

Simplicity constraints in spin foam models

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1. Introduction
2. **Simplicity constraints**
 - 2.1. Continuum
 - 2.2. Discrete
 - 2.3. Geometrical interpretation
 - 2.4. Future work
3. Path integral formulations
4. Importance of the secondary constraints
5. New proposals
 - 5.1. GFT in non-commutative flux variables
 - 5.2. Semi-discrete actions
6. Conclusions

What are spin foams?

- Attempt to write the bare path integral for gravity as

$$\mathcal{Z} = \sum_{\text{(group data)}} \prod_{\text{(building blocks)}} \text{Amplitudes}$$

- Motivated by the need to construct transition amplitudes for canonical LQG

How to construct spin foams?

- Typically from discrete Plebanski action (BC, Reisenberger, EPRL, BO, ...)
- Also attempts from Holst action (Baratin, Flori, Thiemann)

Many encouraging results so far

- Well-defined transition amplitudes for all spin networks (KKL)
- Ultraviolet finite and infrared finite ($\Lambda \neq 0$) (Noui, Roche, Meusburger, Fairbairn, Han, ...)
- Promising large quantum number behaviour (Barrett et al.)

Many big questions and room for further research/researchers

- Summation and continuum limit (e.g. Gurau et al.)
- Proper implementation of the projector (and link with LQG) (e.g. Alesci et al.)
- Explicit evaluation of the quantum corrections (e.g. Riello)

Simplicity constraints in the continuum

- Ensure correct number of degrees of freedom
- Canonical structure: conjugated primary $\Phi(B)$ and secondary $\dot{\Phi}(B) = \Psi(B, \omega)$ second class constraints

Upon discretization

- (some) simplicity constraints second class with themselves
- No clear notion of physical degrees of freedom
- No clear notion of Hamiltonian, time evolution, and appearance of secondary constraints (see Dittrich)

These difficulties lead to important open questions for spin foams

- Torsion-freeness and flatness issue (Hellmann et al.)
- What geometry beyond that of four-simplices?
- What is the geometrical meaning of quantum corrections (offshell contributions)?
- Precise link with canonical theory
- Broader questions: continuum limit, renormalization, extraction of physics, ...

Plebanski action $S[B, \omega, \phi] = \int_{\mathcal{M}} \left(\eta_{IJKL} B^{IJ} \wedge F^{KL}[\omega] + \phi_{IJKL} B^{IJ} \wedge B^{KL} \right)$

Lagrangian picture

- Constraints $\delta_{\phi} S = 0 \rightarrow \mathcal{C} = B^{IJ} \wedge B^{KL} - \sigma \mathcal{V} \varepsilon^{IJKL} = 0$ enforcing

$$B^{IJ} = \pm e^I \wedge e^J \quad \text{or} \quad B^{IJ} = \pm (\star e \wedge e)^{IJ}$$

(a degenerate sector also exist)

- 20 components decomposed into irreps as

$$\underbrace{(\mathbf{2}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{2})}_{\text{reduce internal gauge}} \oplus \underbrace{(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{0})}_{\text{impose left = right Urbantke metric}}$$

- Relaxing the constraints too much yields extra degrees of freedom (Krasnov, SS)

Hamiltonian picture ($\mathcal{M} = \mathbb{R} \times \Sigma$)

- Primary simplicity imposes $B|_{\Sigma} = e \wedge e$
- Secondary simplicity imposes $\omega|_{\Sigma} = \Gamma(e)$ spatial torsion-freeness
- Note however that while $d_{\Gamma} e|_{\Sigma} = 0$,
the Ashtekar–Barbero derivative gives $d_A e|_{\Sigma} = K \wedge e \neq 0$

On the secondary simplicity constraints

- They play a key role in the continuum theory (e.g. Lorentz-covariant analysis of Alexandrov), so why don't we hear much about them in spin foams?
- Being a covariant approach, it is supposed to know about the non-trivial embedding of the AB connection in the Lorentzian phase space
- The secondary constraint requires a Hamiltonian in order to be derived
- Discretization breaks (a priori) diff invariance \rightarrow no clear distinction between primary and secondary

