Simplicity constraints in spin foam models

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What are spin foams?

- Attempt to write the bare path integral for gravity as

$$\mathcal{Z} = \sum_{\text{(group data)}} \prod_{\text{(building blocks)}} \text{Amplitudes}$$

- Motivated by the need to construct transition amplitudes for canonical LQG

How to construct spin foams?

- Typically from discrete Plebanski action (BC, Reisenberger, EPRL, BO, . . .)
- Also attempts from Holst action (Baratin, Flori, Thiemann)

Many encouraging results so far

- Well-defined transition amplitudes for all spin networks (KKL)
- Ultraviolet finite and infrared finite ($\Lambda \neq 0$) (Noui, Roche, Meusburger, Fairbairn, Han, . . .)
- Promising large quantum number behaviour (Barrett et al.)

Many big questions and room for further research/researchers

- Summation and continuum limit (e.g. Gurau et al.)
- Proper implementation of the projector (and link with LQG) (e.g. Alesci et al.)
- Explicit evaluation of the quantum corrections (e.g. Riello)

Simplicity constraints in the continuum

- Ensure correct number of degrees of freedom
- Canonical structure: conjugated primary $\Phi(B)$ and secondary $\dot{\Phi}(B) = \Psi(B, \omega)$ second class constraints

Upon discretization

- (some) simplicity constraints second class with themselves
- No clear notion of physical degrees of freedom
- No clear notion of Hamiltonian, time evolution, and appearance of secondary constraints (see Dittrich)

These difficulties lead to important open questions for spin foams

- Torsion-freeness and flatness issue (Hellmann et al.)
- What geometry beyond that of four-simplices?
- What is the geometrical meaning of quantum corrections (offshell contributions)?
- Precise link with canonical theory
- Broader questions: continuum limit, renormalization, extraction of physics, ...

Plebanski action $S[B,\omega,\phi]=\int_{\mathcal{M}}\Big(\eta_{IJKL}B^{IJ}\wedge F^{KL}[\omega]+\phi_{IJKL}B^{IJ}\wedge B^{KL}\Big)$

Lagrangian picture

- Constraints $\delta_\phi S=0 \quad\rightarrow\quad \mathcal{C}=B^{IJ}\wedge B^{KL}-\sigma\mathcal{V}\varepsilon^{IJKL}=0$ enforcing

$$B^{IJ}=\pm e^I\wedge e^J \qquad \text{or} \qquad B^{IJ}=\pm(\star\, e\wedge e)^{IJ}$$

(a degenerate sector also exist)

- 20 components decomposed into irreps as

$$\underbrace{(\mathbf{2},\mathbf{0})\quad\oplus\quad(\mathbf{0},\mathbf{2})}_{\text{reduce internal gauge}}\quad\oplus\quad\underbrace{(\mathbf{1},\mathbf{1})\quad\oplus\quad(\mathbf{0},\mathbf{0})}_{\text{impose left }=\text{ right Urbantke metric}}$$

- Relaxing the constraints too much yields extra degrees of freedom (Krasnov, SS)

Hamiltonian picture ($\mathcal{M}=\mathbb{R}\times\Sigma$)

- Primary simplicity imposes $B|_\Sigma=e\wedge e$
- Secondary simplicity imposes $\omega|_\Sigma=\Gamma(e)$ spatial torsion-freeness
- Note however that while $\mathrm{d}_\Gamma e|_\Sigma=0$,
  the Ashtekar–Barbero derivative gives $\mathrm{d}_A e|_\Sigma=K\wedge e\neq 0$

On the secondary simplicity constraints

- They play a key role in the continuum theory (e.g. Lorentz-covariant analysis of Alexandrov), so why don't we hear much about them in spin foams?
- Being a covariant approach, it is supposed to know about the non-trivial embedding of the AB connection in the Lorentzian phase space
- The secondary constraint requires a Hamiltonian in order to be derived
- Discretization breaks (a priori) diff invariance $\rightarrow$ no clear distinction between primary and secondary

Discrete Lagrangian (covariant) constraints à la Barrett-Crane

- A geometric four-simplex is uniquely determined by triangle bivectors satisfying

$$\begin{cases} \text{diagonal simplicity} & \mathrm{Tr}(\star X_f X_f)=0 \\ \text{off-diagonal simplicity} & \mathrm{Tr}(\star X_f X_{f'})=0 \\ \text{volume simplicity} & \mathrm{Tr}(\star X_{f_1} X_{f'_1})=\mathrm{Tr}(\star X_{f_2} X_{f'_2}) \end{cases}$$

