Simplicial Group Field Theory models for euclidean quantum gravity: recent developments

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MAX-PLANCK-GESELLSCHAFT

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Outline of the talk.

First Part.

I. Introduction to Simplicial GFT models for 4d euclidean Quantum Gravity.

II. Model building ambiguities. Formal expressions of the (vacuum) amplitudes $\mathcal{A}^\beta$.

III. A new SGFT model (with Duflo map) based on the BO proposal. Explicit formula of the coefficient $w_{\mathrm{BO}}(j^-, j^+, j, \beta)$ encoding the simplicity constraints.

Second Part.

I. Numerical evaluation of the coefficients $w_{\mathrm{BO}}(j^-, j^+, j, \beta)$.

II. Analysis of the results.

Third Part.

I. UV divergences in SGFT and SF models. Why is it an important open issue?

II. Rad. corr. and their large-j scaling behaviour. How to compute the *d.o.d.*? Main strategies and known results.

III. Numerical approach to Rad. Corr. Preliminary results for the BO and EPRL SGFT models. Critical discussion of the results and their implications.

GFTs are combinatorially non-linear and non-local quantum/statistical field theories defined on several copies of a given Lie Group G or its Lie algebra $\mathfrak{g}$. Oriti, Krajewski, Freidel.

$$\Phi : G^{\times 4} \times S^3 \to \mathbb{R} \qquad S[\Phi] = \frac{1}{2}\int \Phi_k \mathcal{K}^{-1} \Phi_k - \frac{\lambda}{5!}\int \Phi_{k_1}\Phi_{k_2}\Phi_{k_3}\Phi_{k_4}\Phi_{k_5}\mathcal{V}$$
$$\Phi_k(G_i) = \Phi_k(G_1, G_2, G_3, G_4) \qquad G = \mathrm{SU}(2),\ \mathrm{Spin}(4),\ \mathrm{SL}(2,\mathbb{C}) \tag{1.1}$$

Two classes of models: Simplicial and Tensorial GFTs. Example of simplicial interaction.

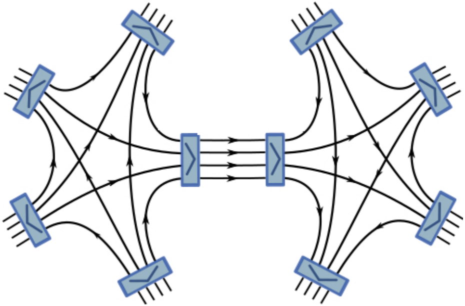


Figure: Two interaction vertices glued along one propagator.

I. SGFTs provides a QFT reformulation and completition of all Spin foam models.

II. The perturbative expansion of such models around the gaussian theory generates the sum on all possible simplicial complexes of all topologies as sum on Feyn graphs.

$$\mathcal{Z}[J] = \int [d\mu_{\mathcal{K}}(\Phi)]\; e^{-S_{\rm int}[\Phi]-J\Phi} = \sum_{\mathcal{G}} \frac{\lambda^N}{sym[\mathcal{G}]}\mathcal{A}(\mathcal{G})$$

III. Advantages. No triangulation dependence. Powerfull QFT methods to study the pert./non-pert. renormalizability and the continuum limit of a large class of models. Oriti, BenGeloun, Carrozza, Lahoche, Benedetti, Samary, Martini, Kowslowski, Duarte; Dittrich, Bahr, Steinhaus.

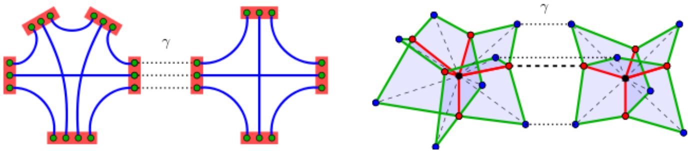


Figure: Gluing of two GFT building blocks.

Key aspect in the formulation of SGFT models for QG: definition and imposition of the required geometricity constraints turning topological simplicial BF theory into simplicial quantum gravity.

