Simplicity constraints in spin foam models

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ILQGS Discussion
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Introduction

What are spin foams?
- Attempt to write the bare path integral for gravity as

\[ Z = \sum_{\text{(group data)}} \prod_{\text{(building blocks)}} \text{Amplitudes} \]

- Motivated by the need to construct transition amplitudes for canonical LQG

How to construct spin foams?
- Typically from discrete Plebanski action (BC, Reisenberger, EPRL, BO, . . .)
- Also attempts from Holst action (Baratin, Flori, Thiemann)

Many encouraging results so far
- Well-defined transition amplitudes for all spin networks (KKL)
- Ultraviolet finite and infrared finite (\( \Lambda \neq 0 \)) (Noui, Roche, Meusburger, Fairbairn, Han, . . .)
- Promising large quantum number behaviour (Barrett et al.)

Many big questions and room for further research/researchers
- Summation and continuum limit (e.g. Gurau et al.)
- Proper implementation of the projector (and link with LQG) (e.g. Alesci et al.)
- Explicit evaluation of the quantum corrections (e.g. Riello)
Simplicity constraints in the continuum

- Ensure correct number of degrees of freedom
- Canonical structure: conjugated primary $\Phi(B)$ and secondary $\dot{\Phi}(B) = \Psi(B, \omega)$ second class constraints

Upon discretization

- (some) simplicity constraints second class with themselves
- No clear notion of physical degrees of freedom
- No clear notion of Hamiltonian, time evolution, and appearance of secondary constraints (see Dittrich)

These difficulties lead to important open questions for spin foams

- Torsion-freeness and flatness issue (Hellmann et al.)
- What geometry beyond that of four-simplices?
- What is the geometrical meaning of quantum corrections (offshell contributions)?
- Precise link with canonical theory
- Broader questions: continuum limit, renormalization, extraction of physics, ...
Plebanski action \[ S[B, \omega, \phi] = \int\mathcal{M} \left( \eta_{IJKL} B^{IJ} \wedge F^{KL}[\omega] + \phi_{IJKL} B^{IJ} \wedge B^{KL} \right) \]

Lagrangian picture

- Constraints \( \delta_{\phi} S = 0 \) \( \implies \mathcal{C} = B^{IJ} \wedge B^{KL} - \sigma \mathcal{V} \varepsilon^{IJKL} = 0 \) enforcing \( B^{IJ} = \pm e^I \wedge e^J \) or \( B^{IJ} = \pm (\ast e \wedge e)^{IJ} \)

  (a degenerate sector also exist)

- 20 components decomposed into irreps as

  \[
  \begin{array}{ccc}
  (2, 0) & \oplus & (0, 2) \\
  \text{(reduce internal gauge)} & & \\
  \end{array}
  \begin{array}{ccc}
  (1, 1) & \oplus & (0, 0) \\
  \text{(impose left = right Urbantke metric)} & & \\
  \end{array}
  \]

- Relaxing the constraints too much yields extra degrees of freedom (Krasnov, SS)

Hamiltonian picture \( (\mathcal{M} = \mathbb{R} \times \Sigma) \)

- Primary simplicity imposes \( B|_{\Sigma} = e \wedge e \)

- Secondary simplicity imposes \( \omega|_{\Sigma} = \Gamma(e) \) spatial torsion-freeness

- Note however that while \( d_{\Gamma} e|_{\Sigma} = 0 \), the Ashtekar–Barbero derivative gives \( d_{\mathcal{A}} e|_{\Sigma} = K \wedge e \neq 0 \)
On the secondary simplicity constraints

- They play a key role in the continuum theory (e.g. Lorentz-covariant analysis of Alexandrov), so why don’t we hear much about them in spin foams?
- Being a covariant approach, it is supposed to know about the non-trivial embedding of the AB connection in the Lorentzian phase space
- The secondary constraint requires a Hamiltonian in order to be derived
- Discretization breaks (a priori) diff invariance $\rightarrow$ no clear distinction between primary and secondary
Discrete Lagrangian (covariant) constraints à la Barrett-Crane

- A geometric four-simplex is uniquely determined by triangle bivectors satisfying
  \[
  \begin{align*}
  \text{diagonal simplicity} & \quad \text{Tr}(\mathbf{*}X_f X_f) = 0 \\
  \text{off-diagonal simplicity} & \quad \text{Tr}(\mathbf{*}X_f X_{f'}) = 0 \\
  \text{volume simplicity} & \quad \text{Tr}(\mathbf{*}X_{f_1} X_{f'_1}) = \text{Tr}(\mathbf{*}X_{f_2} X_{f'_2})
  \end{align*}
  \]

- Volume simplicity involves holonomies (i.e. the connection) and thus can be seen as a secondary constraint.

