LQG as a TQFT with defects
New vacuum with a cosmological constant

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Based on work with B. Dittrich
Motivations

★ There are now different vacua (inequivalent quantum representations) for canonical LQG, spin foams, and group field theory (Ashtekar, Isham, Lewandowski; Koslowski, Sahlmann; Dittrich, MG)

★ How many vacua are there, how are they related, and what can they be used for?

Results

★ We extend the construction of the flat curvature BF vacuum to the case of (2 + 1) Euclidean dimensions for $\Lambda > 0$ (in a sense the most peculiar case)

★ The conjectured underlying mathematical structure is that

\[
\text{vacuum} + \text{excitations} = \text{TQFT} + \text{defects}
\]
1. LQG as a TQFT with defects

2. Contrasting the AL and BF representations

3. Constant curvature vacuum

4. Conclusion and perspectives
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LQG as a TQFT with defects

Kinematical results

* Boundary gravitational degrees of freedom: holonomies and fluxes encoding extrinsic and intrinsic spatial geometry

\[ h_l(A) \in SU(2) \]

\[ X^i_S(A, E) = \int_S h_{(0 \rightarrow x)} E^i(x) h^{-1}_{(0 \rightarrow x)} \]

* Diffeomorphism-invariant Hilbert spaces supporting the holonomy-flux algebra
* Derivation of quantum geometry (Ashtekar, Lewandowski; Rovelli, Smolin)
* Vacuum tell us how to refine states and construct a kinematical continuum limit

Dynamical questions

* How does a smooth diffeomorphism-invariant space-time emerge?
* What is the phase diagram of our “discrete” gravity models?
* What is a vacuum state?
LQG as a TQFT with defects

Is LQG discrete or continuous?

★ Both, since we work on graphs or triangulations, but states live in $\mathcal{H}_\infty$
★ Field theory with arbitrary finite $\#$ of degrees of freedom $=$ TQFT with defects

Classical kinematics as a TQFT with defects (Bianchi; Freidel, Ziprick, MG)

★ LQG phase space on a graph is $\mathcal{P}_\Gamma = \times_{\text{links}} (T^*\text{SU}(2))$
★ Same as that of gravity with almost-everywhere gauge-invariant flat connections

$$\mathcal{P}_\Gamma \simeq T^*\mathcal{A} \parallel (\mathcal{F} \times \mathcal{G})_\Gamma$$

Which TQFT and which defects?

★ In the AL representation, $E = 0$ and defects generate geometry
★ In the BF representation, $F(A) = 0$ and defects generate curvature
★ Classically, these geometries are gauge choices in the reconstruction

$$\mathcal{P}_\Gamma \ni (h_l, X_S) \mapsto (A(x), E(x)) \in T^*\mathcal{A}$$
★ In the quantum theory, these vacua lead to inequivalent representations
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## Contrasting the AL and BF representations

<table>
<thead>
<tr>
<th></th>
<th>AL</th>
<th>BF</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>background TQFT</strong></td>
<td>$E = 0$</td>
<td>$F(A) = 0$</td>
</tr>
<tr>
<td>vacuum state</td>
<td>$</td>
<td>\emptyset\rangle = \text{nothing}$</td>
</tr>
<tr>
<td>excitations</td>
<td>holonomies $h_l \triangleright</td>
<td>exponentiated fluxes $R_i^h \triangleright</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>\emptyset\rangle = h_l$</td>
</tr>
<tr>
<td></td>
<td>dual graphs</td>
<td>$(d - 1)$-simplices</td>
</tr>
<tr>
<td>defects</td>
<td>dual graphs</td>
<td>$(d - 2)$-simplices</td>
</tr>
<tr>
<td>refinement</td>
<td>$j = 0$</td>
<td>flatness</td>
</tr>
<tr>
<td>measure</td>
<td>Haar</td>
<td>discrete</td>
</tr>
<tr>
<td>kin. $C^\infty$ limit</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>generalization</td>
<td>background $E_o$</td>
<td>constant curvature ($\Lambda \neq 0$)</td>
</tr>
</tbody>
</table>
Contrasting the AL and BF representations

Other advantages of the BF representation

⋆ Deals for the first time successfully with the gauge-covariant fluxes $X(A, E)$
⋆ Diffeomorphisms as vertex displacement (in $2 + 1$)
⋆ Coarse-graining of the fluxes
⋆ Encode the interplay between curvature and torsion excitations

\[ X_{e_3 \circ e_2 \circ e_1} \neq 0 \text{ if } g_v \neq 1 \]

