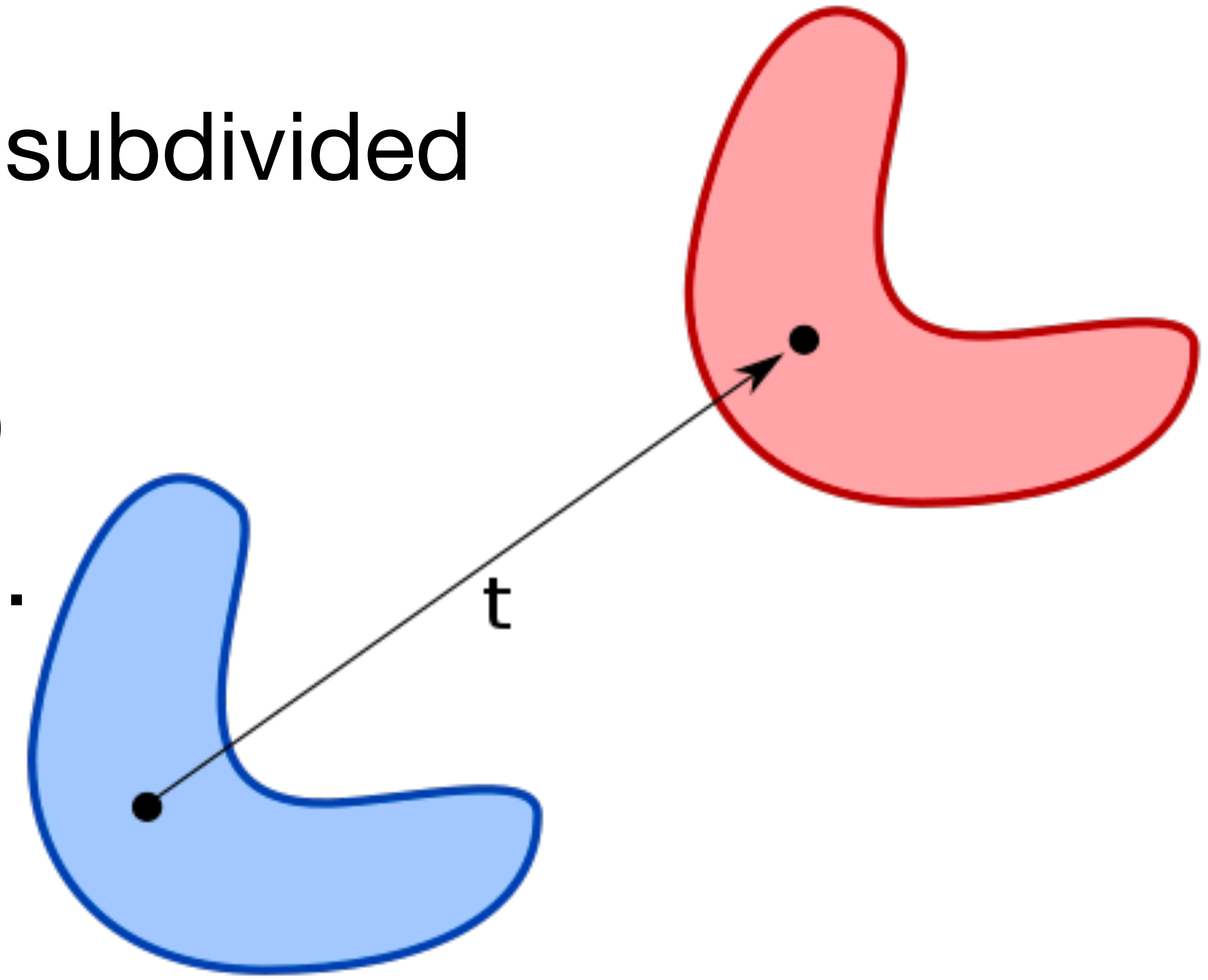
The measure of the n-dimensional hypercube can be subdivided

$1 = \mu([0,1]^n) = 2^n \mu([0,1/2]^n)$ In the limit $n \to \infty$

the subcube has a vanishing measure $\mu([0,1/2]^\infty) = 0$

and so does any subset that we construct from them.

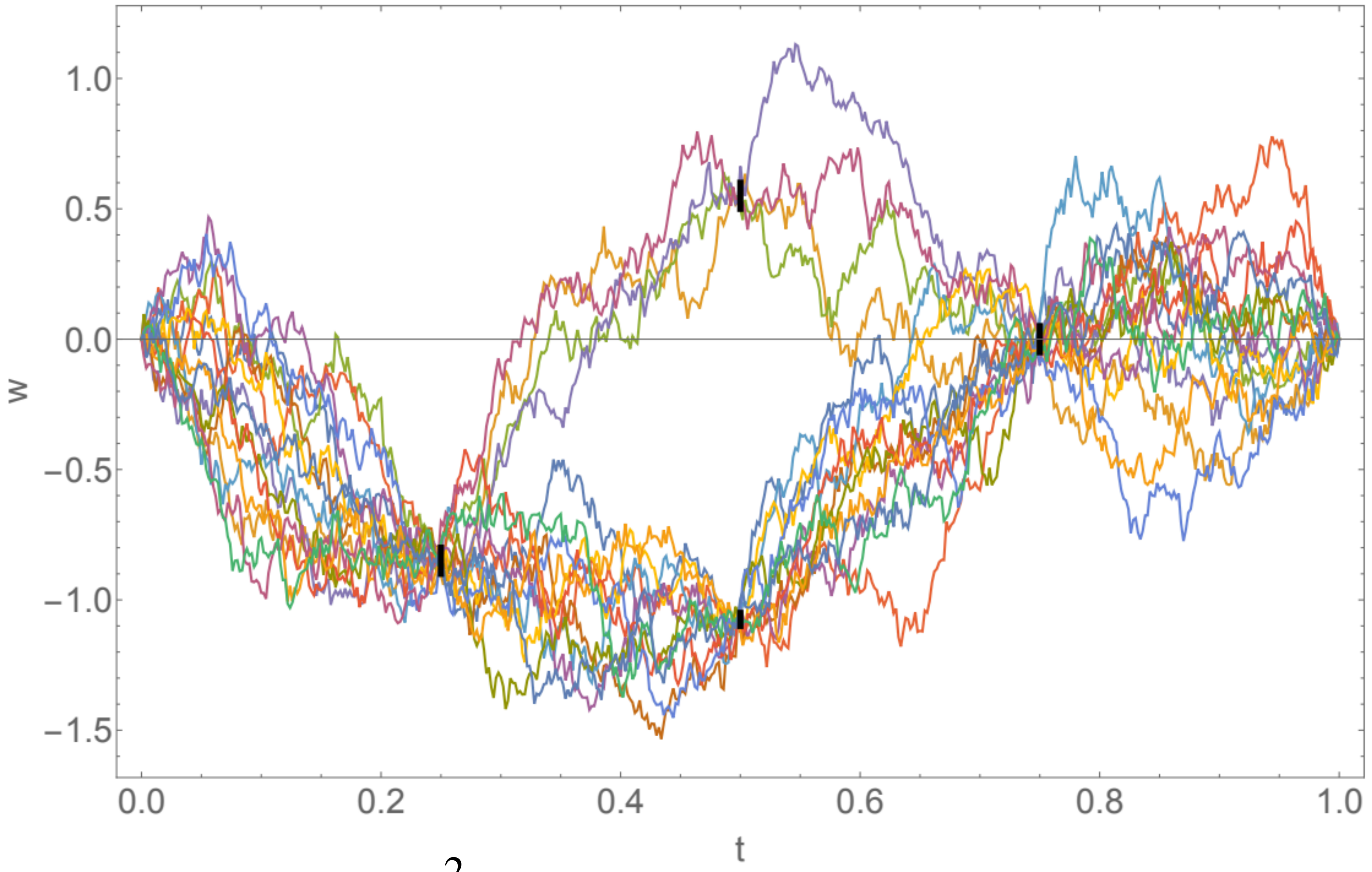
Such measures are useless in physics!



Infinite dimensional integrals

There exist infinite-dimensional measures that that are not translation invariant

Restricted Brownian motion moving between two points, leads to the Brownian bridge measure. When applied to a space of N slits, the measure forms an N-dimensional integral



$$Q = \{w \in \Omega \;\; a_i < w(t_i) < b_i, 0 < t_1 < \ldots < t_N < 1\}$$

$$\mu_B(Q) = \left(\frac{\sqrt{2\pi}W}{\prod_{i=1}^{N+1} (W\sqrt{2\pi}(t_i - t_{i-1}))} \right) \int_{a_1}^{b_1} \ldots \int_{a_N}^{b_N} e^{-\sum_{i=1}^{N+1} \frac{(w_i - w_{i-1})^2}{2W^2(t_i - t_{i-1}}} \mathrm{d}w_1 \ldots \mathrm{d}w_N$$

with stiffness, W. Note that the paths are **not differentiable!**

Feynman-Kac formula

When using a Wick rotation: **interference -> statistical physics**

$$\int e^{\frac{i}{\hbar}\int (m\dot{x}^2/2 - V(x))\mathrm{d}t}\mathscr{D}x \to \int e^{-\frac{1}{\hbar}\int (m\dot{x}^2/2 + V(x))\mathrm{d}t}\mathscr{D}x$$

which is still mathematically **ill-defined**. However, we can define the set of symbols in terms of the **Brownian bridge measure**

$$\frac{\int e^{-\frac{1}{\hbar}\int (m\dot{x}^2/2 + V(x))\mathrm{d}t}\mathscr{D}x}{\int e^{-\frac{1}{\hbar}\int m\dot{x}^2/2\mathrm{d}t}\mathscr{D}x} \equiv \int e^{-\frac{1}{\hbar}\int V(x)\mathrm{d}t}\mathrm{d}\mu_B(x)$$

The smoothing due to the kinetic term and wildness of the infinite product “measure” are beautifully balanced in the Brownian bridge.

$$e^{-\frac{1}{\hbar}\int \frac{m}{2}\dot{x}^2\mathrm{d}t}\mathscr{D}x \equiv \mathrm{d}\mu_B$$

New proposal for real-time QM

When applying Picard-Lefschetz theory to the real-time path integral, can we deform the paths and define the integral using the Brownian bridge measure for each relevant instanton?

$$\int_{\mathbb{R}^n} e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} \equiv \lim_{R\to\infty}\int g_R(\mathbf{x})e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} = \lim_{R\to\infty}\sum_i \int_{\mathcal{J}_i} g_R(\mathbf{x})e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} = \sum_i \int_{\mathcal{J}_i} e^{if(\mathbf{x})}\mathrm{d}\mathbf{x}$$

For a regulator g, that converges to 1 as $R \to \infty$, is analytic in the complex plane, decays rapidly enough that no contributions from infinity are introduced. Extreme paths cancel out and we obtain a unique result:

$$G[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{iS[x]/\hbar}\mathscr{D}x \equiv \sum_{n_C} e^{iS[x_C]/\hbar}\int_{\mathcal{J}_{n_C}} e^{i\theta_{n_C}(\delta x)}\mathrm{d}\mu_{n_C}(\delta x)\ \Theta(T)$$

sum over relevant classical solutions

contour in space of complexities paths associated with the relevant instanton

phase, reduces to Maslow phase in semiclassical limit

Real, positive probability measure

New proposal for real-time QM

The structure of the path integral is completely organized by the classical paths. Note that this formula is exact and not the saddle point approximation. For more details see arXiv:2207.12798 (JF and Neil Turok)

$$G[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{iS[x]/\hbar}\mathscr{D}x \equiv \sum_{n_C} e^{iS[x_C]/\hbar} \int_{\mathscr{J}_{n_C}} e^{i\theta_{n_C}(\delta x)} \mathrm{d}\mu_{n_C}(\delta x)\ \Theta(T)$$

sum over relevant classical solutions

contour in space of complexities paths associated with the relevant instanton

phase, reduces to Maslow phase in semiclassical limit

Real, positive probability measure

where the instantons are defined by

$m\ddot{x} = -V'(x),$ **with** $x(0) = x_0,$ **and** $x(T) = x_1$

This formula should also apply to gravity!

New proposal for real-time QM

The structure of the path integral is completely organized by the classical paths. Note that this formula is exact and not the saddle point approximation. For more details see arXiv:2207.12798 (JF and Neil Turok)

$$G[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{iS[x]/\hbar}\mathscr{D}x \equiv \sum_{n_C} e^{iS[x_C]/\hbar} \int_{\mathscr{J}_{n_C}} e^{i\theta_{n_C}(\delta x)} \mathrm{d}\mu_{n_C}(\delta x)\ \Theta(T)$$

sum over relevant classical solutions

contour in space of complexities paths associated with the relevant instanton

phase, reduces to Maslow phase in semiclassical limit

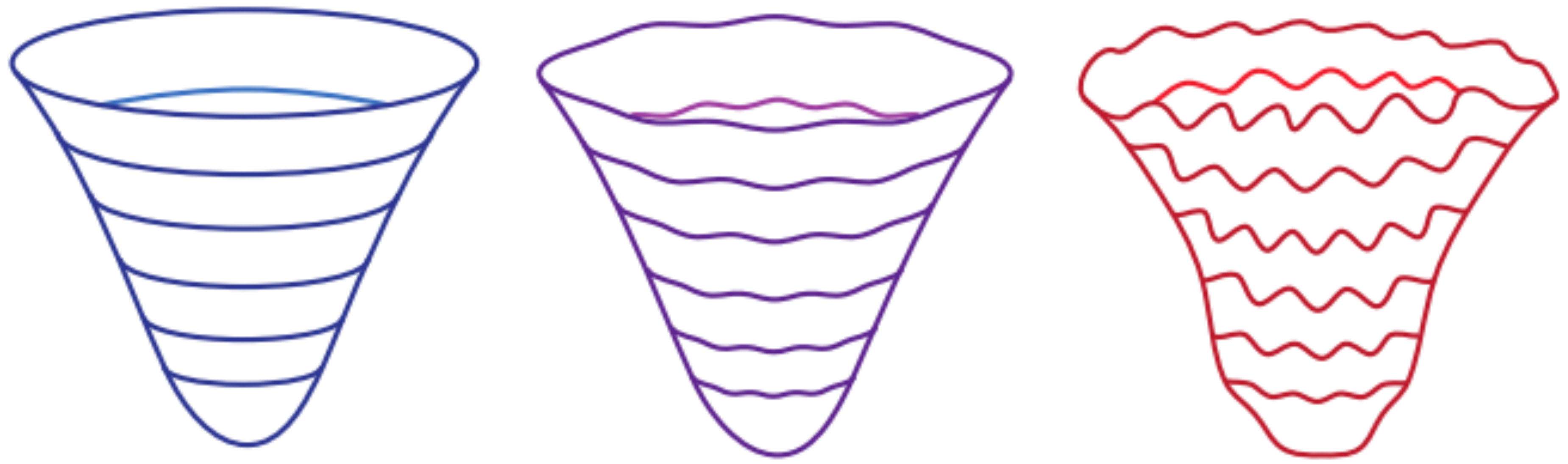
Real, positive probability measure

An instanton is relevant if and only if **there exists a steepest ascent deformation of the saddle point to a real path**

The quantum big bang

The no-boundary proposal

- In the early 1980’s, both Hartle, Hawking and Vilenkin developed famous models for the quantum big bang, known as the no-boundary and the tunnelling proposal.
- Nucleation of a classical closed Lambda-dominated universe out of a forbidden ‘Euclidean’ quantum phase.
- Aim: use the path integral for gravity to construct a predictive model for the initial conditions of our universe.
- In recent work, we used Picard-Lefschetz theory to study these proposals in the Lorentzian formulation.