- It can however be replaced by closure + four-simplex flatness, so one can use
  \[
  \begin{align*}
  \text{diagonal simplicity} & \quad \text{Tr}(\mathbf{*}X_f X_f) = 0 \\
  \text{off-diagonal simplicity} & \quad \text{Tr}(\mathbf{*}X_f X_{f'}) = 0 \\
  \text{closure} & \quad \sum_{f \supset e} X_f = 0
  \end{align*}
  \]
  or
  \[
  \begin{align*}
  \text{linear simplicity} & \quad \mathbf{*}(n_e)_I X^I_J = 0, \quad \forall f \in e \\
  \text{closure} & \quad \sum_{f \supset e} X_f = 0
  \end{align*}
  \]

- The logic here is to define discrete constraints that reproduce the end result of the continuum ones: existence of a unique Riemannian structure.
**Regge geometries**

- Describe discrete Riemannian geometry of manifold using four-simplices and their edge lengths
- There is a notion of Levi–Civita connection $\Gamma$

**Important questions**

- How is the Levi-Civita connection defined in terms of holonomies and fluxes?
- What is the role of the $\mathfrak{su}(2)$ Ashtekar–Barbero connection and the Barbero–Immirzi parameter $\gamma$?

**Remark:** Regge’s key condition of restricting the holonomy on a plane is only consistent for Levi–Civita and not for Ashtekar–Barbero

**Twisted geometries**

- Phase space of covariant LQG and SF in terms of generalized Regge geometries: piecewise-flat polyhedra, discontinuous metric
- Works for arbitrary cellular decomposition
- Crucial information on $K$ that distinguishes $A$ from $\Gamma$ lies in the gauge orbit of the diagonal simplicity constraint
- If, like in the continuum, there are secondary constraints, they provide a non-trivial gauge fixing of these orbits, i.e. a non-trivial embedding of the $\mathfrak{su}(2)$ connection into the Lorentzian one
Discrete secondary constraints

- Lack of secondary constraints often criticized in the Lagrangian approach
- Issue not seen (a priori) by the asymptotics, which is on-shell
- How to define them in the discrete, where there is no canonical analysis?
- Can one discretize directly the $T = 0$ condition?
  - Dittrich, Ryan: yes, and it leads to shape matching conditions
  - Marseille: a torsion-free connection can be defined without shape matching

Toy model (Anzà, SS)

- Single four-simplex and a Hamiltonian imposing flatness
  - Secondary constraints arise, and give the embedding of $\mathfrak{su}(2)$ connection into the Lorentzian one (identical as continuum)
  - Shape matching arises as well, but as a consistency condition for the reality of the connection
Future work

- Inspite of much progress in the understanding of the classical geometry of a spin foam, much work remains to be done: better understanding the extrinsic geometry, with the interplay between torsion, AB and LC connections.
- Adding $T = 0$ as an additional closure constraint? (Wieland)
- Using spinning geometries? (Freidel, Ziprick)

<table>
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<tr>
<th></th>
<th>three-metric</th>
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<th>torsion</th>
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<tr>
<td>Twisted</td>
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</tr>
<tr>
<td>Spinning</td>
<td>continuous</td>
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<td>yes</td>
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- Spinors and twistors
  - Covariant in nature
  - Elegant encoding of primary simplicity as incidence relation
  - Difference between AB and LC shows up in extra phase
  - Tools from complex analysis to evaluate quantum corrections
Path integral formulations