- Volume simplicity involves holonomies (i.e. the connection) and thus can be seen as a secondary constraint
- It can however be replaced by closure + four-simplex flatness, so one can use

$$\begin{cases} \text{diagonal simplicity} & \mathrm{Tr}(\star X_f X_f)=0 \\ \text{off-diagonal simplicity} & \mathrm{Tr}(\star X_f X_{f'})=0 \\ \text{closure} & \sum_{f\supset e} X_f=0 \end{cases}$$

or

$$\begin{cases} \text{linear simplicity} & \star(n_e)_I X_f^{IJ}=0, \quad \forall\, f\in e \\ \text{closure} & \sum_{f\supset e} X_f=0 \end{cases}$$

- The logic here is to define discrete constraints that reproduce the end result of the continuum ones: existence of a unique Riemannian structure

Regge geometries

- Describe discrete Riemannian geometry of manifold using four-simplices and their edge lengths
- There is a notion of Levi–Civita connection Γ

Important questions

- How is the Levi-Civita connection defined in terms of holonomies and fluxes?
- What is the role of the $\mathfrak{su}(2)$ Ashtekar–Barbero connection and the Barbero–Immirzi parameter γ?
  Remark: Regge's key condition of restricting the holonomy on a plane is only consistent for Levi–Civita and not for Ashtekar–Barbero

Twisted geometries

- Phase space of covariant LQG and SF in terms of generalized Regge geometries: piecewise-flat polyhedra, discontinuous metric
- Works for arbitrary cellular decomposition
- Crucial information on K that distinguishes A from Γ lies in the gauge orbit of the diagonal simplicity constraint
- If, like in the continuum, there are secondary constraints, they provide a non-trivial gauge fixing of these orbits, i.e. a non-trivial embedding of the $\mathfrak{su}(2)$ connection into the Lorentzian one

Discrete secondary constraints

- Lack of secondary constraints often criticized in the Lagrangian approach
- Issue not seen (a priori) by the asymptotics, which is on-shell
- How to define them in the discrete, where there is no canonical analysis?
- Can one discretize directly the $T = 0$ condition?
  - Dittrich, Ryan: yes, and it leads to shape matching conditions
  - Marseille: a torsion-free connection can be defined without shape matching

Toy model (Anzà, SS)

- Single four-simplex and a Hamiltonian imposing flatness
  - Secondary constraints arise, and give the embedding of $\mathfrak{su}(2)$ connection into the Lorentzian one (identical as continuum)
  - Shape matching arises as well, but as a consistency condition for the reality of the connection

Future work

- Inspite of much progress in the understanding of the classical geometry of a spin foam, much work remains to be done: better understanding the extrinsic geometry, with the interplay between torsion, AB and LC connections
- Adding $T = 0$ as an additional closure constraint? (Wieland)
- Using spinning geometries? (Freidel, Ziprick)

	three-metric	gluing	torsion
Regge	continuous	piecewise-linear-flat	no
Twisted	discontinuous	piecewise-linear-flat	yes
Spinning	continuous	piecewise-flat	yes

- Spinors and twistors
  - Covariant in nature
  - Elegant encoding of primary simplicity as incidence relation
  - Difference between AB and LC shows up in extra phase
  - Tools from complex analysis to evaluate quantum corrections

What we know from the continuum canonical analysis

- Phase space path integral

$$\mathcal{Z}=\int \mathrm{d}P^a_{IJ}\,\mathrm{d}\omega^{IJ}_a\,\delta(1^{\mathrm{st}})\Delta_1\prod_i\delta(\xi_i)\delta(2^{\mathrm{nd}})\sqrt{\Delta_2}\exp\left(\mathrm{i}\int\mathrm{d}t\int\mathrm{d}^3x\,P^a_{IJ}\partial_0\omega^{IJ}_a\right)$$