Lorentzian Quantum Cosmology

No smooth beginning for spacetime

No Rescue for the No Boundary Proposal: Pointers to the Future of Quantum Cosmology

Inconsistencies of the New No-Boundary Proposal

Quantum Incompleteness of Inflation

No-boundary proposal

Minisuperspace $\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\Omega_3^2$

$$G[a_1; a_0] = \int_0^\infty \mathrm{d}N \int_{a_0}^{a_1} \mathcal{D}a \, e^{iS[N,a]}$$

with the Einstein-Hilbert action for a Λ-dominated universe

$$S = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\,(R - 2\Lambda) = 2\pi^2 \int_0^1 \mathrm{d}t \, N \left[-3a\frac{\dot{a}^2}{N^2} + 3ka - a^3\Lambda\right]$$

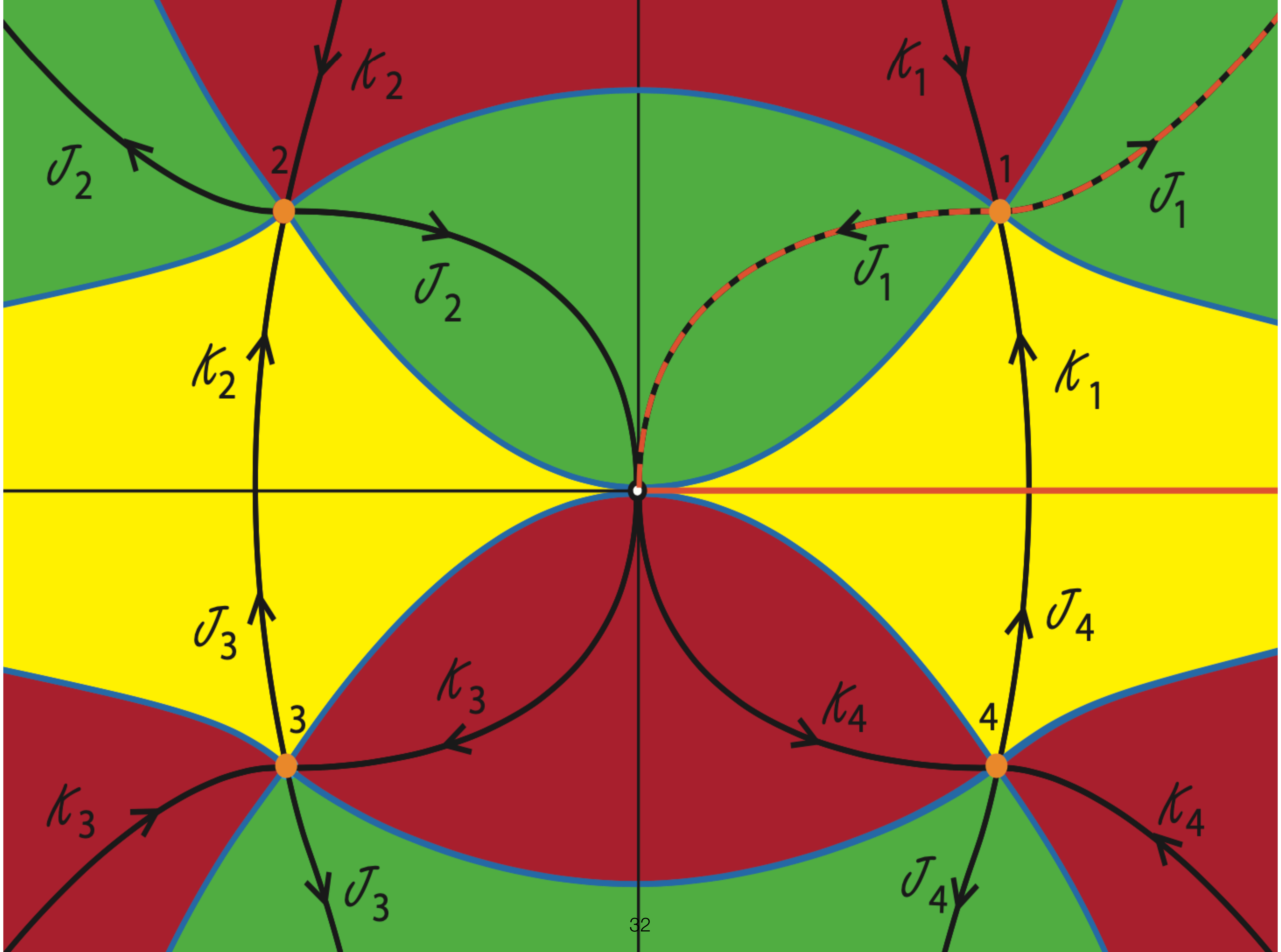
Redefining the lapse $N \mapsto N/a$, the action is quadratic in $q = a^2$

$$S = 2\pi^2 \int_0^1 \mathrm{d}t \left[-\frac{3}{4N}\dot{q}^2 + N(3k - \Lambda q)\right]$$

The propagator $G[q_1; q_0] \propto \int_0^\infty \frac{\mathrm{d}N}{\sqrt{N}} e^{iS_0[q_1;q_0;N]}$ with the classical action

$$S_0[q_1; q_0; N] = N^3\frac{\Lambda^2}{36} + N\left[-\frac{\Lambda}{2}(q_0 + q_1) + 3k\right] + \frac{1}{N}\left[-\frac{3}{4}(q_1 - q_0)^2\right]$$

Halliwell & Louko (1989), Brown & Martinez (1990)



$\mathcal{K}_2$
$\mathcal{K}_1$
$\mathcal{J}_2$
2
1
$\mathcal{J}_1$
$\mathcal{J}_1$
$\mathcal{J}_2$
$\mathcal{K}_2$
$\mathcal{K}_1$
$\mathcal{J}_3$
$\mathcal{J}_4$
$\mathcal{K}_3$
$\mathcal{K}_4$
3
4
$\mathcal{K}_3$
$\mathcal{K}_4$
$\mathcal{J}_3$
$\mathcal{J}_4$

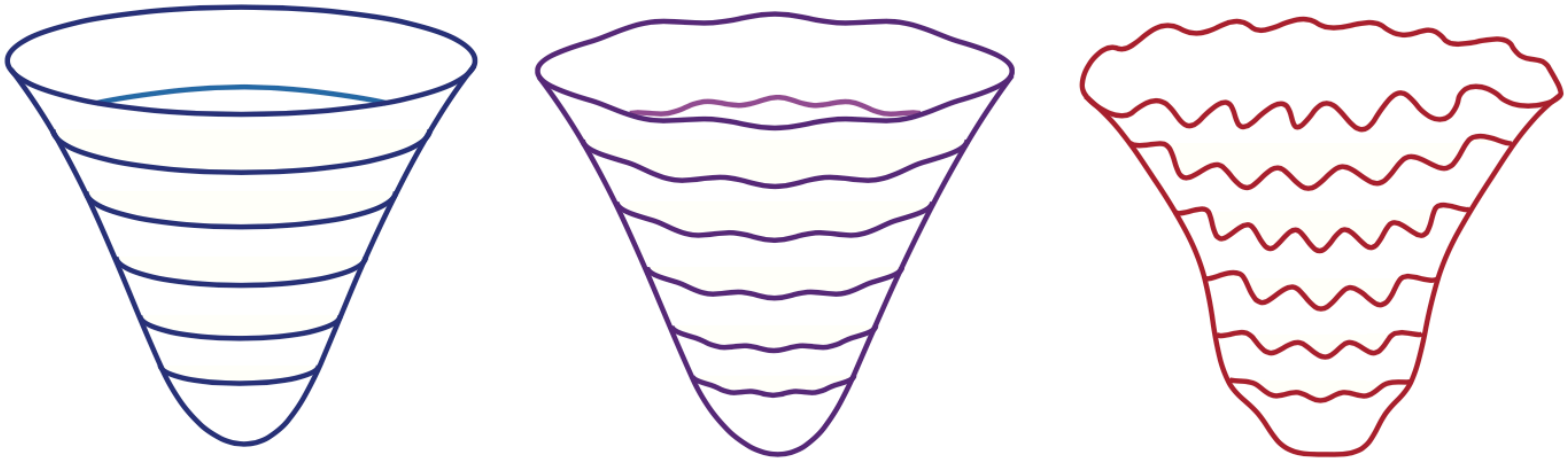
The no-boundary proposal

The total propagator factorizes $G[q_1, \phi_1; 0, 0] = G[q_1; 0] G_\phi[\phi_1; 0]$, with the GW propagator

$$G_\phi[\phi_1; 0] \propto e^{\frac{l(l+1)(l+2)}{2\hbar H^2}\phi_1^2} \times phase$$

This is an inverse Gaussian distribution

$$\left|G_\phi[\phi_1; 0]\right|^2 \propto e^{\frac{l(l+1)(l+2)}{\hbar H^2}\phi_1^2}$$



The propagator in action

$G[x_1, x_0; T]$ $S[x(t)]$

Rosen-Morse potential

- Enough formalism! How to study this all in practice?

- The Teller potential is both solvable and generic

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{V_0}{\cosh^2 x}$$

with the eigenstates

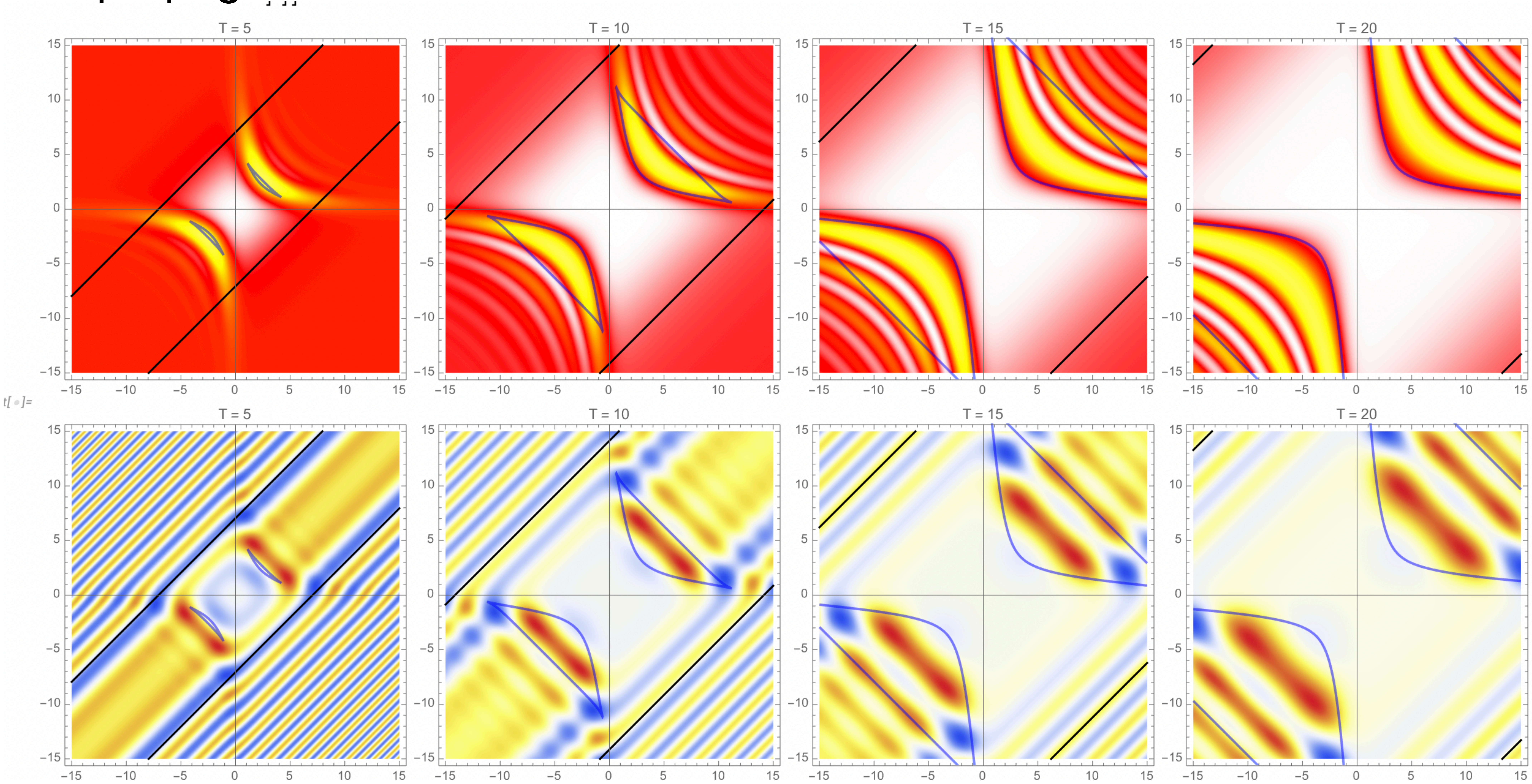
$$\phi_k^{\pm}(x) = \sqrt{\frac{k\sinh(\pi k)}{\cosh(2\pi k) + \cosh(2\pi\nu)}} P_N^{ik}(\pm\tanh x) \qquad N = -\frac{1}{2} + i\nu, \nu = \frac{1}{2\hbar}\sqrt{8mV_0 - \hbar^2}$$

and the path integral in the spectral representation

$$G[x_1, x_0; T] = \Theta(T)\int_0^\infty \left(\phi_k^+(x_1)\phi_k^+(x_0)^* + \phi_k^-(x_1)\phi_k^-(x_0)^*\right) e^{-\frac{i\hbar k^2 T}{2m}}\mathrm{d}k$$

