Continuous formulation of the loop quantum gravity phase space

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Introduction and motivations

Loop quantum gravity has a built-in notion of quantum Riemannian three-geometry. Given a kinematical spin network state on a graph Γ,

- ⋆ links carry information about quanta of area,
- ⋆ nodes carry information about quanta of volume.

For practical applications (cosmology, n-point functions, ...), we work on a truncation of the full LQG Hilbert space $\mathcal{H} = \bigoplus_{\Gamma} \mathcal{H}_\Gamma$.

- ? What is the geometrical interpretation of states on a fixed graph? (Rovelli and Speziale [1005.2927])
- ⋆ Twisted geometries: (Freidel and Speziale [1001.2748])

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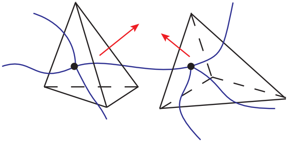

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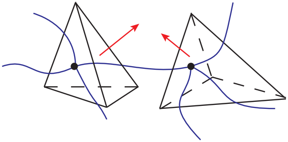


Parametrization of the LQG phase space in terms of continuous geometries?
We are going to relate the holonomy-flux algebra to configurations in the phase space of continuous gravity: piecewise-flat connections and gauge-invariant electric fields.

Outline

1. The continuous phase space of gravity
2. The discrete LQG phase space
3. Relating continuous and discrete geometries
4. Gauge choices
5. Cylindrical consistency
6. Conclusion

The continuous phase space of gravity

The classical starting point of LQG is the formulation of first order gravity in terms of $\mathfrak{su}(2)$-valued phase space variables:

- ⋆ Ashtekar-Barbero connection

$$A_a^i = \Gamma_a^i + \gamma K_a^i.$$

- ⋆ Densitized triad field

$$E_i^a = \frac{1}{2}\varepsilon^{abc}\varepsilon_{ijk}e_b^j e_c^k.$$

These variables form the Poisson algebra

$$\left\{A_a^i(x), E_j^b(y)\right\} = \gamma\delta_j^i\delta_a^b\delta^3(x,y), \qquad \left\{A_a^i(x), A_b^j(y)\right\} = \left\{E_i^a(x), E_j^b(y)\right\} = 0,$$

and span the phase space $\mathcal{P} \equiv T^*\mathcal{A}$, whose symplectic potential is given by

$$\Theta = \int_\Sigma \mathrm{Tr}(E \wedge \delta A).$$

The continuous phase space of gravity

The phase space $\mathcal{P}$ carries an action of the group G of SU(2) gauge transformations. The infinitesimal generator of these transformations is the Gauss constraint

$$\mathcal{G}(\alpha) = \int_\Sigma \alpha^i (\mathrm{d}_A E)_i = 0, \qquad \alpha \in \mathfrak{su}(2),$$

whose action is given by

$$\delta^{\mathcal{G}}_\alpha A = \big\{A, \mathcal{G}(\alpha)\big\} = \mathrm{d}_A \alpha, \qquad\qquad \delta^{\mathcal{G}}_\alpha E = \big\{E, \mathcal{G}(\alpha)\big\} = [E, \alpha].$$

$\mathcal{P}$ also carries an action of diffeomorphisms:

$$\delta^{\mathcal{D}}_\xi A = \big\{A, \mathcal{D}(\xi)\big\} = \mathcal{L}_\xi A, \qquad\qquad \delta^{\mathcal{D}}_\xi E = \big\{E, \mathcal{D}(\xi)\big\} = \mathcal{L}_\xi E,$$

where ξ_a is a vector field.

The continuous phase space of gravity

Given a group G of gauge transformations, with infinitesimal generator $\mathcal{G}$, the symplectic reduction of $\mathcal{P}$ with respect to G is defined as

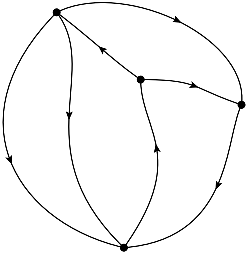
$$\mathcal{P}^G \equiv \mathcal{P} \mathbin{/\mkern-6mu/} G = \mathcal{G}^{-1}(0)/G.$$

The Marsden-Weinstein-Meyer theorem ensures that $\mathcal{P}^G$ is again symplectic manifold. In particular, it carries a symplectic structure.