Discrete Lagrangian (covariant) constraints à la Barrett-Crane

- A geometric four-simplex is uniquely determined by triangle bivectors satisfying

$$\left\{ \begin{array}{ll} \text{diagonal simplicity} & \text{Tr}(\star X_f X_f) = 0 \\ \text{off-diagonal simplicity} & \text{Tr}(\star X_f X_{f'}) = 0 \\ \text{volume simplicity} & \text{Tr}(\star X_{f_1} X_{f'_1}) = \text{Tr}(\star X_{f_2} X_{f'_2}) \end{array} \right.$$

- Volume simplicity involves holonomies (i.e. the connection) and thus can be seen as a secondary constraint
- It can however be replaced by closure + four-simplex flatness, so one can use

$$\left\{ \begin{array}{ll} \text{diagonal simplicity} & \text{Tr}(\star X_f X_f) = 0 \\ \text{off-diagonal simplicity} & \text{Tr}(\star X_f X_{f'}) = 0 \\ \text{closure} & \sum_{f \supset e} X_f = 0 \end{array} \right.$$

or

$$\left\{ \begin{array}{ll} \text{linear simplicity} & \star(n_e)_I X_f^{IJ} = 0, \quad \forall f \in e \\ \text{closure} & \sum_{f \supset e} X_f = 0 \end{array} \right.$$

- The logic here is to define discrete constraints that reproduce the end result of the continuum ones: **existence of a unique Riemannian structure**

Regge geometries

- Describe discrete Riemannian geometry of manifold using four-simplices and their edge lengths
- There is a notion of Levi–Civita connection Γ

Important questions

- How is the Levi-Civita connection defined in terms of holonomies and fluxes?
- What is the role of the $\mathfrak{su}(2)$ Ashtekar–Barbero connection and the Barbero–Immirzi parameter γ ?

Remark: Regge's key condition of restricting the holonomy on a plane is only consistent for Levi–Civita and not for Ashtekar–Barbero

Twisted geometries

- Phase space of covariant LQG and SF in terms of generalized Regge geometries: piecewise-flat polyhedra, discontinuous metric
- Works for arbitrary cellular decomposition
- Crucial information on K that distinguishes A from Γ lies in the gauge orbit of the diagonal simplicity constraint
- If, like in the continuum, there are secondary constraints, they provide a non-trivial gauge fixing of these orbits, i.e. a non-trivial embedding of the $\mathfrak{su}(2)$ connection into the Lorentzian one

Discrete secondary constraints

- Lack of secondary constraints often criticized in the Lagrangian approach
- Issue not seen (a priori) by the asymptotics, which is on-shell
- How to define them in the discrete, where there is no canonical analysis?
- Can one discretize directly the $T = 0$ condition?
 - ▶ Dittrich, Ryan: yes, and it leads to shape matching conditions
 - ▶ Marseille: a torsion-free connection can be defined without shape matching

Toy model (Anzà, SS)

- Single four-simplex and a Hamiltonian imposing flatness
 - ▶ Secondary constraints arise, and give the embedding of $\mathfrak{su}(2)$ connection into the Lorentzian one (identical as continuum)
 - ▶ Shape matching arises as well, but as a consistency condition for the reality of the connection

Future work

- In spite of much progress in the understanding of the classical geometry of a spin foam, much work remains to be done: better understanding the extrinsic geometry, with the interplay between torsion, AB and LC connections
- Adding $T = 0$ as an additional closure constraint? (Wieland)
- Using spinning geometries? (Freidel, Ziprick)

	three-metric	gluing	torsion
Regge	continuous	piecewise-linear-flat	no
Twisted	discontinuous	piecewise-linear-flat	yes
Spinning	continuous	piecewise-flat	yes

- Spinors and twistors
 - Covariant in nature
 - Elegant encoding of primary simplicity as incidence relation
 - Difference between AB and LC shows up in extra phase
 - Tools from complex analysis to evaluate quantum corrections

What we know from the continuum canonical analysis

- Phase space path integral

$$\mathcal{Z} = \int dP_{IJ}^a d\omega_a^{IJ} \delta(1^{\text{st}})\Delta_1 \prod_i \delta(\xi_i) \delta(2^{\text{nd}}) \sqrt{\Delta_2} \exp\left(i \int dt \int d^3x P_{IJ}^a \partial_0 \omega_a^{IJ}\right)$$