Caustics in the Rosen-Morse Barrier

The propagator as a function of time

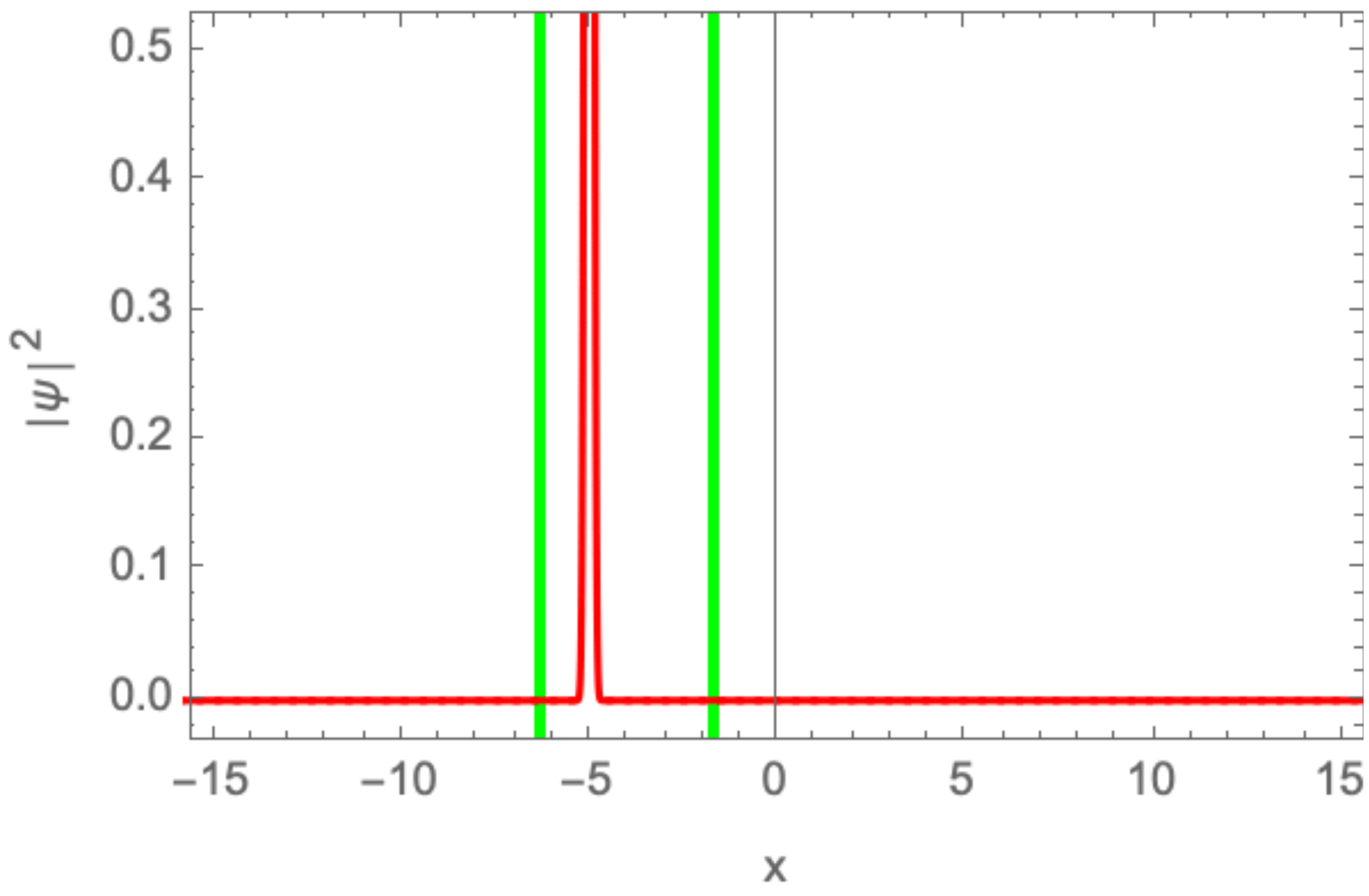


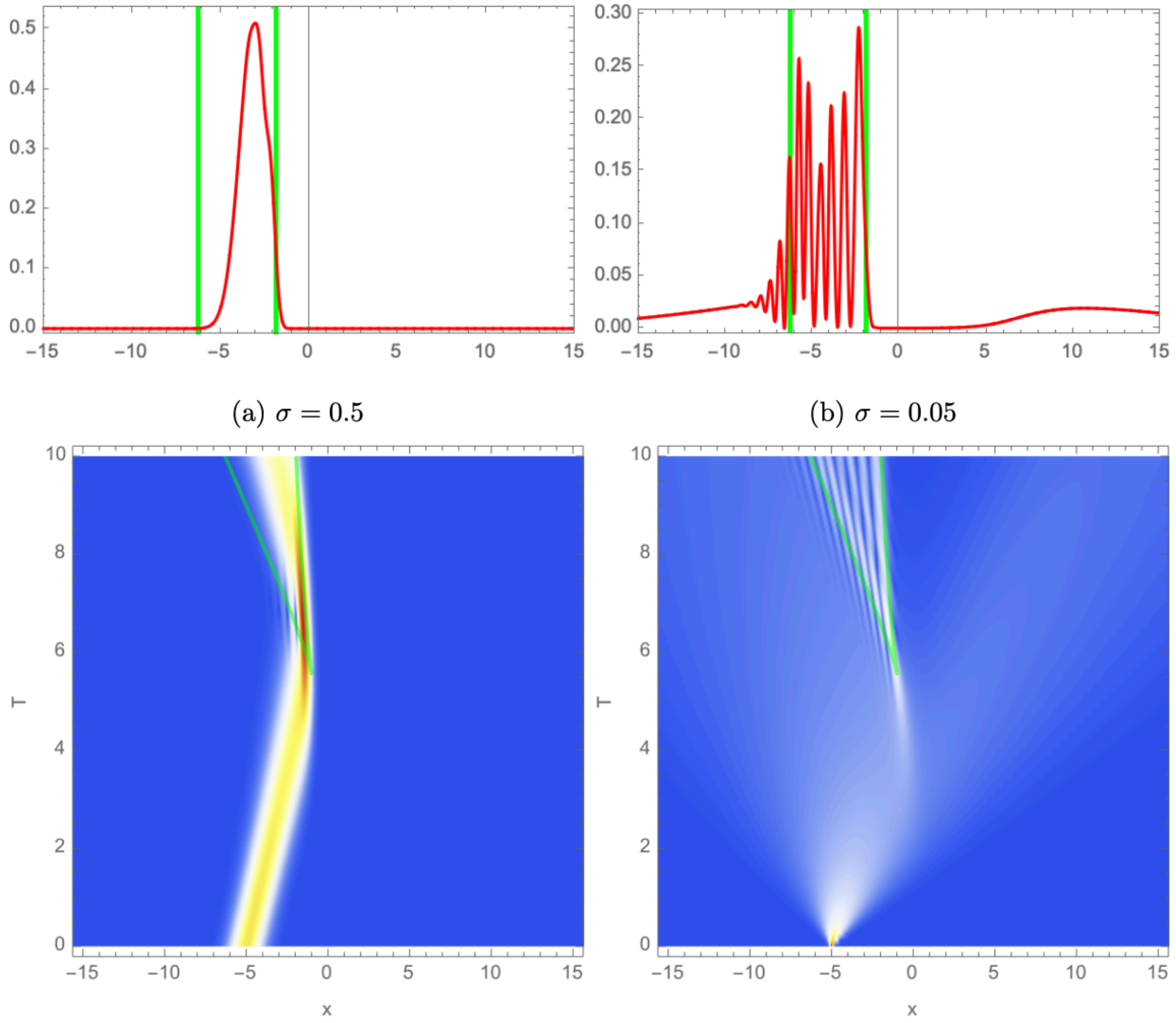
Caustics in the Rosen-Morse Barrier

The potential barrier: there are always either 1 or 3 real classical paths

Caustics in the Rosen-Morse Barrier

Not only in the propagator but also in the Schrödinger equation





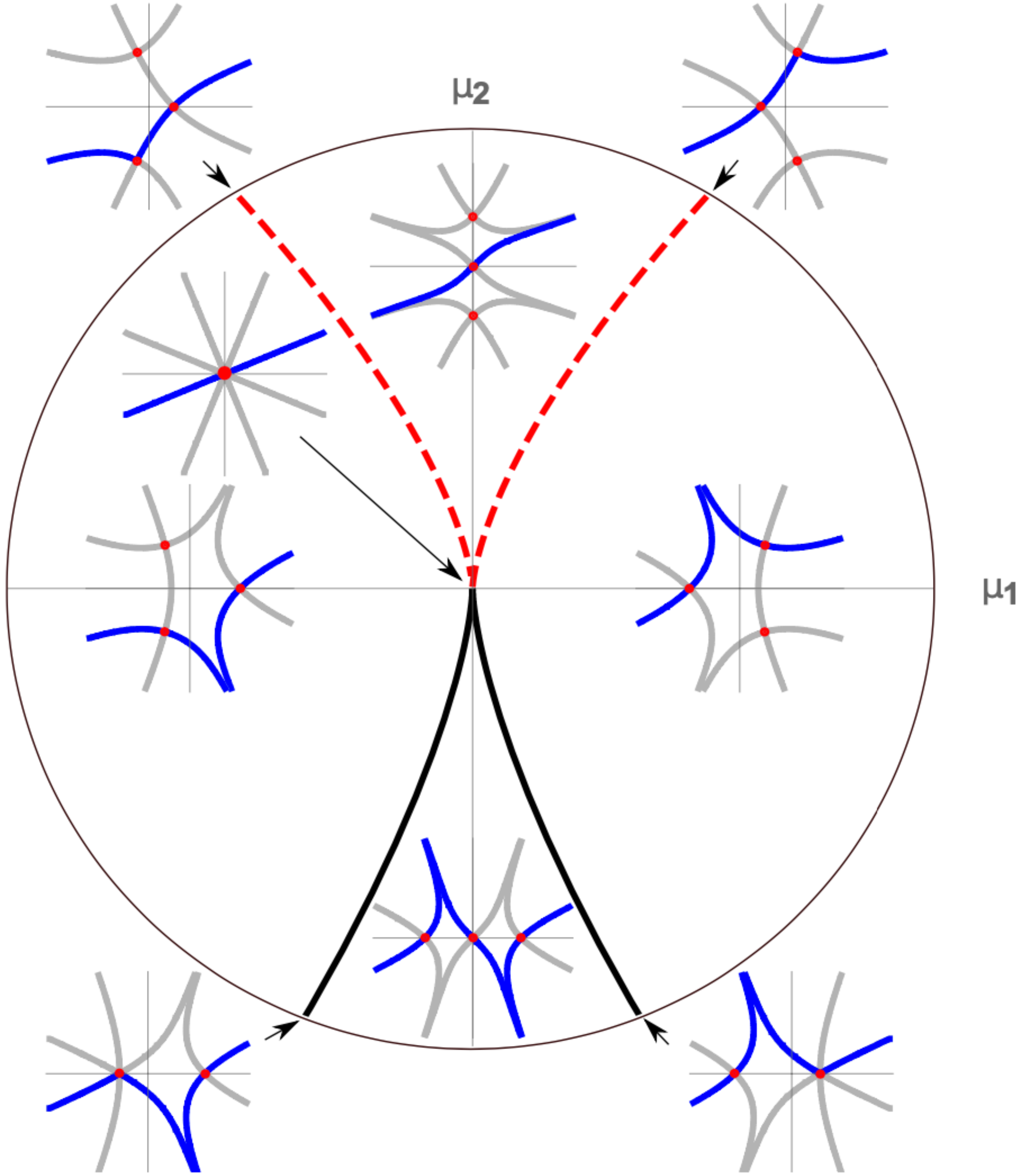
(a) $\sigma = 0.5$

(b) $\sigma = 0.05$

(c) $\sigma = 0.5$

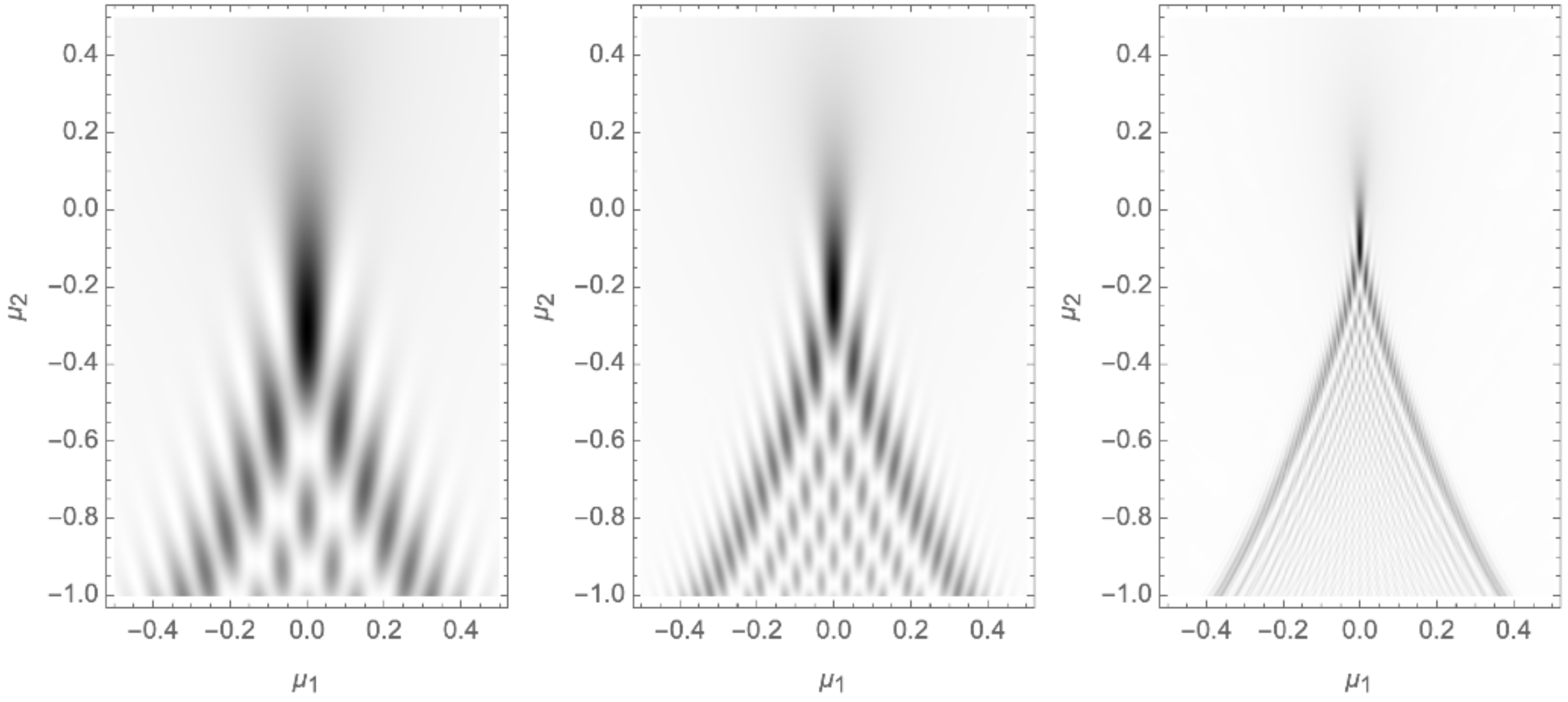
(d) $\sigma = 0.05$

Caustics in the Rosen-Morse Barrier



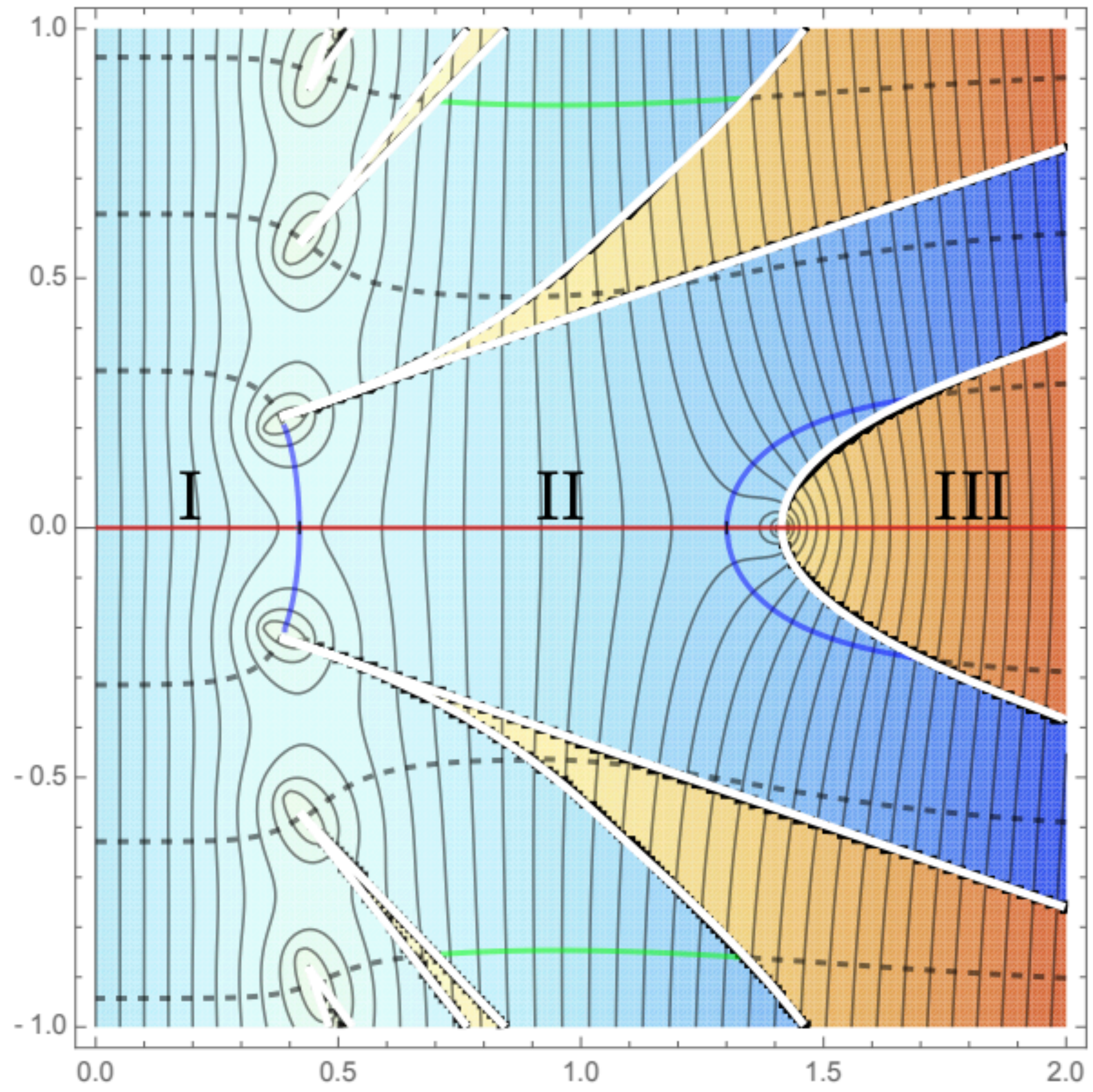
Caustics and Stoke's lines