What we know from the continuum canonical analysis

- Phase space path integral

\[ Z = \int dP^a_{IJ} d\omega^I_J a \delta(1^{\text{st}}) \Delta_1 \prod_i \delta(\xi_i) \delta(2^{\text{nd}}) \sqrt{\Delta_2} \exp \left( i \int dt \int d^3x \; P^a_{IJ} \partial_0 \omega^I_J \right) \]

- Configuration space (Lagrangian) path integral

\[ Z = \int dB^I_J \omega^I_J a \delta(1^{\text{st}}) \Delta'_1 \prod_i \delta(\xi_i) \mu(B) \exp (iS) \]

with \( \mu(B)_{\text{Holst}} = (V_4)^3V_3 \) and \( \mu(B)_{\text{Plebanski}} = (V_4)^9V_3 \)

(Buffenoir, Henneaux, Noui, Roche, Engle, Han, Thiemann)

- Purely geometric Lagrangian path integral (notice no \( \gamma \)) (Shirazi, Engle)

\[ Z = \int dB^I_J \omega^I_J \delta(C) V_4^3V_3 \exp \left( i \int_M \text{Tr}(B \wedge F[\omega(B)]) \right) \]

Most conservative approach

- Discretize either of these expressions, and put it in a spin foam form

Difficulties

- Phase space: how to discretize the secondary second class constraints (reality conditions)?

- Configuration space: how to take into account (discretize) the measure factors? (Bojowald, Perez). Also, Henneaux–Slavnov trick not clear for correlators
Challenge for the linear simplicity constraints

- No continuum formulation of Plebanski with linear constraints $n_J B^{IJ}_{\mu\nu} = 0$ (too few constraints, 18 instead of 20, one needs volume simplicity as well)
- In fact $\mathcal{L} = BF + \phi nB$ describes the degenerate sector of Plebanski (Alexandrov)
- One can replace spatial quadratic simplicity by $n_J B_{ab}^{IJ} = 0$ (9 constraints), and keep quadratic constraints for $B_{0a}$, but then the secondary constraints are again the reality conditions (Alexandrov, MG)

At the discrete level

- For one four-simplex, one can replace discrete quadratic simplicity by linear simplicity, but in a (simplicial) path integral this should lead to measure factors since only the combination

$$\left( \prod_i \delta(C_i) \right) \sqrt{|\det\{C_i, C_j\}|}$$

is invariant under change of parametrization of the constraint surface
Importance of the secondary constraints

**Toy model** (MG, Noui)

- 3d Plebanski = SU(2) gravity covariantly embedded into SO(4), quantization should reproduce the Ponzano–Regge model
  
  ▶ Lagrangian
  \[
  \mathcal{L}[B, \omega, \phi] = \varepsilon^{\mu\nu\rho}(1 + \gamma^{-1} \star)B_{\mu}^{IJ}F_{\nu\rho}^{IJ} + \phi^{\mu}_{I}n_{J}B_{\mu}^{IJ}
  \]

  ▶ Primary simplicity
  \[n_{J}B_{a}^{IJ} = 0 \quad \rightarrow \quad X_{f}^{+} - n_{e}X_{f}^{-}n_{e}^{-1} = 0\]

  ▶ Secondary simplicity
  \[D_{a}n^{I} = 0 \quad \rightarrow \quad (g_{ve}^{-1})^{-1}g_{ve}n_{e}^{-1} = \mathbb{I}\]

  ▶ Use modified measure over holonomies and discrete bivectors

- Vertex amplitude

  \[
  A_{v}[g_{f}] = \sum_{\lambda_{f},j_{ef},i_{e}} \prod_{f \supset v} d_{\lambda_{f}} \prod_{(ef) \supset v} d_{j_{ef}} A_{v}[\lambda_{f}, j_{ef}, i_{e}] S(\Gamma_{v}, \lambda_{f}, j_{ef}, i_{e})[g_{f}, n_{e}]
  \]

  \[
  A_{v}[\lambda_{f}, j_{ef}, i_{e}] = \int_{SO(4)} \prod_{e \supset v} \delta((g_{ve}^{-1})^{-1}g_{ve}n_{e}^{-1}) dg_{ve} S(\Gamma_{v}, \lambda_{f}, j_{ef}, i_{e})[g_{v}^{-1}g_{vu(f)}g_{vd(f)}, n_{e}]
  \]

  \[
  = \{6j\} \prod_{f \supset v} d_{j_{f}}^{-1}\delta j_{u(f)} j_{d(f)}
  \]