Closure constraint - Gauß low.

$$X_f \in \mathfrak{so}(4) \quad X_f \equiv (x_f^-, x_f^+) \quad \sum_{f \supset e} X_f = 0 \quad \mathrm{P}_{\mathrm{cl}}(G_i, \tilde{G}_i) = \int dH \prod_{i=1}^{4} \delta(G_i H \tilde{G}_i^{-1})$$

$$\Phi_k(G_i) = (\mathrm{P}_{\mathrm{cl}} \circ \Phi_{H \triangleright k})(G_i) = \Phi_{H \triangleright k}(HG_i) \quad H \triangleright k = h^+ k (h^-)^{-1} \tag{1.2}$$

Linear simplicity constraints (for the case of models with Barbero-Immirzi parameter).

$$({}^\star X_f^{IJ} - \gamma X_f^{IJ}) k_J = 0 \quad \Leftrightarrow \quad k x_f^- k^{-1} + \beta x_f^+ = 0 \qquad \beta = \frac{\gamma - 1}{\gamma + 1}$$

$$\mathrm{S}_k^\beta(G_i, \tilde{G}_i) = \sum_{J_i j_i} \int \left[du_i\right] \prod_{i=1}^{4} d_{J_i} d_{j_i}\, w(J_i, j_i, \beta) \Theta^{J_i} \left[G_i k^{-1} u_i k (\tilde{G})^{-1}\right] \chi^{j_i}\left[u_i\right] \tag{1.3}$$

Issue: ambiguities in the imposition of the geometricity constraints. Different strategies.

Purpose of the constraint imposition: define models where constraint violating field config. do not contribute to the dynamics. Main strategies and their implication.

I. Insert the constraint operator $\mathsf{C}_k^\beta = \mathsf{P}_{\rm cl} \circ \mathsf{S}_k^\beta$ in the interaction kernel.

II. Implement the constraints in the covariance and thus in the path int measure. Banburski,L.Q.Chen; BenGeloun.

$$\mathcal{K}^\beta = \delta\left[k(k')^{-1}\right](\mathsf{P}_{\rm cl} \circ \mathsf{S}_k^\beta \circ \mathsf{P}_{\rm cl}) \qquad \mathcal{V}^\beta = (\mathsf{S}_{k_i}^\beta \circ \mathsf{P}_{\rm cl} \circ \mathsf{P}_{\rm cl} \circ \mathsf{S}_{k_l}^\beta) \tag{1.4}$$

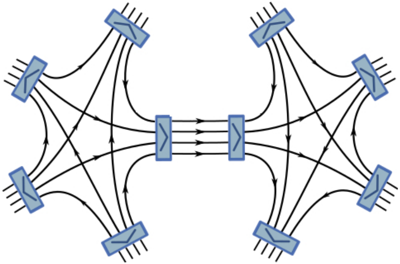
III. Impose the constraints acting with C_k^β on all the fields Φ_k in the GFT action.

IV. At the classical level GFT models resulting from different strategies will have diffent e.o.m and symmetries.

V. At the quantum level the above prescriptions will in general modify the properties of the resulting Feyn. ampl. (Spinfoam ampl.) in particular their large-j scaling behaviour.

VI. When the closure and simplicity constraint operator can be combined into a single orthogonal projector no construction ambiguities apper.