The discrete LQG phase space

In loop quantum gravity, we do not work with the continuous phase space $\mathcal{P}$, but instead with phase spaces P_Γ associated to embedded oriented graphs.

$$\big(A(x),E(x)\big) \quad \xrightarrow{\text{"loop assumption"}} \quad \big(h_e,X_e\big)$$



How do we describe this map from continuous to discrete data?
In other words, how do we construct the holonomies and the fluxes?

The discrete LQG phase space – The holonomies h_e

$$s(e) \bullet \xrightarrow[h_e(A)]{} \bullet\, t(e)$$

The holonomy is given by the path-ordered exponential

$$\mathrm{SU}(2) \ni h_e(A) = \overrightarrow{\exp} \int_e A.$$

- Under finite gauge transformations of the connection,

  $$g \triangleright A = gAg^{-1} + g\mathrm{d}g^{-1}, \qquad g \in \mathrm{SU}(2),$$

  the holonomy becomes

  $$h_e(g \triangleright A) = g_{s(e)} h_e(A) g_{t(e)}^{-1}.$$

- Under diffeomorphisms: $h_e(\Phi^* A) = h_{\Phi(e)}(A)$.
- Under orientation reversal: $h_{e^{-1}}(A) = h_e(A)^{-1}$.

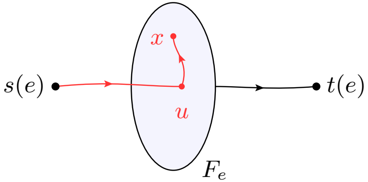
The discrete LQG phase space – The fluxes X_e

To define the flux of the densitized triad field, we need to choose:

- ⋆ A surface F_e intersecting e at the point u.
- ⋆ A system of paths π_e going from the vertex $s(e)$ to the point $x \in F_e$.

With the data (F_e, π_e), we can define the flux

$$X^i_{(F_e,\pi_e)}(A,E) \equiv \int_{F_e} h_{\pi_e}(x) E^i(x) h^{-1}_{\pi_e}(x)$$



where

$$h_{\pi_e}(x) \equiv \overrightarrow{\exp} \int_{s(e)}^{x} A.$$

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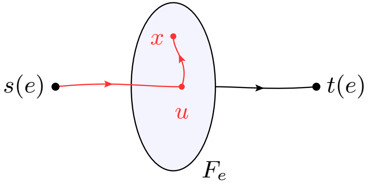


where

$$h_{\pi_e}(x) \equiv \overrightarrow{\exp} \int_{s(e)}^{x} A.$$

It satisfies the following properties:

- ⋆ Gauge-covariance:

$$X_{(F_e,\pi_e)}(g \triangleright A, g \triangleright E) = g_{s(e)} X_{(F_e,\pi_e)}(A,E) g^{-1}_{s(e)}.$$

- ⋆ Orientation reversal:

$$X_{\left(F_{e^{-1}},\pi_{e^{-1}}\right)} = -h_e^{-1} X_{(F_e,\pi_e)} h_e.$$

These flux elements are naturally non-commuting, because they carry information about both A and E.

The discrete LQG phase space – The holonomy-flux algebra

The discrete elements $(h_e, X_e) \in \mathrm{SU}(2) \times \mathfrak{su}(2)$ associated to each edge of the graph Γ satisfy the Poisson algebra

$$\{X_e^i, X_{e'}^j\} = \delta_{ee'}\epsilon^{ij}{}_k X_e^k, \qquad \{X_e^i, h_{e'}\} = -\delta_{ee'}\tau^i h_e + \delta_{ee'^{-1}} h_e \tau^i, \qquad \{h_e, h_{e'}\} = 0.$$

This is the Poisson structure on $T^*\mathrm{SU}(2)$.

The spin network phase space associated to the graph Γ is therefore given by

$$P_\Gamma \equiv \underset{e}{\times} T^*\mathrm{SU}(2)_e,$$

and the symplectic potential is

$$\Theta_\Gamma = \sum_e \mathrm{Tr}\left(X_e \mathrm{d}h_e h_e^{-1}\right).$$

The Hilbert space $\mathcal{H}_\Gamma$ of LQG on a graph is the quantization of P_Γ.