- Configuration space (Lagrangian) path integral

$$\mathcal{Z}=\int \mathrm{d}B^{IJ}_{\mu\nu}\,\mathrm{d}\omega^{IJ}_\mu\,\delta(1^{\mathrm{st}})\Delta_1'\prod_i\delta(\xi_i)\mu(B)\exp\left(\mathrm{i}S\right)$$

  with $\mu(B)_{\mathrm{Holst}}=(V_4)^3V_3$ and $\mu(B)_{\mathrm{Plebanski}}=(V_4)^9V_3$
  (Buffenoir, Henneaux, Noui, Roche, Engle, Han, Thiemann)
- Purely geometric Lagrangian path integral (notice no γ) (Shirazi, Engle)

$$\mathcal{Z}=\int \mathrm{d}B^{IJ}_{\mu\nu}\,\delta(\mathcal{C})V_4^3V_3\exp\left(\mathrm{i}\int_{\mathcal{M}}\mathrm{Tr}\big(B\wedge F\big[\omega(B)\big]\big)\right)$$

Most conservative approach

- Discretize either of these expressions, and put it in a spin foam form

Difficulties

- Phase space: how to discretize the secondary second class constraints (reality conditions)?
- Configuration space: how to take into account (discretize) the measure factors? (Bojowald, Perez). Also, Henneaux–Slavnov trick not clear for correlators

Challenge for the linear simplicity constraints

- No continuum formulation of Plebanski with linear constraints $n_J B^{IJ}_{\mu\nu} = 0$ (too few constraints, 18 instead of 20, one needs volume simplicity as well)
- In fact $\mathcal{L} = BF + \phi nB$ describes the degenerate sector of Plebanski (Alexandrov)
- One can replace spatial quadratic simplicity by $n_J B^{IJ}_{ab} = 0$ (9 constraints), and keep quadratic constraints for B_{0a}, but then the secondary constraints are again the reality conditions (Alexandrov, MG)

At the discrete level

- For one four-simplex, one can replace discrete quadratic simplicity by linear simplicity, but in a (simplicial) path integral this shoud leads to measure factors since only the combination

$$\left(\prod_i \delta(\mathcal{C}_i)\right) \sqrt{|\det\{\mathcal{C}_i, \mathcal{C}_j\}|}$$

is invariant under change of parametrization of the constraint surface

Toy model (MG, Noui)

- 3d Plebanski = SU(2) gravity covariantly embedded into SO(4), quantization should reproduce the Ponzano–Regge model
  - Lagrangian $\mathcal{L}[B,\omega,\phi] = \varepsilon^{\mu\nu\rho}(1+\gamma^{-1}\star)B^{IJ}_{\mu}F^{IJ}_{\nu\rho} + \phi^{\mu}_{I}n_J B^{IJ}_{\mu}$
  - Primary simplicity $n_J B^{IJ}_a = 0 \quad \rightarrow \quad X^+_f - n_e X^-_f n_e^{-1} = 0$
  - Secondary simplicity $D_a n^I = 0 \quad \rightarrow \quad (g^+_{ve})^{-1} g^-_{ve} n_e^{-1} = \mathbb{I}$
  - Use modified measure over holonomies and discrete bivectors
- Vertex amplitude

$$
\begin{aligned}
A_v[g_f] &= \sum_{\lambda_f, j_{ef}, i_e} \prod_{f\supset v} d_{\lambda_f} \prod_{(ef)\supset v} d_{j_{ef}} A_v[\lambda_f, j_{ef}, i_e] \mathcal{S}_{(\Gamma_v,\lambda_f,j_{ef},i_e)}[g_f, n_e] \\
A_v[\lambda_f, j_{ef}, i_e] &= \int_{\mathrm{SO}(4)} \prod_{e\supset v} \delta\big((g^+_{ve})^{-1} g^-_{ve} n_e^{-1}\big) \mathrm{d}g_{ve}\, \mathcal{S}_{(\Gamma_v,\lambda_f,j_{ef},i_e)}\left[g^{-1}_{vu(f)} g_{vd(f)}, n_e\right] \\
&= \{6j\} \prod_{f\supset v} d^{-1}_{j_f} \delta_{j_{u(f)f} j_{d(f)f}}
\end{aligned}
$$

- After transforming back to the bivector representations, gluing all the vertex amplitudes $A_v[X_f]$ together using the measure $\delta\big(X^+_f - n_e X^-_f n_e^{-1}\big)\mathrm{d}X_f$ leads to the Ponzano–Regge model