Key point: regardless of the above construction choices, the amplitudes $\mathcal{A}_{\mathcal{G}}$ have always the same general structure. In the spin picture, they only differ for the explicit form and the number of insertions of the coefficient encoding the simplicity constraints.
Notation: v, e, f denote the vertices, the edges (four-strands) and faces of an arbitrary vavuum GFT graph $\mathcal{G}$.



$$\mathcal{A}^{\beta}(\mathcal{G}) = \sum_{J_f j_{ef}} \sum_{I_{ve} i_e} \prod_{f\in\mathcal{G}} d_{J_f} \prod_{e\in\partial f} d_{j_{ef}} \prod_{e\in\mathcal{G}} \mathcal{A}_e^{\beta}(J_f, j_{ef}, I_{ve}, I_{v'e}, I_e) \prod_{v\in\mathcal{G}} \{15 J_f\}_v \quad (1.5)$$

$$\mathcal{A}_e^{\beta} = d_{I_{ve}} d_{I_{v'e}} d_{I_e} \left[\prod_{f\ni e} w^n(J_f, j_{ef}, \beta)\right] \mathscr{E}(J_f, j_{ef}, I_{ve}, I_{v'e}, I_e) \qquad J = (j^-, j^+) \quad I = (i^-, i^+) \quad (1.6)$$

$$\mathscr{E} = \begin{Bmatrix} j_1^- & i^- & j_2^- \\ j_1^+ & i^+ & j_2^+ \\ j_1 & i & j_2 \end{Bmatrix} \begin{Bmatrix} j_3^- & i^- & j_4^- \\ j_3^+ & i^+ & j_4^+ \\ j_3 & i & j_4 \end{Bmatrix} \begin{Bmatrix} j_1^- & i'^- & j_2^- \\ j_1^+ & i'^+ & j_2^+ \\ j_1 & i & j_2 \end{Bmatrix} \begin{Bmatrix} j_3^- & i'^- & j_4^- \\ j_3^+ & i'^+ & j_4^+ \\ j_3 & i & j_4 \end{Bmatrix} \quad (1.7)$$

The same amplitudes can be rewritten in the group representation, exploiting the duality between the spin and the holonomy formulation of the GFT framework. This is achieved either by taking the inverse Fourier transform giving the pure lattice gauge theory formulation of the model's amplitudes or directly from the corresponding Feyn. rules in group picture. Dittrich, Bahr, Steinhaus, Hellmann, Kaminski.

$$\mathcal{A}^{\beta}(\mathcal{G}) = \int \Big[\prod_{v\in\mathcal{G}}\prod_{e\ni v} dH_{ve}\Big]\Big[\prod_{e\in\mathcal{G}} dk_e\Big] \prod_{f\in\mathcal{F}_{\mathrm{cl}}} \mathcal{A}^{\beta}_f(H_{ve},k_e) \tag{1.8}$$

$$\mathcal{A}^{\beta}_f(H_{ve},k_e) = \sum_{J_f j_{ef}} d_{J_f}\int\Big[\prod_{e\in f} du_{ef}\Big]\Theta^{J_f}\Big[\prod_{e\in f} H_{ve}U_{ef}(H_{v'e})^{-1}\Big]\Big[\prod_{e\in f} d_{j_{ef}}\, w^{q}(J_f,j_{ef},\beta)\chi^{j_{ef}}(u_{ef})\Big] \tag{1.9}$$

Simplicity constraints: relation between the selfdual/anti-selfdual parts of a bivector. Standard criteria: translate this condition into a restriction on spins, Perelomov states, spinors. Perez.

EPRL Model: Engle,Pereira,Rovelli,Livine; Engle,Zipfel; S.Alexandrov; KKL.

$$w_{\rm EPRL}(j^-, j^+, j, \beta) = \delta_{j^- \, |\beta| j^+} \delta_{j\,(1-\beta) j^+} \qquad |\beta| \leq 1 \tag{1.10}$$

BO Model: Baratin,Oriti;FO (to appear). Exploiting the lie algebra formulation, the simplicity constraints are imposed directly on flux variables via $\star$ multiplication with a NC $\delta^\star$.

$$\Phi^\beta_{\rm BO}(X_i, k) = \prod_{i=1}^{4} S^\beta_{\rm BO}(X_i, k) \star \Phi(X_1, X_2, X_3, X_4, k) \tag{1.11}$$

$$S^\beta_{\rm BO}(x^-, x^+, k) = \delta_{-kx^- k^{-1}}\left(\beta x^+\right) = \int du\, E_{k^{-1}uk}(x^-) E_u(\beta x^+) \qquad u \in {\rm SU}(2) \tag{1.12}$$

We need S^β_k to be in the image of the NC Fourier Transform. This requirement implies:

$$E_u(\beta x^+) = \Omega(\psi_u, \beta) E_{u^\beta}(x^+) \tag{1.13}$$

The above Eq. admits a unique non-trivial solution defining the group element u^β.