The discrete LQG phase space – Gauge invariance

The discrete spin network phase space P_Γ carries a natural action of the gauge group $\mathrm{SU}(2)^{|V_\Gamma|}$:

$$g \triangleright h_e = g_{s(e)} h_e g_{t(e)}^{-1}, \qquad g \triangleright X_e = g_{s(e)} X_e g_{s(e)}^{-1},$$

and it is therefore possible to define the gauge-invariant phase space

$$P_\Gamma^G \equiv P_\Gamma \mathbin{/\!\!/} \mathrm{SU}(2)^{|V_\Gamma|} = G_v^{-1}(0)/\mathrm{SU}(2)^{|V_\Gamma|}.$$

We are going to identify later on the generator G_v of these transformations. It is the discrete analogue of the continuous Gauss law.

The discrete LQG phase space

Given an embedding of Γ into Σ, a surface F_e and a system of paths π_e, we have described a map

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & P_\Gamma \\ (A,E) & \longmapsto & \big(h_e(A), X_{(F_e,\pi_e)}(A,E)\big) \end{array}$$

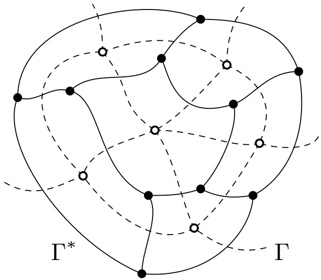
from continuous to discrete data.

To what extent can this map be inverted?

A priori there are many ambiguities, and it is not possible to uniquely reconstruct the continuous data starting from h_e and X_e.

Relating continuous and discrete geometries

Let us consider a cellular decomposition Δ of Σ, with one-skeleton Γ^*, together with its dual graph Γ.



continuous data

$\big(A(x), E(x)\big) \in \mathcal{P}$

discrete data

$(h_e, X_e) \in P_\Gamma$

Relating continuous and discrete geometries

We are going to construct the symplectic reduction of $\mathcal{P}$ with respect to two constraints:

Flatness outside of Γ^*

Generated by

$$\mathcal{F}(\phi) = \int_\Sigma \phi_i \wedge F^i(A),$$

$$\delta^{\mathcal{F}}_\phi A = 0, \qquad \delta^{\mathcal{F}}_\phi E = \mathrm{d}_A \phi,$$

for $\phi \in \Omega^1(\Sigma, \mathfrak{su}(2))$ vanishing on Γ^*.

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Gauss law outside of the vertices of Γ

Generated by

$$\mathcal{G}(\alpha) = \int_\Sigma \alpha^i (\mathrm{d}_A E)_i,$$

$$\delta^{\mathcal{G}}_\alpha A = \mathrm{d}_A \alpha, \qquad \delta^{\mathcal{G}}_\alpha E = [E, \alpha],$$

for $\alpha \in \Omega^0(\Sigma, \mathfrak{su}(2))$ vanishing on the vertices V_Γ of Γ.

Relating continuous and discrete geometries

With these two constraints, we can define the symplectic reduction

$$\begin{aligned}\widetilde{\mathcal{P}} &\equiv T^*\mathcal{A} /\!\!/ \big(\mathcal{G}\times\mathcal{F}\big)\\ &= \mathcal{C}/\big(\mathcal{G}\times\mathcal{F}\big),\end{aligned}$$

where the constraint space is

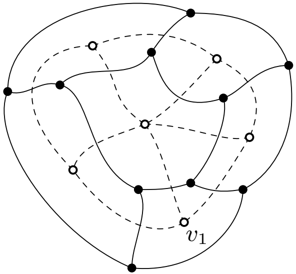
$$\mathcal{C}\equiv\big\{(A,E)\in T^*\mathcal{A}\mid F(A)=0\text{ outside of }\Gamma^*,\ \mathrm{d}_AE=0\text{ outside of }V_\Gamma\big\}.$$

Claim: The orbit space $\widetilde{\mathcal{P}}$ is finite-dimensional. It is the continuous analogue of the discrete spin network phase space P_Γ.