- Configuration space (Lagrangian) path integral

$$\mathcal{Z} = \int dB_{\mu\nu}^{IJ} d\omega_{\mu}^{IJ} \delta(1^{\text{st}})\Delta'_1 \prod_i \delta(\xi_i) \mu(B) \exp(iS)$$

with $\mu(B)_{\text{Holst}} = (V_4)^3 V_3$ and $\mu(B)_{\text{Plebanski}} = (V_4)^9 V_3$

(Buffenoir, Henneaux, Noui, Roche, Engle, Han, Thiemann)

- Purely geometric Lagrangian path integral (notice no γ) (Shirazi, Engle)

$$\mathcal{Z} = \int dB_{\mu\nu}^{IJ} \delta(\mathcal{C}) V_4^3 V_3 \exp\left(i \int_{\mathcal{M}} \text{Tr}(B \wedge F[\omega(B)])\right)$$

Most conservative approach

- Discretize either of these expressions, and put it in a spin foam form

Difficulties

- Phase space: how to discretize the secondary second class constraints (reality conditions)?
- Configuration space: how to take into account (discretize) the measure factors? (Bojowald, Perez). Also, Henneaux–Slavnov trick not clear for correlators

Challenge for the linear simplicity constraints

- No continuum formulation of Plebanski with linear constraints $n_J B_{\mu\nu}^{IJ} = 0$ (too few constraints, 18 instead of 20, one needs volume simplicity as well)
- In fact $\mathcal{L} = BF + \phi nB$ describes the degenerate sector of Plebanski (Alexandrov)
- One can replace spatial quadratic simplicity by $n_J B_{ab}^{IJ} = 0$ (9 constraints), and keep quadratic constraints for B_{0a} , but then the secondary constraints are again the reality conditions (Alexandrov, MG)

At the discrete level

- For one four-simplex, one can replace discrete quadratic simplicity by linear simplicity, but in a (simplicial) path integral this should lead to measure factors since only the combination

$$\left(\prod_i \delta(\mathcal{C}_i) \right) \sqrt{|\det\{\mathcal{C}_i, \mathcal{C}_j\}|}$$

is invariant under change of parametrization of the constraint surface

Toy model (MG, Noui)

- 3d Plebanski = SU(2) gravity covariantly embedded into SO(4), quantization should reproduce the Ponzano–Regge model

- ▶ Lagrangian $\mathcal{L}[B, \omega, \phi] = \varepsilon^{\mu\nu\rho} (1 + \gamma^{-1} \star) B_\mu^{IJ} F_{\nu\rho}^{IJ} + \phi_I^\mu n_J B_\mu^{IJ}$
- ▶ Primary simplicity $n_J B_a^{IJ} = 0 \rightarrow X_f^+ - n_e X_f^- n_e^{-1} = 0$
- ▶ Secondary simplicity $D_a n^I = 0 \rightarrow (g_{ve}^+)^{-1} g_{ve}^- n_e^{-1} = \mathbb{I}$
- ▶ Use modified measure over holonomies and discrete bivectors

- Vertex amplitude

$$A_v[g_f] = \sum_{\lambda_f, j_{ef}, i_e} \prod_{f \supset v} d\lambda_f \prod_{(ef) \supset v} d_{j_{ef}} A_v[\lambda_f, j_{ef}, i_e] \mathcal{S}_{(\Gamma_v, \lambda_f, j_{ef}, i_e)}[g_f, n_e]$$

$$\begin{aligned} A_v[\lambda_f, j_{ef}, i_e] &= \int_{\text{SO}(4)} \prod_{e \supset v} \delta((g_{ve}^+)^{-1} g_{ve}^- n_e^{-1}) dg_{ve} \mathcal{S}_{(\Gamma_v, \lambda_f, j_{ef}, i_e)}[g_{vu}^{-1}(f) g_{vd}(f), n_e] \\ &= \{6j\} \prod_{f \supset v} d_{j_f}^{-1} \delta_{j_{u(f)} f j_{d(f)} f} \end{aligned}$$