- After transforming back to the bivector representations, gluing all the vertex amplitudes \(A_{v}[X_{f}]\) together using the measure \(\delta(X_{f}^{+} - n_{e}X_{f}^{-}n_{e}^{-1})dX_{f}\) leads to the Ponzano–Regge model
Importance of the secondary constraints

Lessons

• Barbero–Immirzi parameter $\gamma$ plays absolutely no role

• Trivially true since we are just computing

\[ Z = \int \delta(X^+ - nX^- n^{-1})dX \delta((g^+)^{-1}g^- n^{-1})dg \exp (i(1 + \gamma^{-1} \star)XG) \]

• Sum over redundant label $\lambda_f$ in $A_v[g_f]$ reduce the boundary (projected spin network) states the usual SU(2) spin networks

• The correct dynamics after imposing the secondary constraints, while the primary ones only play a role at the very end for the gluing
  
  ▶ Natural: the secondary constraints are $\Psi \sim \{H, \Phi\}$, so they probe the dynamics. Moreover, while the primary constraints constrain the $X_f$ living on the boundaries of tetrahedra (3d) or four-simplices (4d), the secondary ones constrain the holonomies $g_{ve}$ living inside the simplices

• Agrees by construction with canonical quantization

• BC, FK, or EPRL imposition of the constraints do not lead to the Ponzano–Regge model in this example, but to something non-topological

• Exactly soluble because $\Psi(B, \omega) = \Psi(\omega)$, which is not true in 4d
Choice of connection

- In 4d, one could think of rewriting the path integral in terms of the covariant AB connection

\[
A_a^{IJ} = \mathcal{R}_K^I (1 + \gamma^{-1}) \omega_a^{KL} + 2(1 - \gamma^{-1}) n^{[I} \partial_a n^{J]} \\
= \omega_a^{IJ} + 2(1 - \gamma^{-1}) n^{[I} D_a n^{J]}
\]

Then \( \Psi(B, \omega) \to \Psi(A) = \mathcal{B}_K^I A_a^{KL} - 2n^{[I} \partial_a n^{J]} = 0 \), which can be discretized as \((g_{ve}^+)^{-1} g_{ve} n_e^{-1} = I\)

- But we just saw that this leads to SU(2) BF theory
- Reason: AB connection is not a spacetime connection, and \( A_{2\text{nd class}} = 0 \neq \omega \)
Inducing constraints on the holonomies (Baratin, Oriti)

- Non-commutative simplicial path integral via group Fourier transform

\[ \hat{\phi}(X_1, \ldots, X_4) = \int_G [dg]^4 \phi(g_1, \ldots, g_4) E_{g_1}(X_1) \cdots E_{g_4}(X_4), \quad E_{g_1}(X)E_{g_1}(X) \]

- Propagator and vertex

\[ P(X_1^{\tau}, \ldots, X_4^{\tau}, X_1^{\tau'}, \ldots, X_4^{\tau'}) = \prod_{t=1}^4 \delta^*_{-X_t^{\tau}}(X_t^{\tau'}) \]
\[ V(X_1^{\tau}, \ldots, X_{10}^{\tau}, X_1^{\tau'}, \ldots, X_{10}^{\tau'}) = \int [dg_{\tau\sigma}]^5 \prod_{t=1}^{10} \left( \delta^*_{-X_t^{\tau}} \ast E_{g_{\tau\sigma}g_{\sigma\tau'}} \right) (X_t^{\tau'}) \]

- Amplitude for triangle (face \( \in \Delta^* \)) bounded by \( N \) tetrahedra with reference \( \tau_0 \)

\[ A_t[g_{\sigma\tau}] = \int [dX_t^{\tau}]^N \star \prod_{i=0}^N \left( \delta^*_{X_t^{\tau_i}} \ast E_{g_{\tau_i\tau_{i+1}}} \right) (X_{\ell}^{\tau_{i+1}}) \]