$$\Psi(\mu,\nu) = \int e^{i\nu(x^4/4+\mu_2 x^2/2+\mu_1 x)}\mathrm{d}x$$

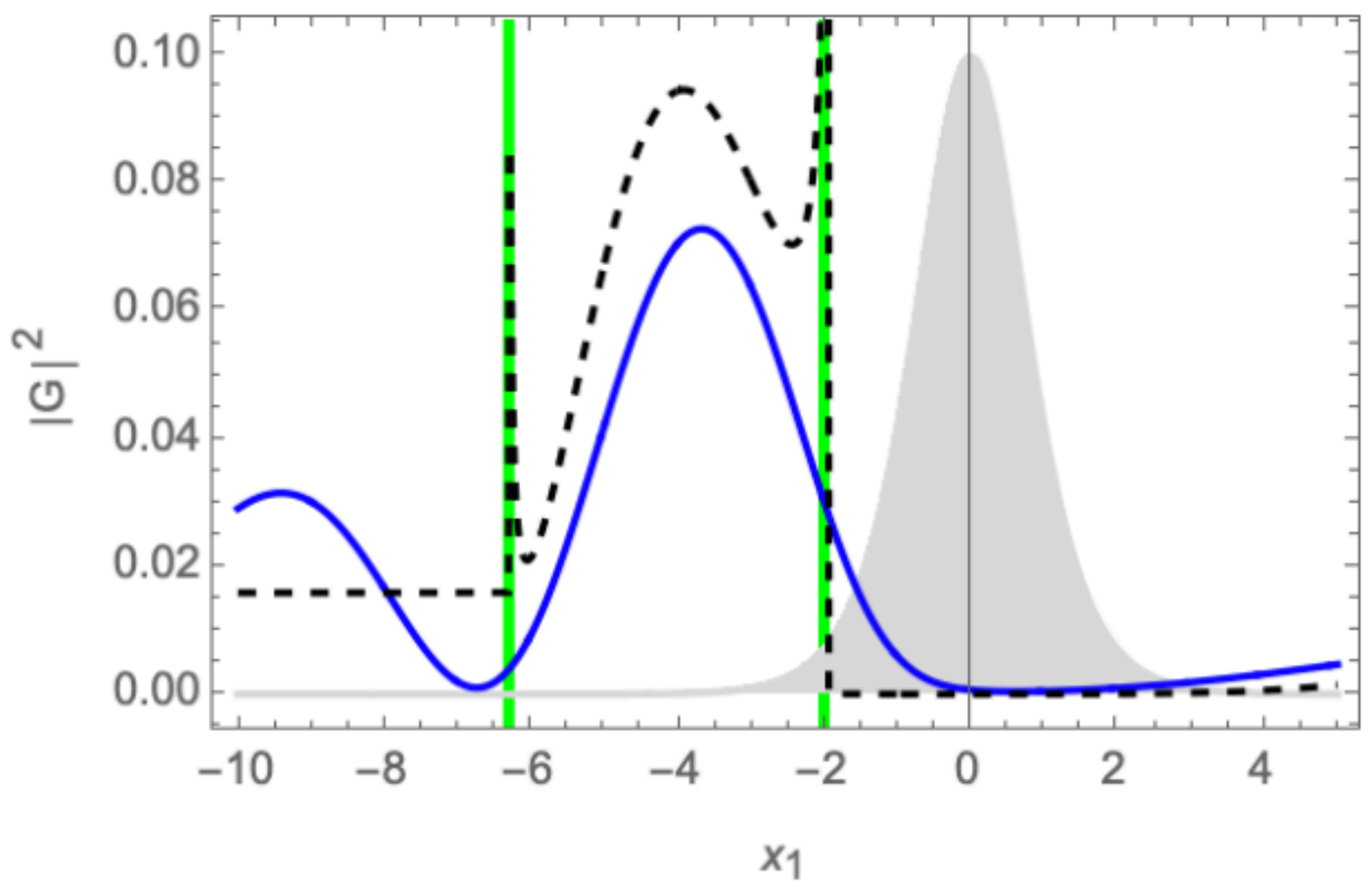


(a) $\nu = 50$ (b) $\nu = 100$ (c) $\nu = 500$

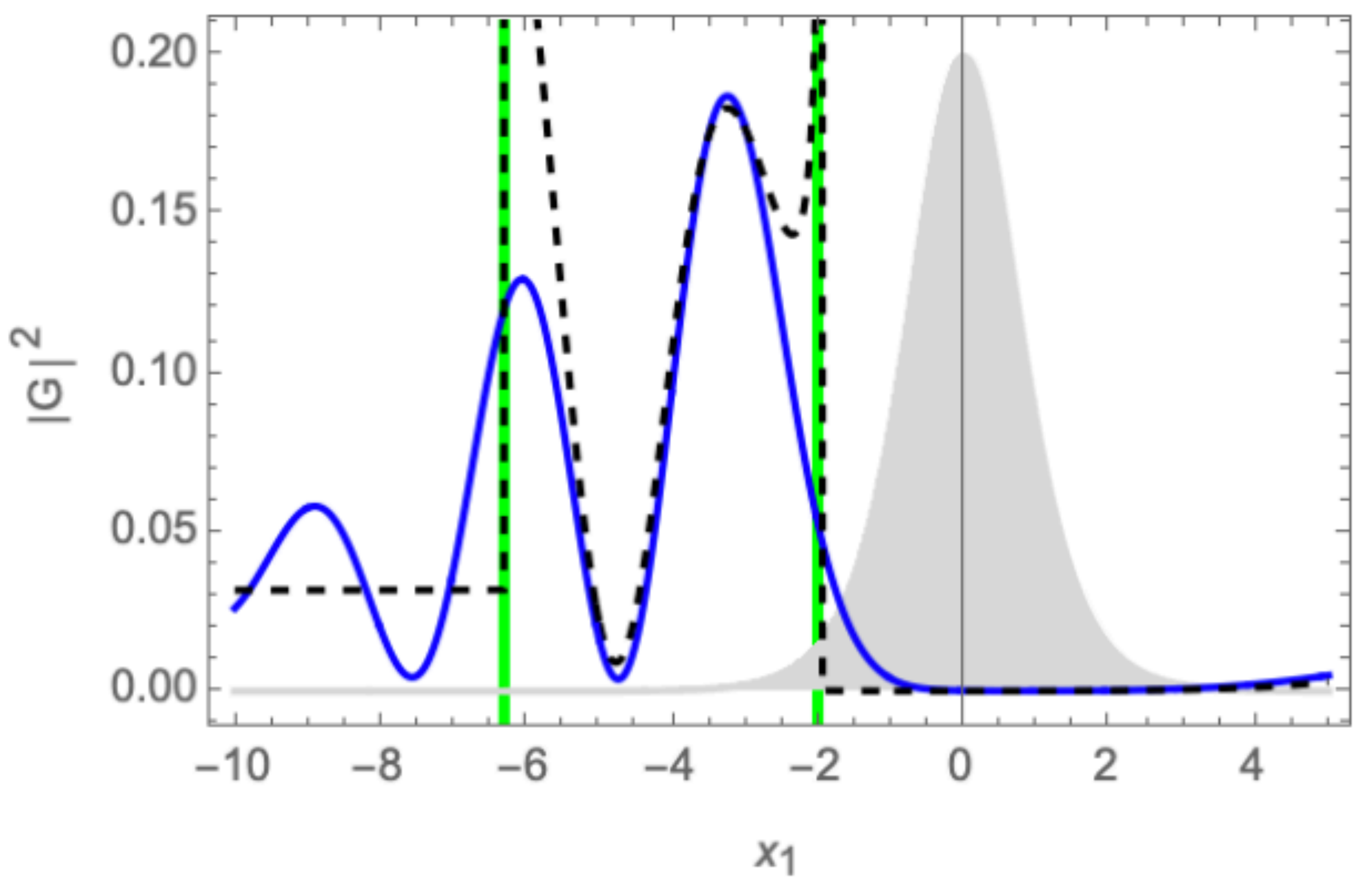
Caustics in the Rosen-Morse Barrier



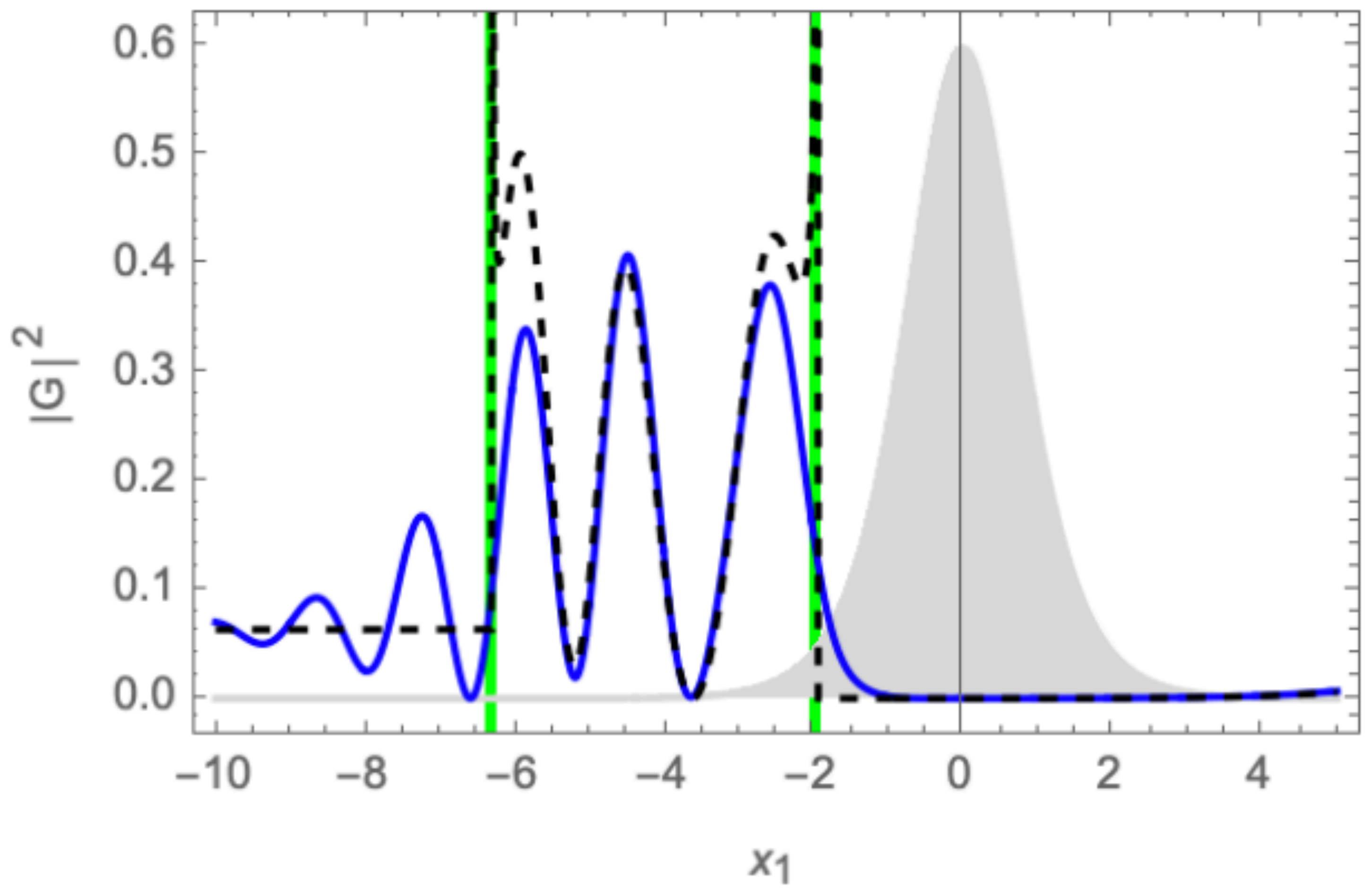
(a) x_1



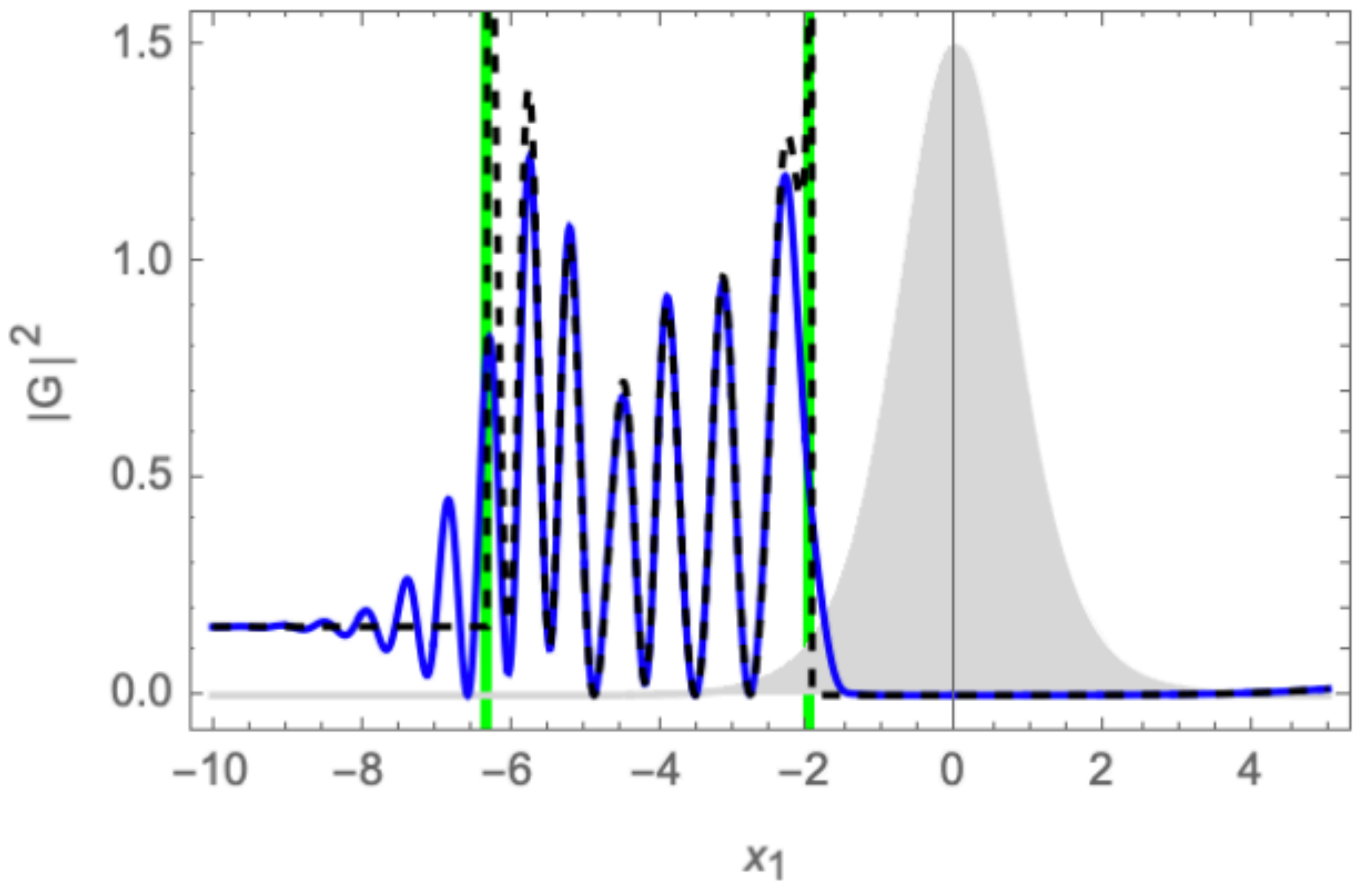
(a) $\hbar = 1$



(b) $\hbar = 0.5$

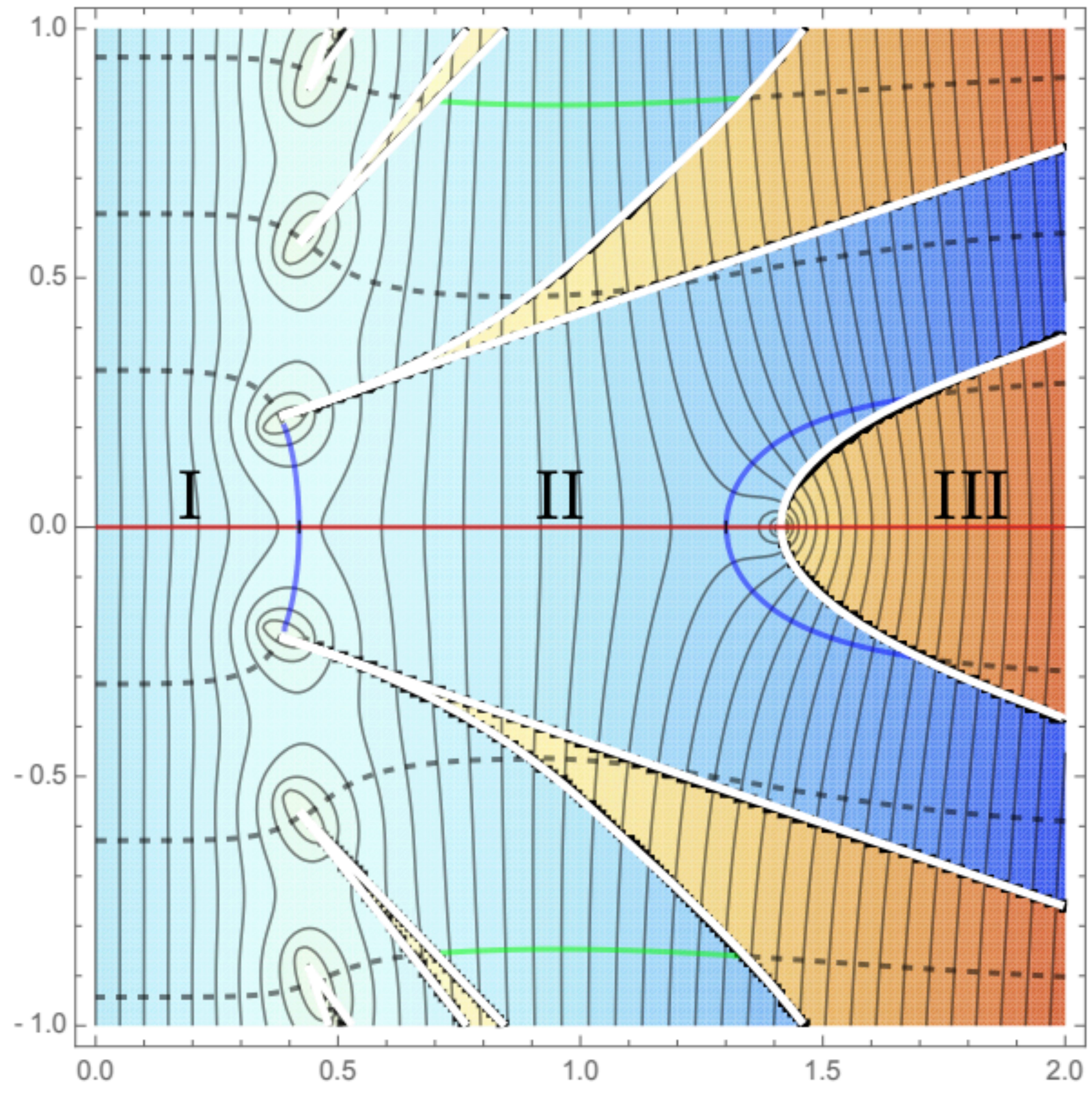


(c) $\hbar = 0.25$

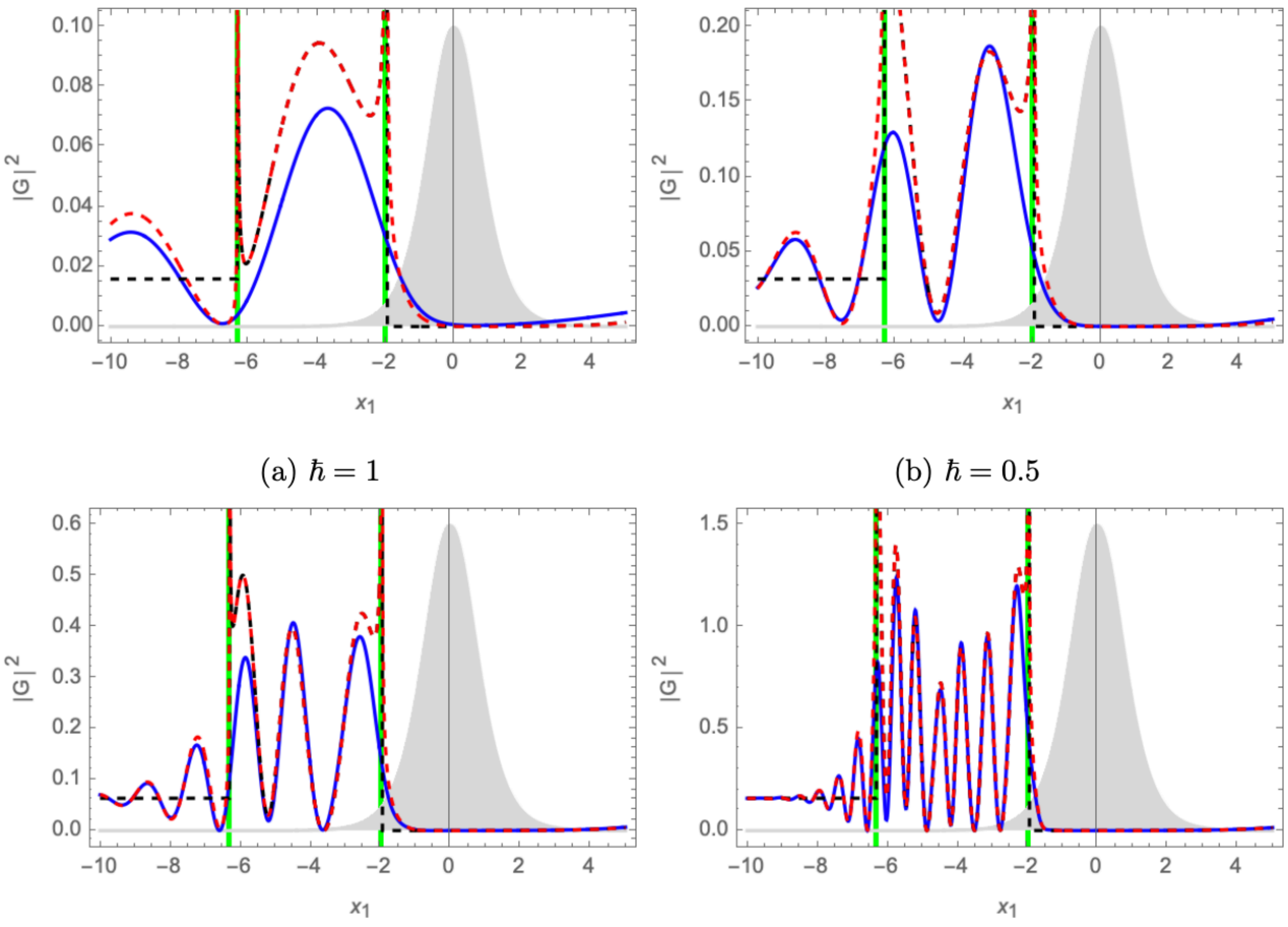


(d) $\hbar = 0.1$

Caustics in the Rosen-Morse Barrier



(a) x_1



(a) $\hbar = 1$

(b) $\hbar = 0.5$

(c) $\hbar = 0.25$

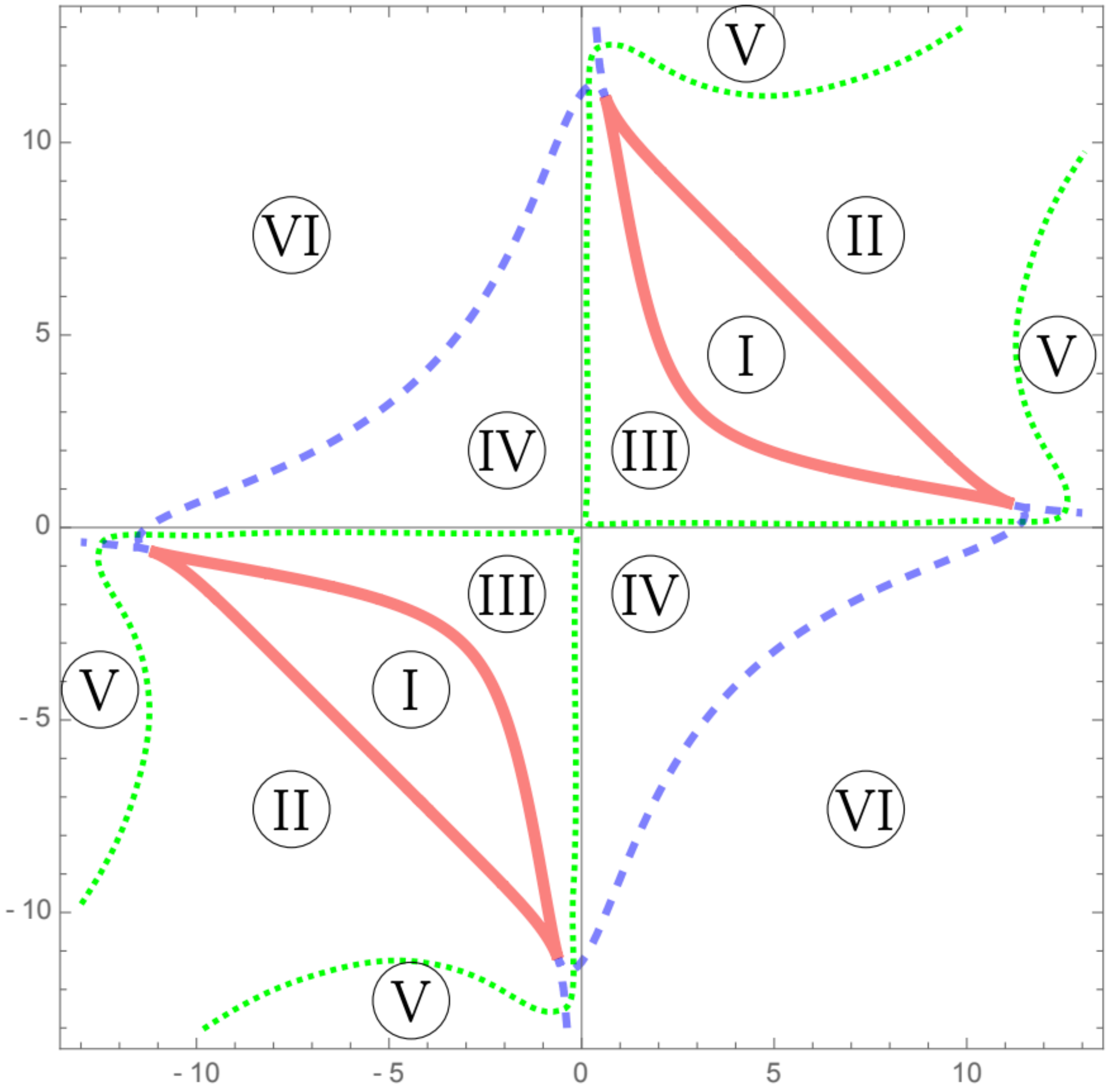
(d) $\hbar = 0.1$

Caustics in the Rosen-Morse Barrier

Caustics, **Stoke's phenomena** and **singularity crossings**, organize the classical paths solving the boundary value problem corresponding to the real-time Feynman path integral

By tracking the global structure, we define the path integral

We find six qualitatively different regions



Energy propagator

Classically forbidden behaviour is most easily seen in the energy propagator

$$K[x_1, x_0; E] \text{ “=” } \int_0^\infty \int_{x(0)=x_0}^{x(T)=x_1} e^{i(S[x]+ET)/\hbar} \mathcal{D}x \, \mathrm{d}T \qquad \frac{\delta S}{\delta x} = 0\,, \qquad E + \frac{\partial S}{\partial T} = 0\,,$$