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$$u^\beta = e^{i\frac{\psi_\beta}{2}\hat{n}_\beta\cdot\vec{\sigma}} \quad \psi_\beta = |\beta|\psi \quad \psi \in [0, 2\pi[\quad \hat{n}_\beta = \mathrm{sign}(\beta)\hat{n} \quad \Omega(\beta,\psi) = \frac{\sin\frac{|\beta|\psi}{2}}{\beta\sin\frac{\psi}{2}} \tag{1.14}$$

Few important remarks:

I. The simplicity constraints S_k^β encodes the choice of the Q. map via the parametric deformation of the group element u^β dictated by the form of the NC plane waves.

II. For generic values of β, S_k^β it is not an orthogonal projector unless $\beta = 0, 1$.

III. It commutes with the closure constraint projector $\mathrm{P}_{\mathrm{cl}} \circ S^\beta_{H^{-1}\triangleright k} = S_k^\beta \circ \mathrm{P}_{\mathrm{cl}}$ up to a rotation of the normal.

IV. It does not impose any rationality condition on the Immirzi parameter.

Upon group Fourier transform and Peter-Weyl decomposition we find:

$$S^{j^-j^+\beta}_{m^-m^+n^-n^+} = \int d\mathscr{U}\, D^{j^-}_{m^-n^-}(u) D^{j^+}_{m^+n^+}(u^\beta) = \sum_{jm} C^{j^-j^+j}_{m^-m^+m} C^{j^-j^+j}_{n^-n^+m} w_{\mathrm{BO}}(j^-, j^+, j, \beta) \tag{1.15}$$

The coefficient $w_{\rm BO}(j^-, j^+, j, \beta)$ encoding the simpl. constraints is defined as:

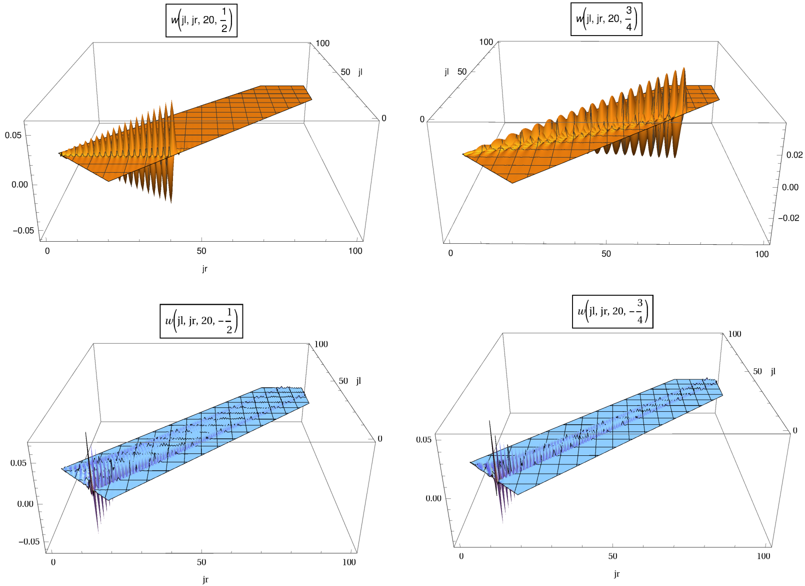
$$w_{\rm BO}(j^-, j^+, j, \beta) = \frac{(-1)^{j^-+j^++j}}{\pi\sqrt{(2j^-+1)(2j^++1)}} \sum_{a=0}^{\lambda} (\mathrm{sign}(\beta))^a \begin{Bmatrix} a & j^- & j^- \\ j & j^+ & j^+ \end{Bmatrix} \mathcal{T}_a^{j^-j^+}(|\beta|) \quad (1.16)$$