Lessons

- Barbero–Immirzi parameter γ plays absolutely no role
- Trivially true since we are just computing

$$“\mathcal{Z}=\int\delta(X^+-nX^-n^{-1})\mathrm{d}X\,\delta\big((g^+)^{-1}g^-n^{-1}\big)\mathrm{d}g\,\exp\big(\mathrm{i}(1+\gamma^{-1}\star)XG\big)”$$

- Sum over redundant label λ_f in $A_v[g_f]$ reduce the boundary (projected spin network) states the usual SU(2) spin networks
- The correct dynamics after imposing the secondary constraints, while the primary ones only play a role at the very end for the gluing
  - Natural: the secondary constraints are $\Psi\sim\{H,\Phi\}$, so they probe the dynamics. Moreover, while the primary constraints constrain the X_f living on the boundaries of tetrahedra (3d) or four-simplices (4d), the secondary ones constrain the holonomies g_{ve} living inside the simplices
- Agrees by construction with canonical quantization
- BC, FK, or EPRL imposition of the constraints do not lead to the Ponzano–Regge model in this example, but to something non-topological
- Exactly soluble because $\Psi(B,\omega)=\Psi(\omega)$, which is not true in 4d

Choice of connection

- In 4d, one could think of rewriting the path integral in terms of the covariant AB connection

$$
\begin{aligned}
A_a^{IJ} &= \mathcal{R}^{IJ}_{KL}(1+\gamma^{-1}\star)\omega_a^{KL} + 2(1-\gamma^{-1}\star)n^{[I}\partial_a n^{J]} \\
&= \omega_a^{IJ} + 2(1-\gamma^{-1}\star)n^{[I}D_a n^{J]}
\end{aligned}
$$

  Then $\Psi(B,\omega) \to \Psi(A) = \mathcal{B}^{IJ}_{KL}A_a^{KL} - 2n^{[I}\partial_a n^{J]} = 0$, which can be discretized as $(g^+_{ve})^{-1}g^-_{ve}n_e^{-1} = \mathbb{I}$

  - But we just saw that this leads to SU(2) BF theory
  - Reason: AB connection is not a spacetime connection, and $A|_{2^{\text{nd}}\ \text{class}\ =0} \neq \omega$

Inducing constraints on the holonomies (Baratin, Oriti)

- Non-commutative simplicial path integral via group Fourier transform

$$\widehat{\phi}(X_1,\ldots,X_4)=\int_G [\mathrm{d}g]^4\,\phi(g_1,\ldots,g_4)\mathrm{E}_{g_1}(X_1)\ldots\mathrm{E}_{g_4}(X_4), \qquad \mathrm{E}_{g_1}(X)\mathrm{E}_{g_1}(X)$$

- Propagator and vertex

$$P(X_1^\tau,\ldots,X_4^\tau,X_1^{\tau'},\ldots,X_4^{\tau'}) = \prod_{t=1}^{4}\delta^\star_{-X_t^\tau}(X_t^{\tau'})$$

$$V(X_1^\tau,\ldots,X_{10}^\tau,X_1^{\tau'},\ldots,X_{10}^{\tau'}) = \int [\mathrm{d}g_{\sigma\tau}]^5\prod_{t=1}^{10}\left(\delta^\star_{-X_t^\tau}\star\mathrm{E}_{g_{\tau\sigma}g_{\sigma\tau'}}\right)(X_t^{\tau'})$$

- Amplitude for triangle (face $\in\Delta^*$) bounded by N tetrahedra with reference τ_0

$$A_t[g_{\sigma\tau}]=\int [\mathrm{d}X_t^\tau]^N \mathop{\bigstar}_{i=0}^{N}\left(\delta^\star_{X_t^{\tau_i}}\star\mathrm{E}_{g_{\tau_i\tau_{i+1}}}\right)(X_\ell^{\tau_{i+1}})$$

- Final partition function

$$\mathcal{Z}_{\mathrm{BF}}=\int\prod_{(\sigma\tau)}\mathrm{d}g_{\sigma\tau}\prod_t\Big(\mathrm{d}X_t\,\mathrm{E}_{G_t}(X_t)\Big)=\int\prod_{(\sigma\tau)}\mathrm{d}g_{\sigma\tau}\prod_t\mathrm{d}X_t\,e^{\mathrm{i}\sum_t\mathrm{Tr}(X_tG_t)}$$