- Final partition function

\[ Z_{BF} = \int \prod_{(\sigma\tau)} dg_{\sigma\tau} \prod_t (dX_t E_{G_t}(X_t)) = \int \prod_{(\sigma\tau)} dg_{\sigma\tau} \prod_t dX_t e^{i \sum_t \text{Tr}(X_t G_t)} \]
Models for gravity

- Extend the fields to incorporate the normals \( n \)
- Impose linear simplicity \((nX_t^{-n^{-1}} + \beta X_t^+ = 0)\), with \( \beta = (\gamma - 1)/(\gamma + 1) \), using

\[
S^\beta_n(X_t) := \delta^*_{nX_t^{-n^{-1}}(\beta X_t^+)}
\]

in propagator or vertex (not a projector for \( \beta \neq 1 \))

- For \( \beta = 1 \), the upshot is

\[
\mathcal{Z} = \int \prod_{(\sigma \tau)} dg_{\sigma \tau} \prod_{\tau} \prod_{t} dX_t \left( dX_t \overset{N_t}{\star} S_{g_0i \triangleright n_{\tau_i}}(X_t) \right) \star e^{i \sum_t \text{Tr}(X_t G_t)}
\]

- Separating \( i = 0 \) from the rest yields

\[
\mathcal{Z} = \int \prod_{\tau} d\tau \prod_{t} dX_t \left[ \mathcal{D}^{(X_t, k_\tau)}[g_{\sigma \tau}] \right] \star \prod_{t} \left( e^{i \text{Tr}(X_t G_t)} \star \delta^*_{k_{\tau_0}^{-1}X_t^{-1}X_t^+} \right)
\]

\[
\mathcal{Z} = \mathcal{Z}_{BC}
\]
General result

- Non-commutative simplicial path integral with measure over the bivectors (imposing primary simplicity) and over the holonomies → interpreted as imposing secondary constraints coming from the consistent imposition in all tetrahedron frames of the primary ones

For $\beta \neq 1$

- Resulting model is different from EPRL/FK

Difficulty with the factorization of the constraints

- It appears too simple to be correct. The 3d Plebanski toy model (MG, Noui) shows that it would not reproduce the Ponzano–Regge model: the constraints on the holonomies are not the full set of secondary constraints coming from the imposition at all times of linear simplicity
New proposals

Semi-discrete actions

Decomposition of a dual face \( f \in \Delta^* \)

![Decomposition Diagram]

Semi-discrete action (Wieland)

\[
S_{(N)} = \sum_{f \in \Delta^*} \sum_{e \subset f} \sum_{i=1}^{N} \text{Tr} \left( X_f H_{(fe)i} \right) \xrightarrow{N \to \infty} S = \sum_{f \in \Delta^*} S_f = \sum_{e \in \Delta^*} S_e = \sum_{v \in \Delta^*} S_v
\]

- Edge action \( S_e = \sum_{f \supset e} \int_0^1 dt \text{Tr} \left( X_f(t)D_t h_f(t)^{-1} h_f(t) \right) \)
- Covariant symplectic potential on \( T^*G \), continuous (time?) parameter \( t \) along the edge \( e \), and covariant derivative \( D_t := \partial_t + \omega_e(t) \)
- Possible classical version of the construction of Baratin and Oriti
- Need to add simplicity constraints and study canonical analysis (done by Wieland with spinors)
- How should these prescriptions distinguish between gravity and the toy models (4d degenerate and 3d Plebanski) if both are of the form BF + linear simplicity?
Conclusions

Achievements

- We have a well-defined spin foam framework (BC, EPR, FK, LS, EPRL, BO, Barrett et al.), fully covariant (DL, RS), provides amplitudes to all spinnets (KKL)
- Very good control on primary simplicity
- Good control on simplicial large quantum number asymptotics
- Common framework for first order discrete actions
- Improved understanding of role of secondary constraints

Future questions

- Clarify geometricity and torsion-lessness off the Regge framework?
- Understand how to implement this relation in spin foam models?
- Improve evaluation of quantum corrections

Methods

- Twistors and spinors
- Self-dual variables
- GFT methods