- After transforming back to the bivector representations, gluing all the vertex amplitudes $A_v[X_f]$ together using the measure $\delta(X_f^+ - n_e X_f^- n_e^{-1}) dX_f$ leads to the Ponzano–Regge model

Lessons

- Barbero–Immirzi parameter γ plays absolutely no role
- Trivially true since we are just computing

$$\mathcal{Z} = \int \delta(X^+ - nX^- n^{-1}) dX \delta((g^+)^{-1} g^- n^{-1}) dg \exp(i(1 + \gamma^{-1} \star) XG)$$

- Sum over redundant label λ_f in $A_v[g_f]$ reduce the boundary (projected spin network) states the usual $SU(2)$ spin networks
- The correct dynamics after imposing the secondary constraints, while the primary ones only play a role at the very end for the gluing
 - ▶ Natural: the secondary constraints are $\Psi \sim \{H, \Phi\}$, so they probe the dynamics. Moreover, while the primary constraints constrain the X_f living on the boundaries of tetrahedra (3d) or four-simplices (4d), the secondary ones constrain the holonomies g_{ve} living inside the simplices
- Agrees by construction with canonical quantization
- BC, FK, or EPRL imposition of the constraints do not lead to the Ponzano–Regge model in this example, but to something non-topological
- Exactly soluble because $\Psi(B, \omega) = \Psi(\omega)$, which is not true in 4d

Choice of connection

- In 4d, one could think of rewriting the path integral in terms of the covariant AB connection

$$\begin{aligned} A_a^{IJ} &= \mathcal{R}_{KL}^{IJ} (1 + \gamma^{-1} \star) \omega_a^{KL} + 2(1 - \gamma^{-1} \star) n^{[I} \partial_a n^{J]} \\ &= \omega_a^{IJ} + 2(1 - \gamma^{-1} \star) n^{[I} D_a n^{J]} \end{aligned}$$

Then $\Psi(B, \omega) \rightarrow \Psi(A) = \mathcal{B}_{KL}^{IJ} A_a^{KL} - 2n^{[I} \partial_a n^{J]} = 0$, which can be discretized as $(g_{ve}^+)^{-1} g_{ve}^- n_e^{-1} = \mathbb{I}$

- ▶ But we just saw that this leads to SU(2) BF theory
- ▶ Reason: AB connection is not a spacetime connection, and $A|_{2^{\text{nd}} \text{ class} = 0} \neq \omega$

Inducing constraints on the holonomies (Baratin, Oriti)

- Non-commutative simplicial path integral via group Fourier transform

$$\widehat{\phi}(X_1, \dots, X_4) = \int_G [dg]^4 \phi(g_1, \dots, g_4) E_{g_1}(X_1) \dots E_{g_4}(X_4), \quad E_{g_1}(X) E_{g_1}(X)$$

- Propagator and vertex

$$P(X_1^\tau, \dots, X_4^\tau, X_1^{\tau'}, \dots, X_4^{\tau'}) = \prod_{t=1}^4 \delta_{-X_t^\tau}^*(X_t^{\tau'})$$

$$V(X_1^\tau, \dots, X_{10}^\tau, X_1^{\tau'}, \dots, X_{10}^{\tau'}) = \int [dg_{\sigma\tau}]^5 \prod_{t=1}^{10} \left(\delta_{-X_t^\tau}^* \star E_{g_{\tau\sigma} g_{\sigma\tau'}} \right) (X_t^{\tau'})$$

- Amplitude for triangle (face $\in \Delta^*$) bounded by N tetrahedra with reference τ_0

$$A_t[g_{\sigma\tau}] = \int [dX_t^\tau]^N \star_{i=0}^N \left(\delta_{X_t^{\tau_i}}^* \star E_{g_{\tau_i \tau_{i+1}}} \right) (X_t^{\tau_{i+1}})$$

- Final partition function

$$\mathcal{Z}_{\text{BF}} = \int \prod_{(\sigma\tau)} dg_{\sigma\tau} \prod_t \left(dX_t E_{G_t}(X_t) \right) = \int \prod_{(\sigma\tau)} dg_{\sigma\tau} \prod_t dX_t e^{i \sum_t \text{Tr}(X_t G_t)}$$