Energy propagator

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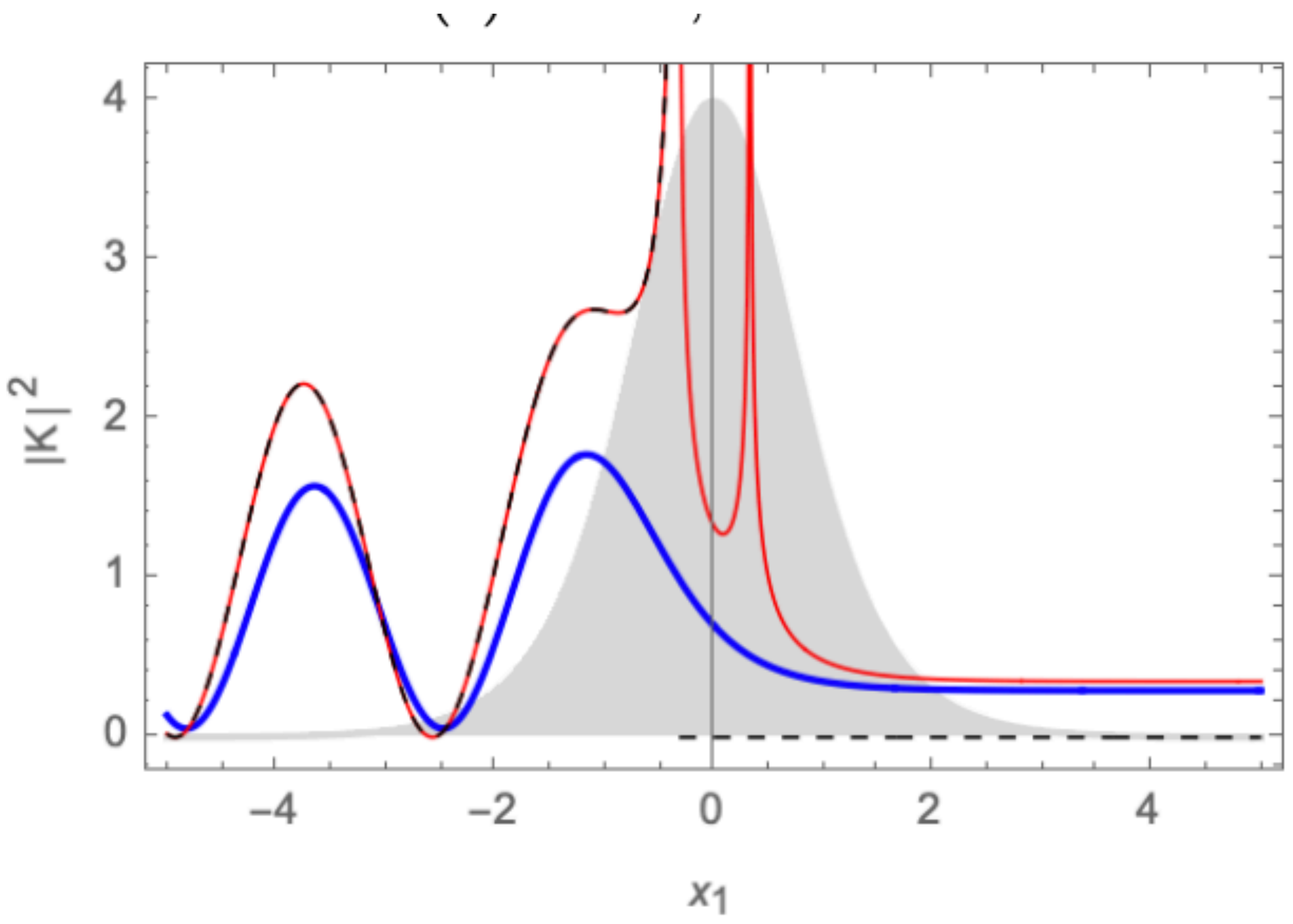
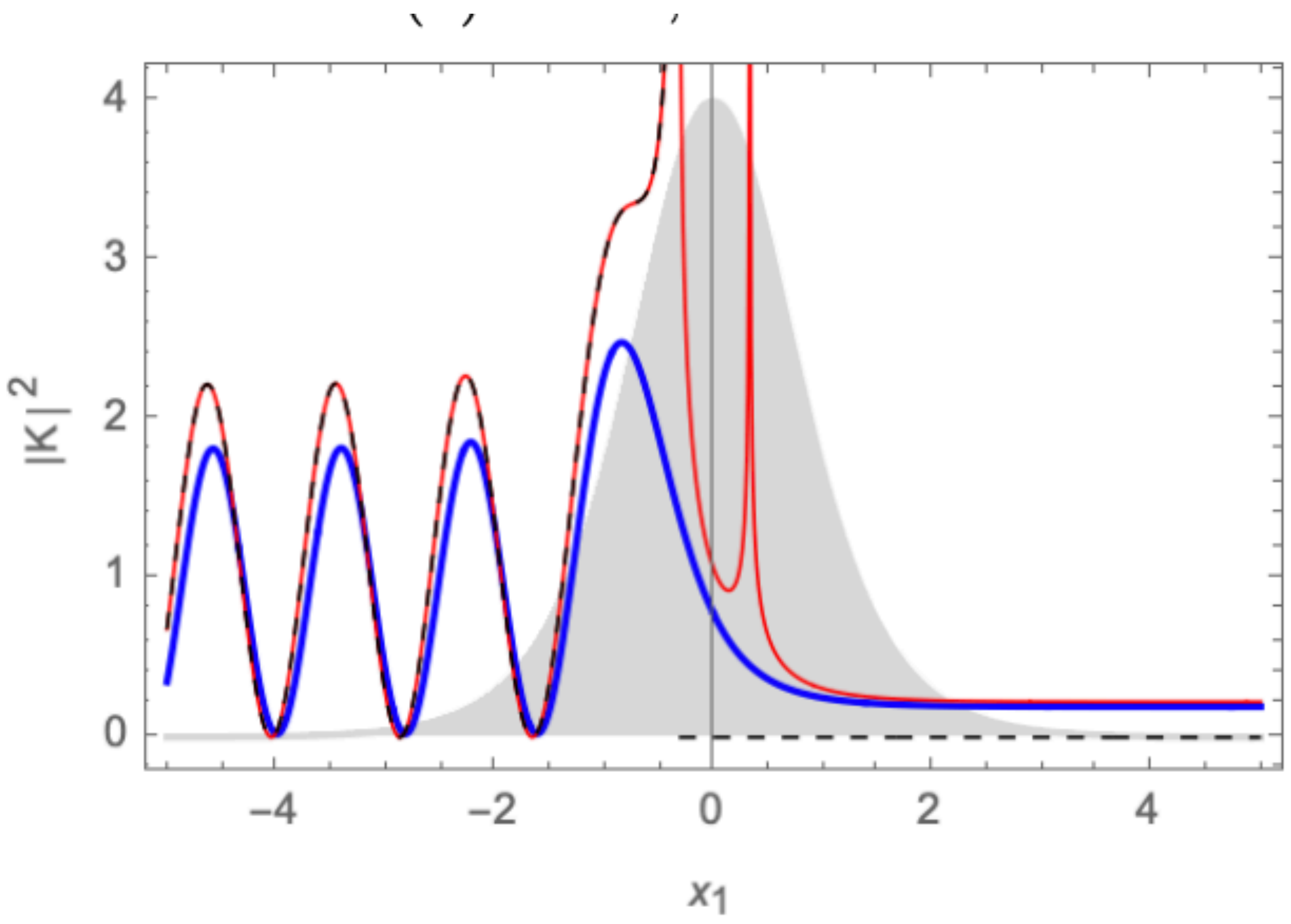
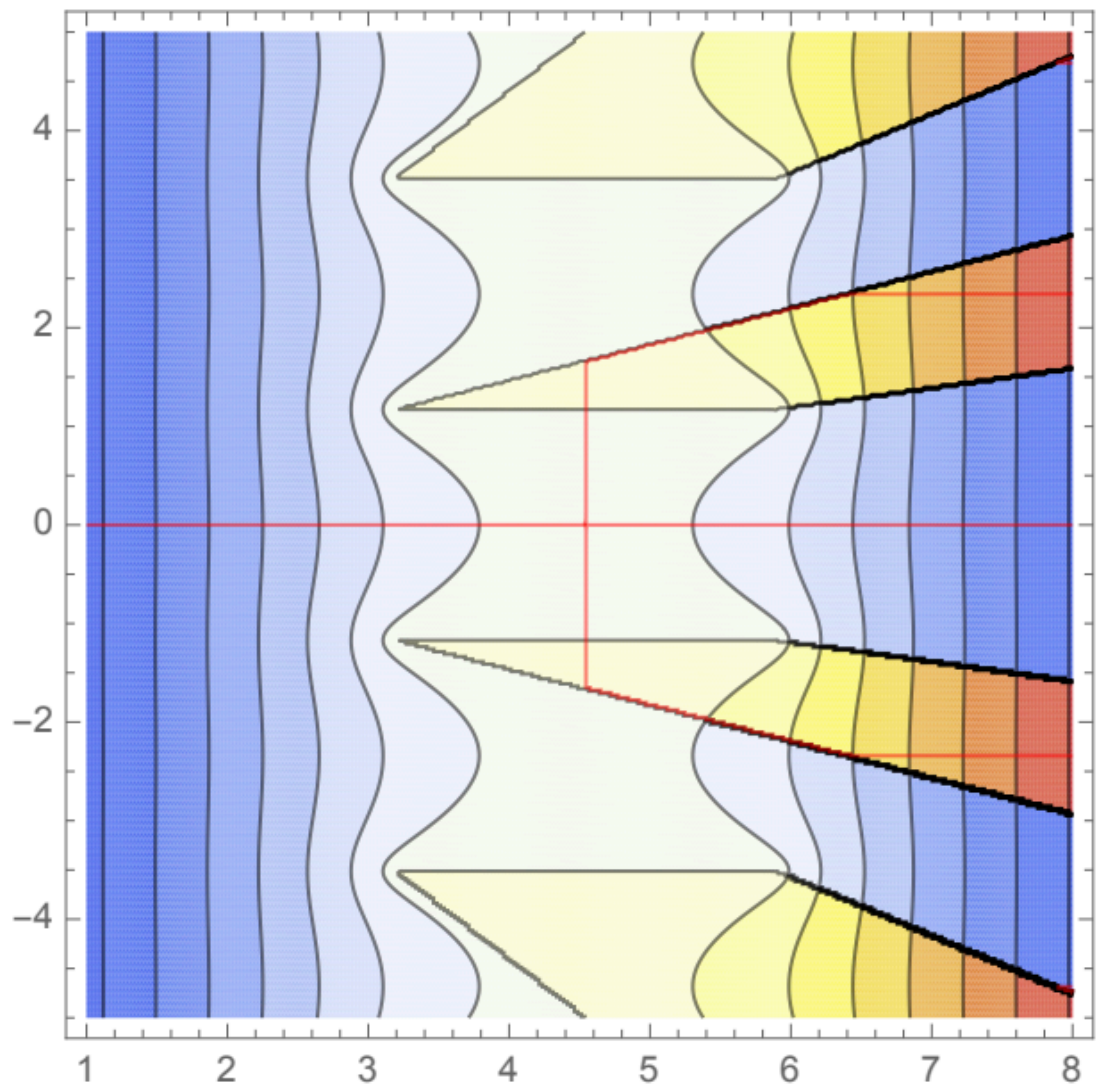
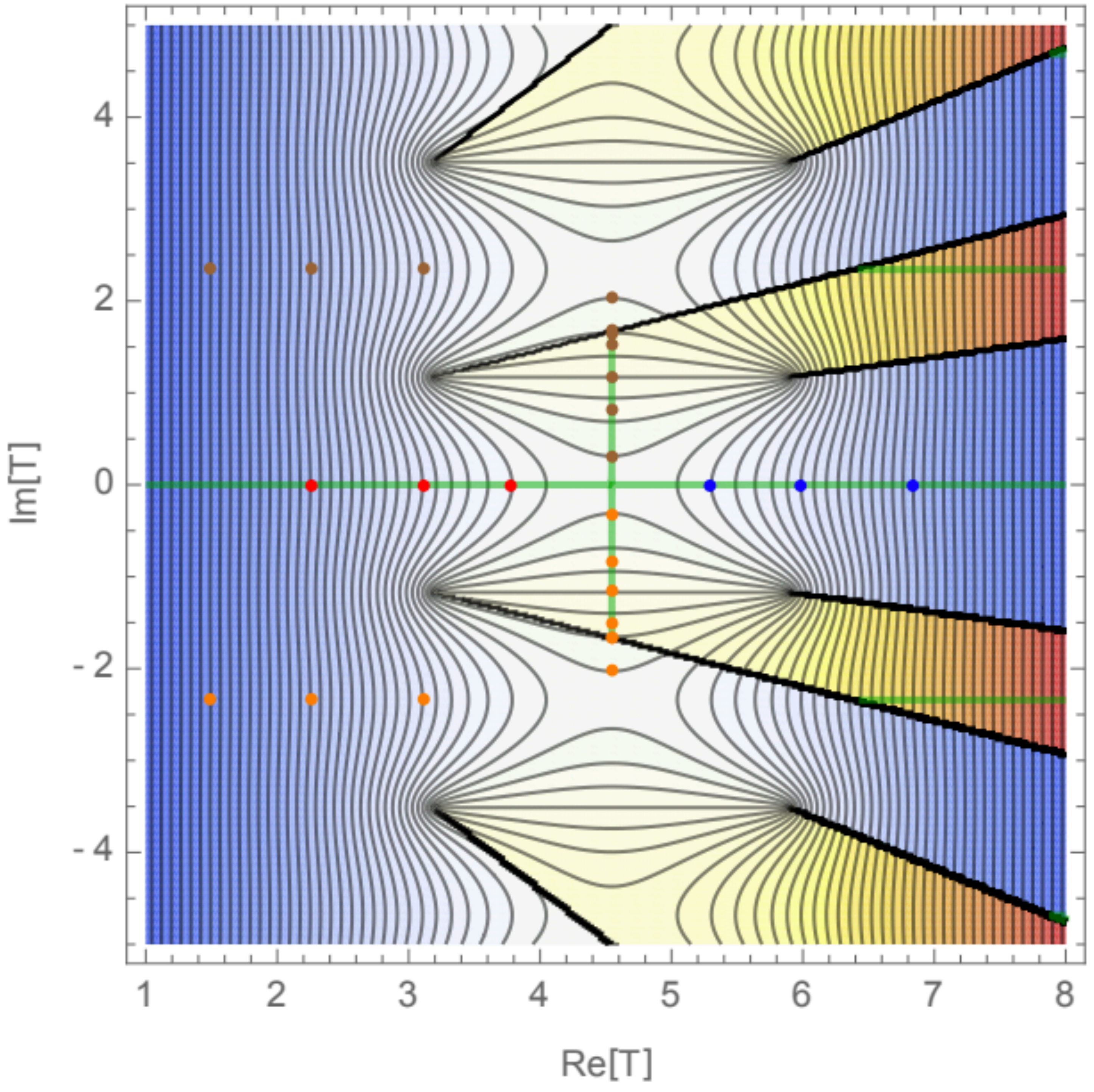