Relation with point particles and quantum double structure

⋆ Exponentiated fluxes and gauge transformations respectively generate curvature and torsion excitations
⋆ In $(2 + 1)$ gravity coupled to point particles, curvature and torsion are related to mass and spin, and these label unitary irreps of the Drinfeld double $DSU(2)$
1. LQG as a TQFT with defects

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4. Conclusion and perspectives
Constant curvature vacuum

Setup

* (2 + 1) spacetime dimensions with $\Lambda \neq 0$
* It is known that quantum groups emerge in the quantum theory

What we will not (but could/should) do

* Dynamical emergence: take flat SU(2) BF vacuum and solve $F[A] = \pm \Lambda(e \wedge e)$
* Kinematical emergence: construct BF vacuum with non-commutative holonomies $h(A \pm \sqrt{\Lambda}e)$ (Noui, Perez, Pranzetti)

Strategy and goal

* Assume quantum group structure from the onset
* For Euclidean $\Lambda < 0$, we could construct a vacuum based on the deformed phase space $\text{SL}(2, \mathbb{C}) \xrightarrow{\Lambda \to 0} \text{ISU}(2)$ (Bonzom, Dupuis, Girelli, Livine)
* For Euclidean $\Lambda > 0$, the quantum group is $\mathcal{U}_q(\mathfrak{su}(2))$ at root of unity, the TQFT is known (Turaev, Viro), but there is no group picture!
* We will show that the Turaev–Viro TQFT with defects is like a $\text{BF}_\Lambda$ vacuum with excitations

What we need to construct

1) A way of writing down and manipulating states (since there is no group picture)
2) A Hilbert space which contains flat states (vacuum) and excited states
3) A creation/excitation operator
1) Writing and manipulating states: graphical calculus

\* \( \mathcal{U}_q(\mathfrak{su}(2)) \) at root of unity is a modular braided fusion category \( \mathcal{C} \)

- **Category**: irreps given by spins \( j \in \{0, 1/2, \ldots, k/2\} \), with \( k = (G_N \hbar \sqrt{\Lambda})^{-1} \)
- **Fusion**: there are fusion coefficients such that \( i \otimes j = \bigoplus_k N_{ikj}^k \)

- **Braided**: there is an \( R \)-matrix such that

\[
\begin{array}{c}
\begin{array}{c}
\text{j} \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i \\
k
\end{array}
\end{array}
= R_{ik}^{ij}
\]

- **Modular**: \( \det(S) \neq 0 \) with \( S_{ij} = \frac{1}{D} \bigg| \begin{array}{c}
\begin{array}{c}
i \\
\cdots
\end{array}
\end{array} \bigg|_j \) and \( \mathcal{D}^2 = \sum_j d_j^2 \)

\* Dynamics (topological invariance) encoded in \( \{6j\}_q \) symbol

\[
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
m \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
= \sum_n F_{kln}^{ijm}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{j}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
n \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\]
2) Ribbon graph Hilbert space

- Consider a $p$-punctured $2d$ manifold $\Sigma_p$
- A puncture is obtained by removing a disc and placing a marked point on $\partial \Sigma$
- Allow for graphs in $\Sigma_p$ to have one link ending at each puncture
- Define $\mathcal{H}_p$ as the span of such graphs modulo the local equivalence relations

\[
\begin{align*}
    j \quad &\quad = \quad j \\
    \begin{array}{c}
    i \\
    j \quad m \quad l \\
    j \\
    \end{array} &\quad = \quad \sum_n F_{kln}^{ijm} \\
    \begin{array}{c}
    i \\
    k \quad j \\
    j \\
    l \\
    \end{array} &\quad = \quad \sqrt{\frac{d_i d_j}{d_k}} \delta_{kl} N_{ij}^k
\end{align*}
\]
Examples

* 2-sphere with 0 punctures: $\dim \mathcal{H}_0 = 1$

* 2-sphere with 1 puncture: $\dim \mathcal{H}_1 = 1$
Constant curvature vacuum

Examples

* 2-sphere with 2 punctures $\cong$ topologically a cylinder
* Non-trivial states corresponding to allowed spin labelings of

![Diagram of a sphere with punctures](image)

* Introduce $Q$ basis given by

$$Q^{ij}_{rs} = \begin{array}{ccc}
& & \\
& j & \\
i & & r \\
& & s \\
& & j
\end{array}$$

* $\dim \mathcal{H}_2 = \sum_{ijrs} N^r_{ij} N^s_{ij}$ depends on the level $k$

* Punctures can carry curvature (non-contractible cycles) and torsion (open links)
* Graphs can be seen as dual to degenerate triangulations
* Basis found for arbitrary topology and $p$ with pants and tree decomposition
Vacuum state