$$\mathcal{T}_a^{j^-j^+}(|\beta|) = (2a+1) \sum_{p=-j^-}^{j^-} \sum_{q=-j^+}^{j^+} C^{j^-\,a\,j^-}_{p\,0\,p}\, C^{j^+\,a\,j^+}_{q\,0\,q}\, \Upsilon_{pq}(|\beta|) \quad (1.17)$$

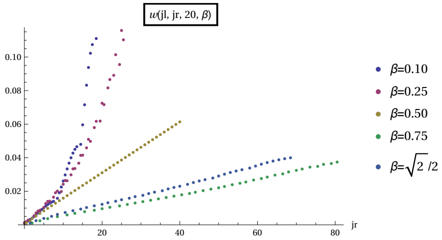
$$\Upsilon_{pq}(|\beta|) = \int_0^{2\pi} d\psi\, \frac{1}{|\beta|} \sin\frac{\psi}{2} \sin\frac{|\beta|\psi}{2}\, e^{-i(p+|\beta|q)\psi}$$

$$= \begin{cases} \frac{i-ie^{2i\pi|\beta|}+2\pi|\beta|(|\beta|-1)}{4\beta^2(|\beta|-1)} & \forall\, p,q; \quad 2(p+|\beta|q) = 1-|\beta| \\ \frac{i-ie^{2i\pi|\beta|}-2\pi|\beta|(|\beta|+1)}{4\beta^2(|\beta|+1)} & \forall\, p,q; \quad 2(p+|\beta|q) = -1-|\beta| \\ -\frac{i-ie^{-2i\pi|\beta|}+2\pi|\beta|(|\beta|+1)}{4\beta^2(|\beta|+1)} & \forall\, p,q; \quad 2(p+|\beta|q) = 1+|\beta| \\ -\frac{i-ie^{-2i\pi|\beta|}-2\pi|\beta|(|\beta|-1)}{4\beta^2(|\beta|-1)} & \forall\, p,q; \quad 2(p+|\beta|q) = -1+|\beta| \\ \frac{8i|\beta|(p+|\beta|q)e^{-2i\pi(p+|\beta|q)}\cos|\beta|\pi - 2\left(-1+\beta^2+(p+|\beta|q)^2\right)e^{-2i\pi(p+|\beta|q)}\sin|\beta|\pi + 8i|\beta|(p+|\beta|q)}{|\beta|(2|\beta|q+|\beta|+2p-1)(2|\beta|q+|\beta|+2p+1)(2|\beta|q-|\beta|+2p-1)(2|\beta|q-|\beta|+2p+1)} & \end{cases} \quad (1.18)$$

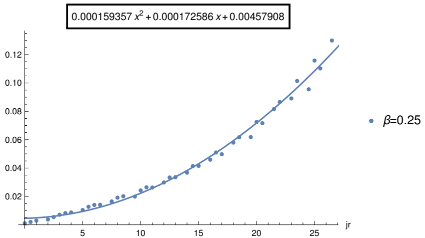
Next step: numerical analysis of the above coefficient. Oriti, Celoria, Finocchiaro.