Models for gravity

- Extend the fields to incorporate the normals n
- Impose linear simplicity ($nX_t^- n^{-1} + \beta X_t^+ = 0$), with $\beta = (\gamma - 1)/(\gamma + 1)$, using

$$S_n^\beta(X_t) := \delta^\star_{nX_t^- n^{-1}}(\beta X_t^+)$$

  in propagator or vertex (not a projector for $\beta \neq 1$)
- For $\beta = 1$, the upshot is

$$\mathcal{Z} = \int \prod_{(\sigma\tau)} \mathrm{d}g_{\sigma\tau} \prod_\tau \mathrm{d}n_\tau \prod_t \left(\mathrm{d}X_t \mathop{\bigstar}_{i=0}^{N_t} S_{g_{0i} \triangleright n_{\tau_i}}(X_t) \right) \star e^{\mathrm{i} \sum_t \mathrm{Tr}(X_t G_t)}$$

- Separating $i = 0$ from the rest yields

$$\mathcal{Z} = \int \prod_\tau \mathrm{d}n_\tau \prod_t \mathrm{d}X_t \underbrace{\mathcal{D}^{(X_t, k_\tau)}[g_{\sigma\tau}]}_{\substack{\text{simplicity in frames} \\ \text{around the dual face}}} \star \prod_t \left(e^{\mathrm{i}\mathrm{Tr}(X_t G_t)} \star \underbrace{\delta^\star_{k_{\tau 0} X_t^- k_{\tau 0}^{-1}}(X_t^+)}_{\substack{\text{simplicity in reference} \\ \text{tetrahedron frame}}} \right)$$

$$= \mathcal{Z}_{\mathrm{BC}}$$

General result

- Non-commutative simplicial path integral with measure over the bivectors (imposing primary simplicity) and over the holonomies $\rightarrow$ interpreted as imposing secondary constraints coming from the consistent imposition in all tetrahedron frames of the primary ones

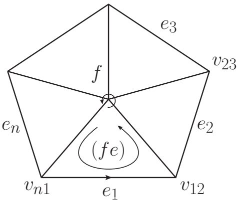
For $\beta \neq 1$

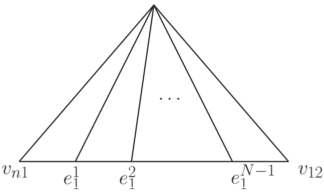
- Resulting model is different from EPRL/FK

Difficulty with the factorization of the constraints

- It appears too simple to be correct. The 3d Plebanski toy model (MG, Noui) shows that it would not reproduce the Ponzano–Regge model: the constraints on the holonomies are not the full set of secondary constraints coming from the imposition at all times of linear simplicity

Decomposition of a dual face $f \in \Delta^*$





Semi-discrete action (Wieland)

$$S_{(N)} = \sum_{f\in\Delta^*}\sum_{e\subset f}\sum_{i=1}^{N} \mathrm{Tr}\Big(X_f H_{(fe)^i}\Big) \xrightarrow{N\to\infty} S = \sum_{f\in\Delta^*} S_f = \sum_{e\in\Delta^*} S_e = \sum_{v\in\Delta^*} S_v$$

- Edge action $\qquad S_e = \displaystyle\sum_{f\supset e}^{d}\int_0^1 \mathrm{d}t\, \mathrm{Tr}\Big(X_f(t)\mathrm{D}_t h_f(t)^{-1} h_f(t)\Big)$
- Covariant symplectic potential on T^*G, continuous (time?) parameter t along the edge e, and covariant derivative $\mathrm{D}_t := \partial_t + \omega_e(t)$
- Possible classical version of the construction of Baratin and Oriti
- Need to add simplicity constraints and study canonical analysis (done by Wieland with spinors)
- How should these prescriptions distinguish between gravity and the toy models (4d degenerate and 3d Plebanski) if both are of the form BF + linear simplicity?

Achievements

- We have a well-defined spin foam framework (BC, EPR, FK, LS, EPRL, BO, Barrett et al.), fully covariant (DL, RS), provides amplitudes to all spinnets (KKL)
- Very good control on primary simplicity
- Good control on *simplicial* large quantum number asymptotics
- Common framework for first order discrete actions
- Improved understanding of role of secondary constraints

Future questions

- Clarify geometricity and torsion-lessness off the Regge framework?
- Understand how to implement this relation in spin foam models?
- Improve evaluation of quantum corrections

Methods

- Twistors and spinors
- Self-dual variables
- GFT methods