* The vacuum is selected by two projections imposing (Levin, Wen)
  
  - Gauge-invariance: \( B_v \triangleright \text{vertex}(i, j, k) = N_{i,j}^k \text{ vertex}(i, j, k) \)
  
  - Flatness: \( B_p \triangleright \bullet = \frac{1}{D} \sum_j d_j \bigcirc j := \bigcirc \)

* These dotted (vacuum) lines have the property that i.e. make the punctures invisible, i.e. remove the curvature that they carry

* In the SU(2) case, \( B_p \) becomes \( \sum_j d_j \chi_j(g) = \delta(g) \)

* If \( \Gamma \) is dual to a triangulation, one gets the Turaev–Viro invariant (Kirillov Jr.)

\[
\langle \Gamma | \prod_{\text{faces}} B_p | \Gamma \rangle = \text{TV}(\Sigma \times [0, 1])
\]

* The embedding map adds a puncture with no curvature, i.e. \( \bigcirc \)
The tube algebra

\* States $Q$ on the cylinder (2-punctured 2-sphere) can be thought of as operators
\* Stacking two cylinders (matching the marked points) gives back a cylinder, and in terms of $Q$’s this multiplication defines the tube algebra (Ocneanu; Müger)

\[
Q_{rs}^{ij} Q_{su}^{kl} = \sum_{mn} \alpha(F(\text{spins})) Q_{ru}^{mn}
\]

\* The $Q$ states therefore define both a vector space and an algebra
\* The quasi-particle excitations we are looking for are representations of (or modules over) this algebra (Lan, Wen)
\* More precisely, we look for states $\xi$ with a stability property $Q \triangleright \xi = \xi$

Remark: back to the group picture

\* Consider the idempotent (or projector) fluxes $X_h$ such that $X_h X_{h'} = \delta_{hh'} X_h$
\* Gauge transformations act on the fluxes by conjugation, so $g X_h = X_{ghg^{-1}} g$
\* Then the algebra satisfies the multiplication rule of the Drinfeld double

\[
X_h g X_{h'} g' = \delta_{h,ghg^{-1}} X_h g g'
\]
**Quantum double**

- Irreducible representations of the tube algebra (or the category of modules) form a quantum double category $\mathcal{Z}(\mathcal{C})$, and because here $\mathcal{C}$ is modular, $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \otimes \mathcal{C}^*$

- Basis states $O_{rs}^{ij} = \begin{array}{c}
\bullet
\end{array}$ satisfy the projection condition $O \propto O$, are orthonormal (König, Kuperberg, Reichardt), and are stable in the sense that

\[ \sum_{pq} \Omega^{ij}_{rl,qp} = \sum_{pq} \Omega^{ij}_{rp,ql} \]

- The objects

\[ \Omega^{ij}_{rl,qp} = \sum_{mn} \sqrt{\frac{d_m d_n}{d_r d_l^2}} R^{il}_{rm} R^{lj}_{ln} F^{lmr}_{ij} F^{qlp}_{ij} F^{qlp}_{ij} \]

are called half-braiding tensors and label elements of the quantum double $\mathcal{Z}(\mathcal{C})$
3) Excitations

* The excitation operators are given by the following oriented ribbons

\[
\xi \colon 1 \rightarrow D \sum_k \sqrt{\frac{d_k}{d_i d_j}}
\]

* By definition, the half-braiding tensors have to satisfy so-called naturality conditions \(\leftrightarrow\) fixed point conditions for the \(\mathcal{Q}\) algebra \(\leftrightarrow\) sliding property

* E.g., acting on the cylinder vacuum state \(\bullet \circ \circ\) gives a basis state \(\mathcal{O}^{ij}_{l_1 l_2}\)

* Open/closed ribbon operators correspond respectively to fluxes/holonomies

* Automatically describes the Turaev–Viro model coupled to point particles
Conclusion and perspectives

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Conclusion and perspectives

New framework and results

* New physical BF vacuum for SU(2) in $d = 2, 3$ spatial dimensions
* Full continuum Hilbert space supporting curvature excitations
* Allows for geometrical coarse graining of the fluxes
* New take on the dynamics and extraction of physics
* Same structures found in wider class of TQFTs with defects (allows $\Lambda \neq 0$)
* Establishes link with other areas (condensed matter, extended TQFTs)
* Right mathematical framework to describe fixed points of coarse-graining flow

Generalizations and applications

* Other quantum groups, signature, and signs of $\Lambda$
* Extension to $(3 + 1)$
* Dynamics of the defects (ILQGS by Wolfgang Wieland on April 5$^{th}$)
* Non-commutative flux representation
* Non-compact gauge groups
* Cosmology
* Black holes