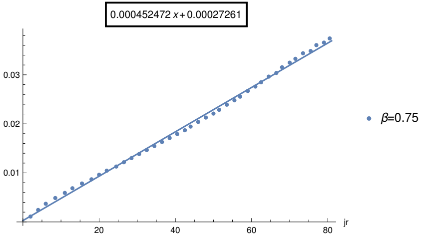
$w\left(\text{jl}, \text{jr}, 20, \frac{1}{2}\right)$
$w\left(\text{jl}, \text{jr}, 20, \frac{3}{4}\right)$
$w\left(\text{jl}, \text{jr}, 20, -\frac{1}{2}\right)$
$w\left(\text{jl}, \text{jr}, 20, -\frac{3}{4}\right)$
jr
jl
0.05
0.00
−0.05
0.02
−0.02
0
50
100



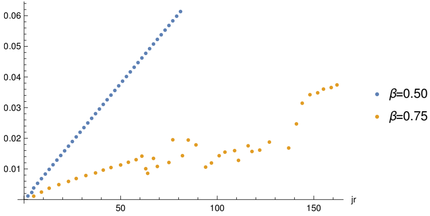
w(jl, jr, 20, β)
jr
β=0.10
β=0.25
β=0.50
β=0.75
β=√2 /2



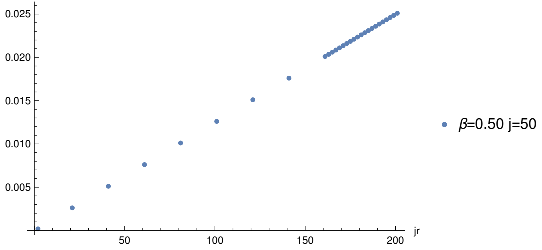
0.000159357 x² + 0.000172586 x + 0.00457908
jr
β=0.25

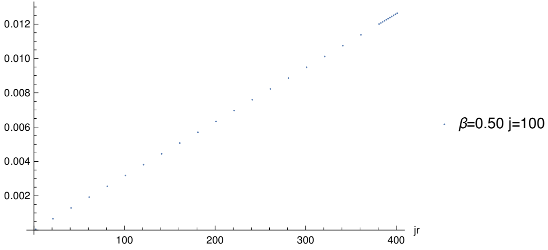


0.000452472 x + 0.00027261
jr
β=0.75



jr
β=0.50
β=0.75


0.025
0.020
0.015
0.010
0.005
50
100
150
200
jr
β=0.50 j=50


0.012
0.010
0.008
0.006
0.004
0.002
100
200
300
400
jr
β=0.50 j=100

I. The analysis of the divergences allows us to understand in a quantitative way how different model building prescriptions (e.g. different choices of the face weights, different way of imposing the constraint) affect the UV scaling beahviour of the resulting amplitudes.

II. It provides a criterion to put constraints on the ambiguities involved in the models'definitions. E.g. tuning the choice of the face weights to obtain convergent results for specific families of bubbles.

III. It allows to extract an approximate exstimate of the degree of divergence of specific classes of Feyn. graphs useful in searching for power counting renorm. models. More in general it represents a preliminary exercise providing important indication about the large-N limit analysis of geometric 4d SGFT models.

IV. It allows us to formulate a practical euristic ansatz for the theory space, e.g. by keeping those interactions required to consistently renormalize the model up to a given order in perturbation theory. This is needed if we want to study the FRG of SGFT.

V. Moreover divergences are expected to be related to the existence of residual gauge symmetries (diffeomorphisms).Therefore the study of the divergences can give us an hint about the presence of such residual symmetries.

Let us mention the know results simultaneously giving an overview of the method used.

I. Radiative corrections to the self-energy and interaction vertex of the topological SU(2) BF and EPRL Spinfoam models both in the Euclidean and Lorentzian cases (only the self energy in the second case). Perini,Rovelli,Speziale;Riello.

II. Exact degree of divergence of arbitrary connected 2-complexes in the Riemannian Dupuis-Livine (holomorphic) Spinfoam model. L.Q.Chen;Banburski,L.Q.Chen; Freidel,Hnybida,Dupuis,Livine,Speziale,Tambornino.

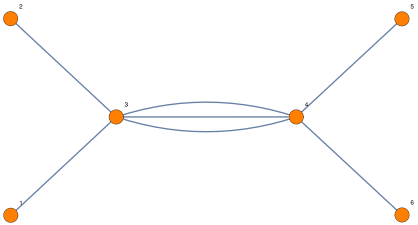
III. Holonomy Spinfoam models: single and multiple bubble contributions in the BC model. Divergences and the Wave Front Set analysis. Dittrich,Bonzom, Dittrich, Bahr,Hellmann,Kaminski. Mathematical characterization of (bubble) divergences in Spinfoam models; Bonzom,Smerlak.