Models for gravity

- Extend the fields to incorporate the normals n
- Impose linear simplicity ($nX_t^- n^{-1} + \beta X_t^+ = 0$), with $\beta = (\gamma - 1)/(\gamma + 1)$, using

$$S_n^\beta(X_t) := \delta_{nX_t^- n^{-1}}^\star(\beta X_t^+)$$

in propagator or vertex (not a projector for $\beta \neq 1$)

- For $\beta = 1$, the upshot is

$$\mathcal{Z} = \int \prod_{(\sigma\tau)} dg_{\sigma\tau} \prod_{\tau} dn_{\tau} \prod_t \left(dX_t \star_{i=0}^{N_t} S_{g_{0i} \triangleright n_{\tau_i}}(X_t) \right) \star e^{i \sum_t \text{Tr}(X_t G_t)}$$

- Separating $i = 0$ from the rest yields

$$\begin{aligned} \mathcal{Z} &= \int \prod_{\tau} dn_{\tau} \prod_t dX_t \underbrace{\mathcal{D}^{(X_t, k_{\tau})}[g_{\sigma\tau}]}_{\text{simplicity in frames around the dual face}} \star \prod_t \left(e^{i \text{Tr}(X_t G_t)} \star \underbrace{\delta_{k_{\tau_0} X_t^- k_{\tau_0}^{-1}}^\star(X_t^+)}_{\text{simplicity in reference tetrahedron frame}} \right) \\ &= \mathcal{Z}_{\text{BC}} \end{aligned}$$

General result

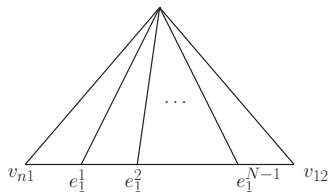
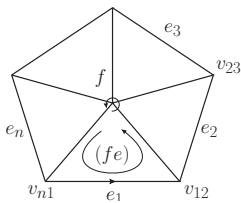
- Non-commutative simplicial path integral with measure over the bivectors (imposing primary simplicity) and over the holonomies \rightarrow interpreted as imposing secondary constraints coming from the consistent imposition in all tetrahedron frames of the primary ones

For $\beta \neq 1$

- Resulting model is different from EPRL/FK

Difficulty with the factorization of the constraints

- It appears too simple to be correct. The 3d Plebanski toy model (MG, Noui) shows that it would not reproduce the Ponzano–Regge model: the constraints on the holonomies are not the full set of secondary constraints coming from the imposition at all times of linear simplicity

Decomposition of a dual face $f \in \Delta^*$ 

Semi-discrete action (Wieland)

$$S_{(N)} = \sum_{f \in \Delta^*} \sum_{e \subset f} \sum_{i=1}^N \text{Tr} \left(X_f H_{(fe)^i} \right) \xrightarrow{N \rightarrow \infty} S = \sum_{f \in \Delta^*} S_f = \sum_{e \in \Delta^*} S_e = \sum_{v \in \Delta^*} S_v$$

- Edge action
$$S_e = \sum_{f \supset e} \int_0^1 dt \text{Tr} \left(X_f(t) D_t h_f(t)^{-1} h_f(t) \right)$$
- Covariant symplectic potential on T^*G , continuous (time?) parameter t along the edge e , and covariant derivative $D_t := \partial_t + \omega_e(t)$
- Possible classical version of the construction of Baratin and Oriti
- Need to add simplicity constraints and study canonical analysis (done by Wieland with spinors)
- How should these prescriptions distinguish between gravity and the toy models (4d degenerate and 3d Plebanski) if both are of the form BF + linear simplicity?

Achievements

- We have a well-defined spin foam framework (BC, EPR, FK, LS, EPRL, BO, Barrett et al.), fully covariant (DL, RS), provides amplitudes to all spinnets (KKL)
- Very good control on primary simplicity
- Good control on *simplicial* large quantum number asymptotics
- Common framework for first order discrete actions
- Improved understanding of role of secondary constraints

Future questions

- Clarify geometricity and torsion-lessness off the Regge framework?
- Understand how to implement this relation in spin foam models?
- Improve evaluation of quantum corrections

Methods

- Twistors and spinors
- Self-dual variables
- GFT methods