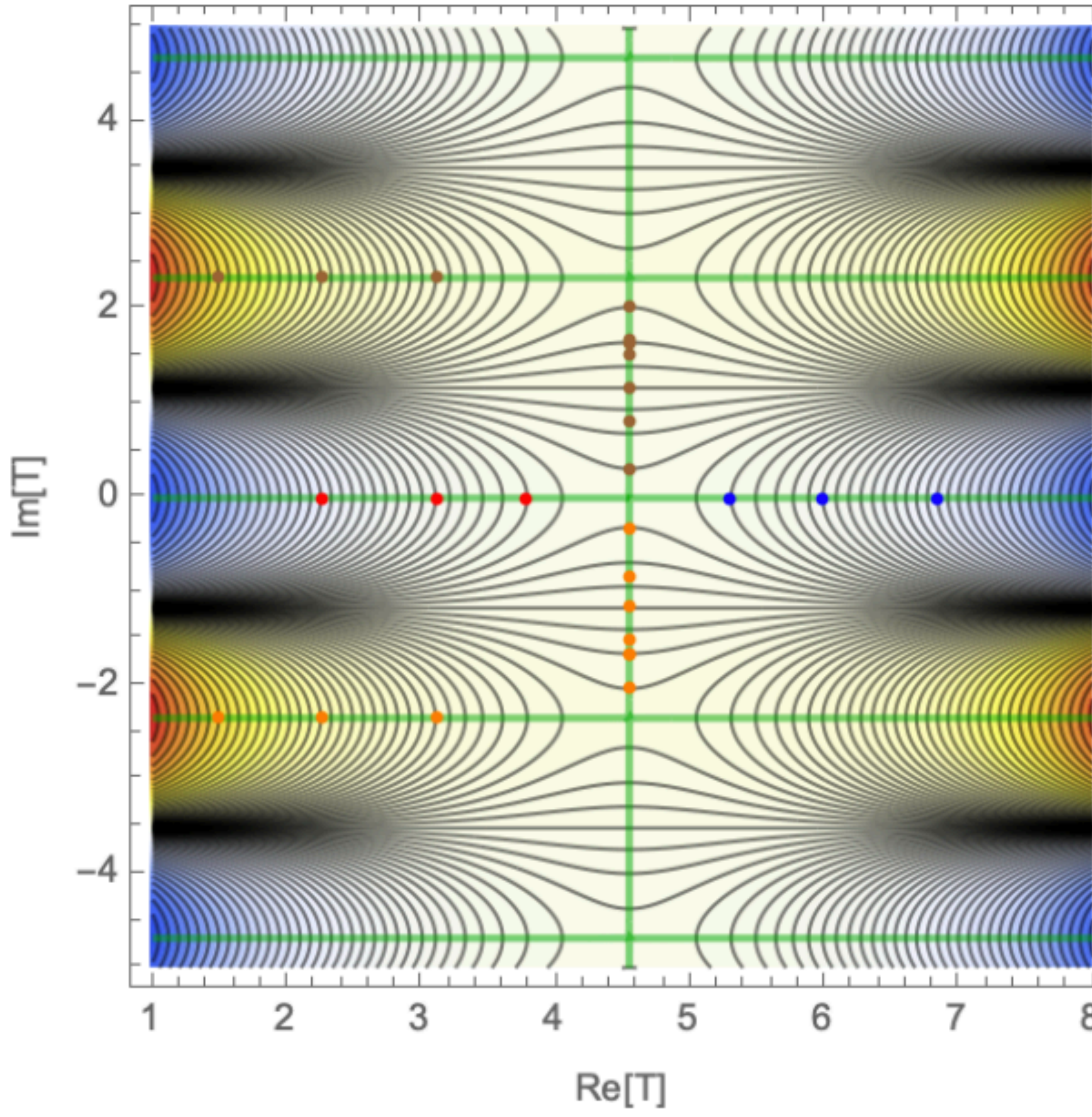


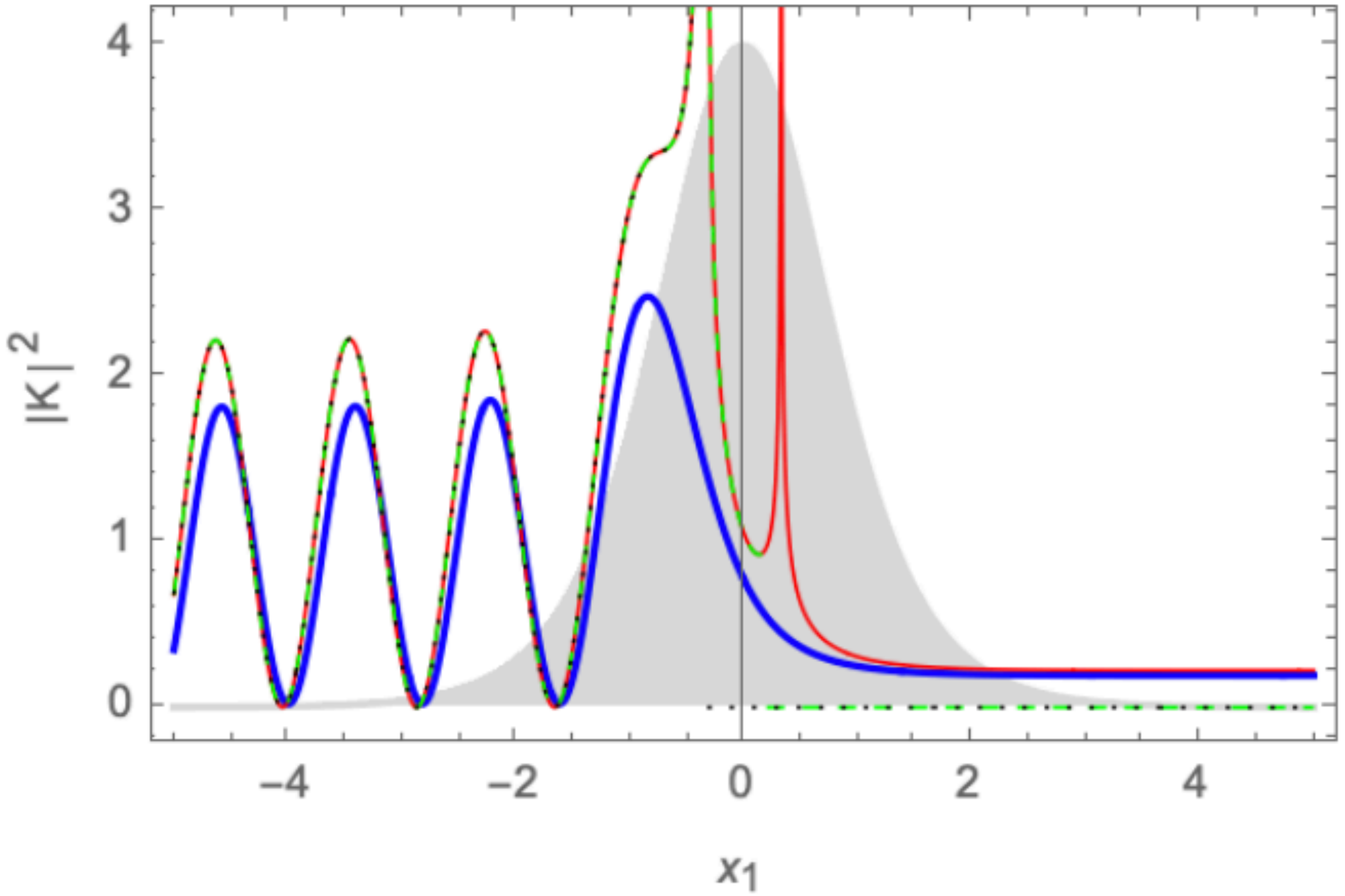


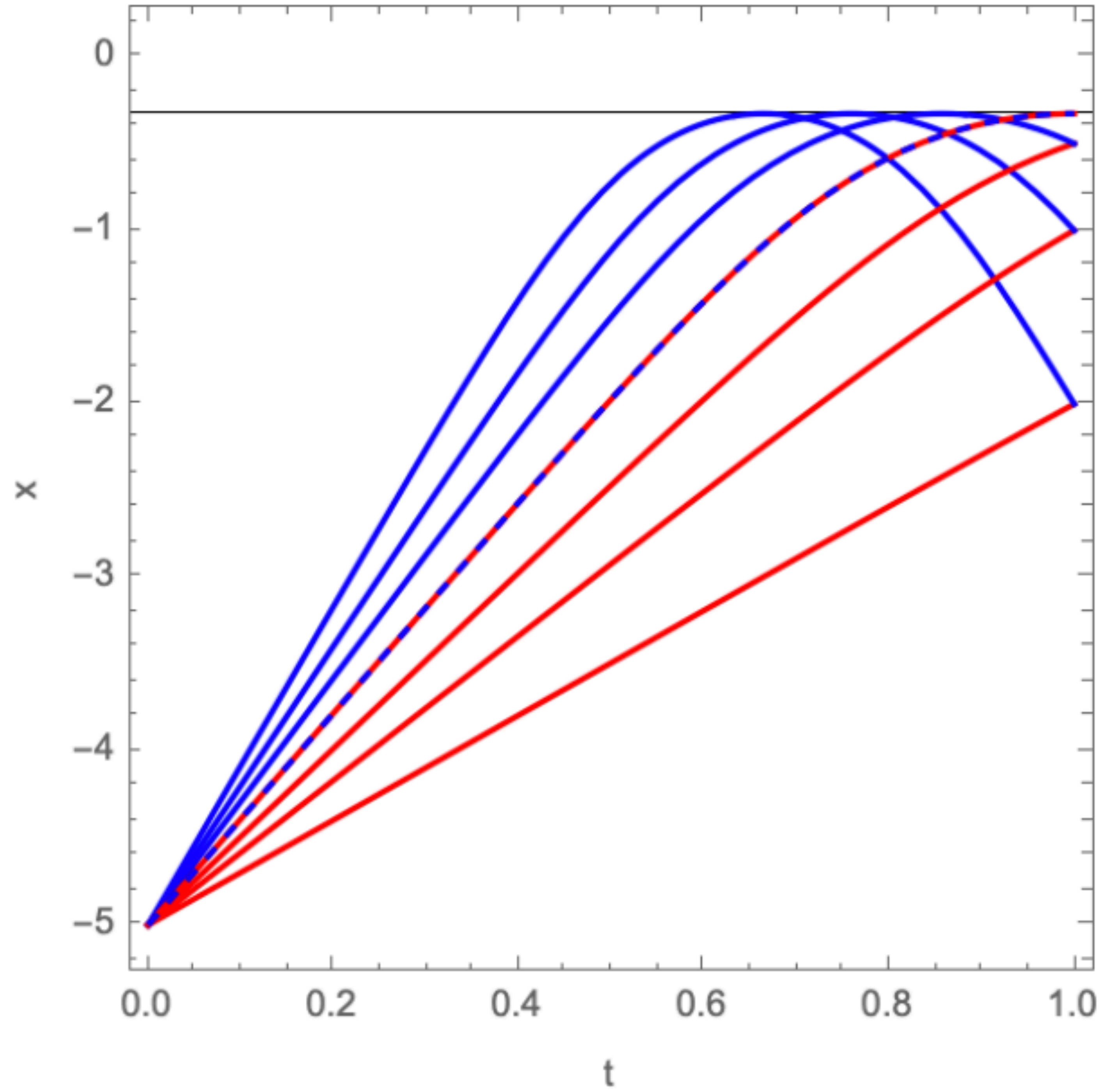
Energy propagator

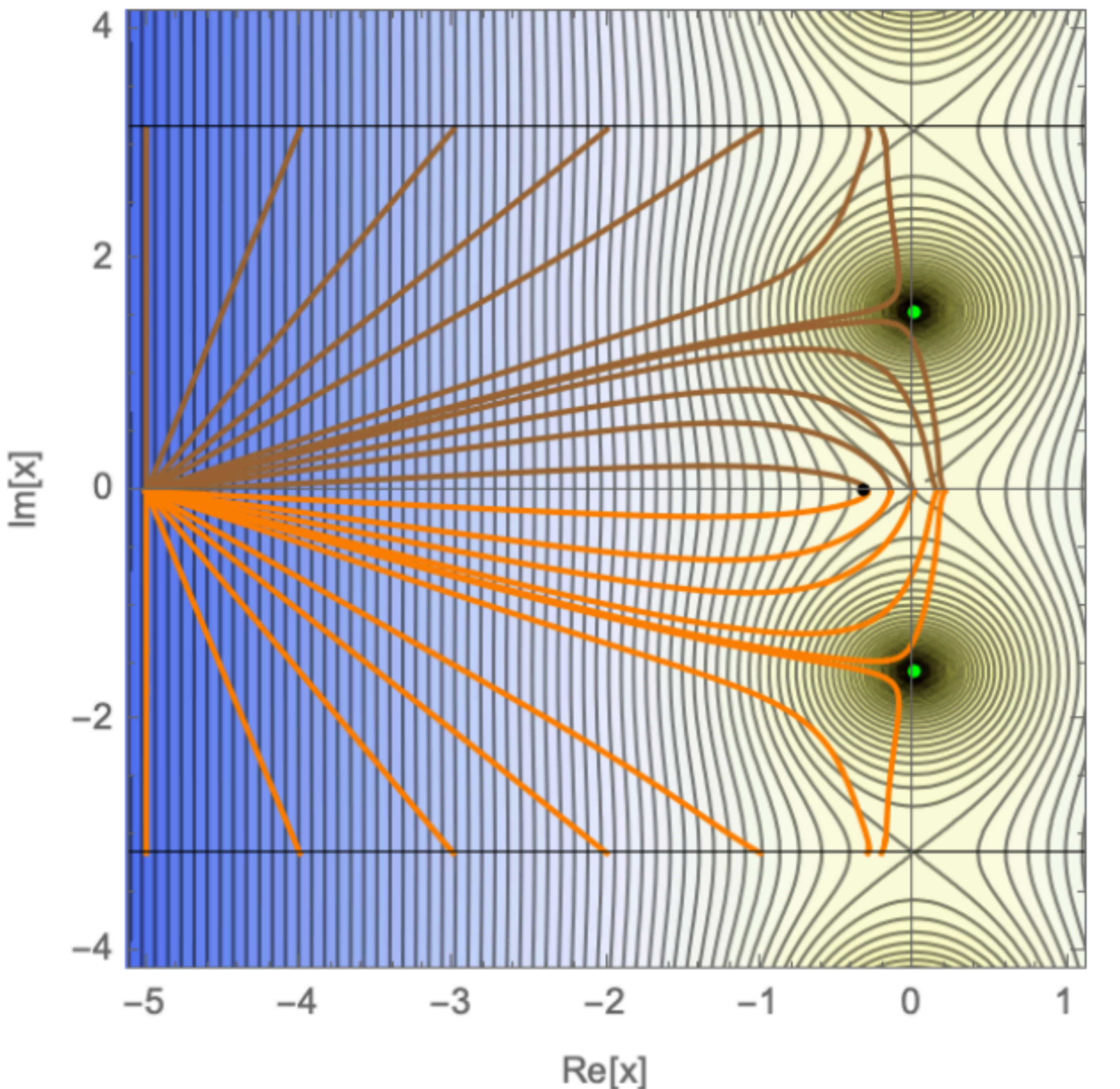
As we change the final position, the classical paths coalesce in a caustic and subsequently undergo a singularity crossing and complex caustic after with the classical path no longer solves the boundary value problem











Quantum tunnelling rate

Classically forbidden behaviour is most easily seen in the energy propagator

$$x_C(\lambda) = x_0 - i\pi\lambda$$

$$S_C = -\frac{m\pi^2}{2T} + i\sqrt{2mV_0}\pi$$

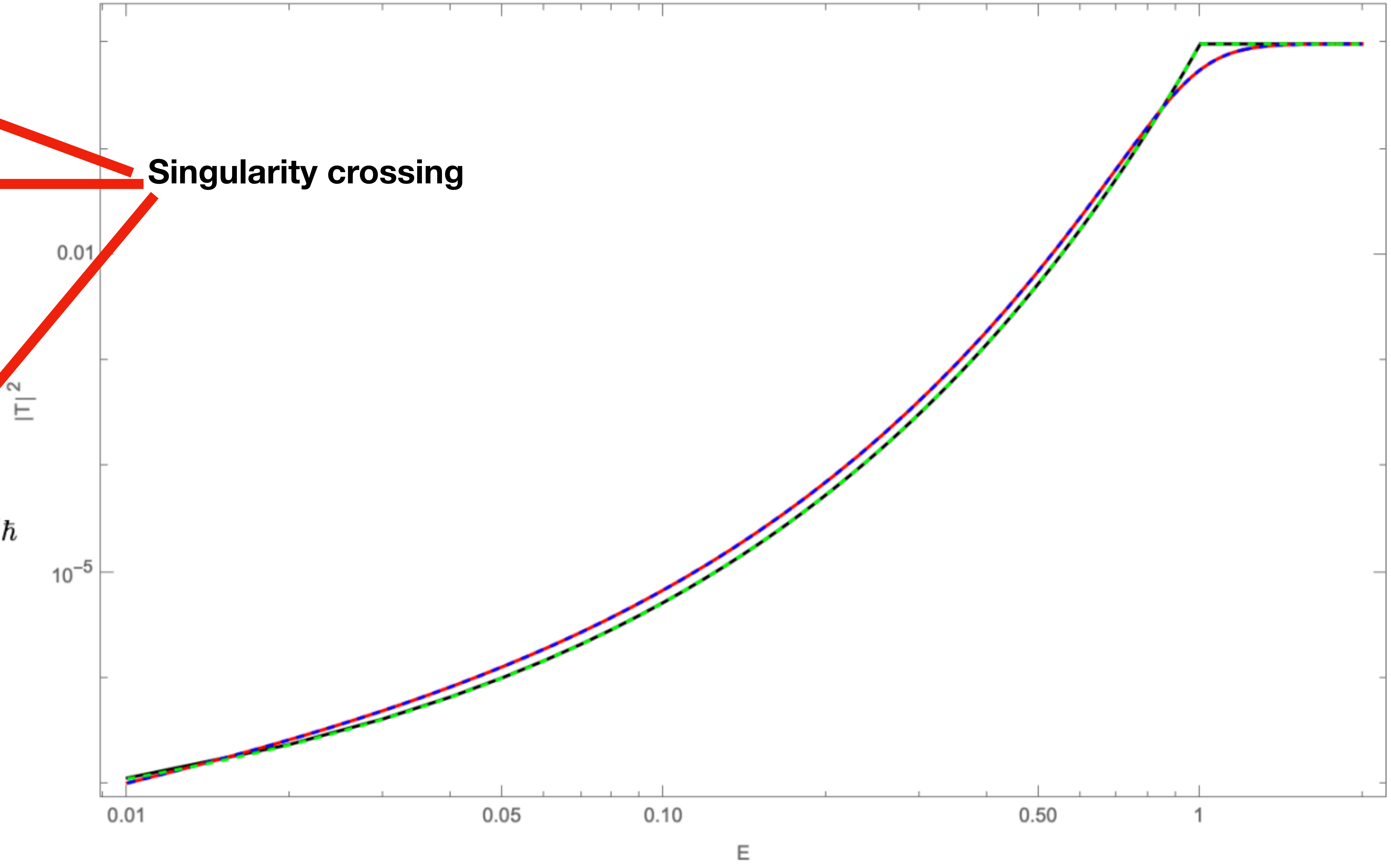
$$T_C = -i\sqrt{\frac{m\pi^2}{2E}}$$

Singularity crossing

We recover the WKB tunnelling rate:

$$\left|e^{i(S_C+ET_C)/\hbar}\right|^2 = e^{-2\pi\sqrt{2m}(\sqrt{V_0}-\sqrt{E})/\hbar}$$

Complex classical paths are indeed responsible for quantum tunnelling and and quantum reflections!



Caustics in the Rosen-Morse Well

The propagator consists of an **interference pattern** structured by **caustics**!

Summary

- **Interference** is central to our understanding of the **quantum universe**
- We **propose a new definition** of the real-time path integral using **Picard-Lefschetz theory**
- Instantons go **beyond** complex classical paths! Singularity crossings are central to a real-time description of quantum tunnelling
- We hope that this will be useful in **quantum mechanics, quantum field theory, and Lorentzian quantum cosmology**