IV. Radiative correction to the two-point function of Boulatov-Ooguri GFT model via heat kernel regularization. BenGeloun,Bonzom.

V. Radiative corrections to the euclidean EPRL GFT model with geometricity constraint enforced either in the propagator or in the Vertex. BenGeloun,Gurau, Rivasseau;Krajewski,Magnen,Rivasseau,Tanasa,Vitale.

VI. Scaling bounds, powercounting results and perturbative renormalizability of topological TGFTs in 3d and $4d$. Oriti,Gurau,Freidel,BenGeloun,Carrozza,Carrozza.

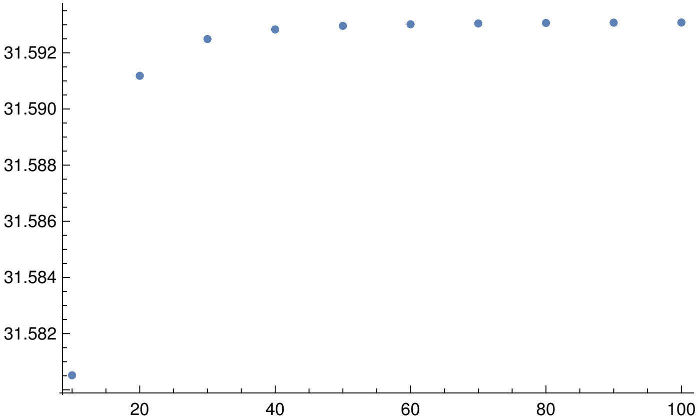
Radiative correction to the four-point function. Leading order term. Comparing EPRL and BO.



Evaluating the scaling behaviour of the bulk amplitude.

$$\mathcal{A}^{\mathrm{EPRL}}_{\mathrm{bulk}}(\mathcal{G}^4,\beta)=\sum_{j^+}\frac{[2(1-|\beta|)j^+ +1]^3}{(2|\beta|j^+ +1)(2j^+ +1)} \tag{1.19}$$

$$\mathcal{A}^{\mathrm{BO}}_{\mathrm{bulk}}(\mathcal{G}^4,\beta)=\sum_{j_1^- j_1^+}\sum_{j_1 j_2 j_3}\frac{d_{j_1}d_{j_2}d_{j_3}}{d_{j_1^-}d_{j_1^+}}w_{\mathrm{BO}}(j_1^-,j_1^+,j_1,\beta)w_{\mathrm{BO}}(j_1^-,j_1^+,j_2,\beta)w_{\mathrm{BO}}(j_1^-,j_1^+,j_3,\beta) \tag{1.20}$$



β=0.50

Figure: Plot of the Bulk amplitude as function of the Cutoff. BO model.

Summary

- I In this talk I presented a new GFT-Spinfoam model. Oriti,Finocchiaro (to appear)
- II We provided the explicit formula of the coefficient encoding the simplicity constraints. We studied its behaviour numerically and discussed the results. Oriti, Celoria,Finocchiaro (to appear)
- III With a very simple example we showed how to use the above results (BO model) to study the scaling behaviour of radiative corrections via numerical methods, comparing our result to the EPRL model.

Outlook

- I Perform a systematic analysis of the leading order radiative corrections to the n-point functions $n \leq 5$ for different values of the β and different prescription for implementig the simplicity constraints. Perform a similar analysis for the EPRL model
- II Extract an approximate estimate of the degree of divergences of specific classes of Feyn. graphs. Try to formulate an ansatz for the theory space of SGFT models.
- III Define the analog of the BO model with $SL(2, \mathbb{C})$ group. Oriti,Rosati (to appear).