Let us now construct the explicit parametrization of $\widetilde{\mathcal{P}}$.

Relating continuous and discrete geometries

We are going to solve $\mathcal{F}$ and $\mathcal{G}$ on the three-cells C_v of $\Delta \setminus \Gamma^*$, and then glue the solutions consistently.



Relating continuous and discrete geometries

On a three-dimensional cell C_v, we have the following parametrization:

Solution to the curvature constraint

In terms of the SU(2) element

$$a_v(x) \equiv \overrightarrow{\exp} \int_x^v A,$$

the flat connection is given by $A(x) = a_v(x)\mathrm{d}a_v(x)^{-1}$.
Note that the integration does not depend on the path.

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Note that the integration does not depend on the path.

Solution to the Gauss constraint outside of the vertex v

For a flat connection, the covariant derivative of the electric field becomes

$$\mathrm{d}_A E = \mathrm{d}E + [a_v \mathrm{d}a_v^{-1}, E] = a_v \mathrm{d}X_v a_v^{-1},$$

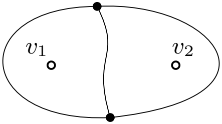
where

$$X_v \equiv a_v^{-1} E a_v \in \mathfrak{su}(2).$$

Therefore, the Gauss law $\mathrm{d}_A E = 0$ implies that X_v is a closed two-form.

Now we have to glue these solutions with the neighboring cells...

Relating continuous and discrete geometries



cell C_1

$a_1(x)$

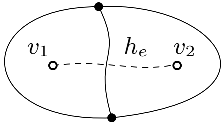
$X_1(x)$

cell C_2

$a_2(x)$

$X_2(x)$

Relating continuous and discrete geometries



cell C_1	gluing condition	cell C_2
$a_1(x)$	$a_2(x) = a_1(x)h_e$	$a_2(x)$
$X_1(x)$	$X_2(x) = h_e^{-1}X_1(x)h_e$	$X_2(x)$

The transition element h_e is the holonomy

$$h_e(A) = a_{s(e)}(x)^{-1}a_{t(e)}(x) = \overrightarrow{\exp}\int_e A.$$

This shows that the constraint space $\mathcal{C}$ is spanned by the data $\big(a_v, X_v, h_e\big)$.
Now, we have to divide $\mathcal{C}$ by the action of the gauge groups $\mathcal{F}$ and $\mathcal{G}$.

Relating continuous and discrete geometries

The action of $\mathcal{F}$ and $\mathcal{G}$ on the discrete data $\big(a_v, X_v, h_e\big)$ is given by

$$a_v(x) \longrightarrow g a_v(x), \qquad\qquad X_v(x) \longrightarrow X_v(x) + \mathrm{d}\big(a_v^{-1} \phi a_v\big),$$

with $\phi = 0$ on Γ^* and $g = 1$ on V_Γ.

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with $\phi = 0$ on Γ^* and $g = 1$ on V_Γ.

The observables that are invariant under $\big(\mathcal{F} \times \mathcal{G}\big)$ are therefore:

- ⋆ Holonomies

$$h_e(A) \equiv \overrightarrow{\exp} \int_e A = a_{s(e)}(x)^{-1} a_{t(e)}(x).$$

- ⋆ Fluxes

$$X_e(A, E) = \int_{F_e} h_{\pi_e}(x) E(x) h_{\pi_e}^{-1}(x) = \int_{F_e} a_v(v) a_v(x)^{-1} E(x) a_v(x) a_v(v)^{-1} = \int_{F_e} X_v.$$

The continuous Gauss law becomes

$$\int_{C_v} a_v(x)^{-1} \mathrm{d}_A E(x) a_v(x) = \int_{C_v} \mathrm{d}X_v = \int_{\cup_e F_e = \partial C_v} X_{s(e)} = \sum_{e | s(e) = v} X_e = G_v.$$

Relating continuous and discrete geometries

We are going to show that the map

$$\begin{array}{rcccl} [\mathcal{I}] & : & \widetilde{\mathcal{P}} & \longrightarrow & P_\Gamma \\ & & (A,E) & \longmapsto & \big(h_e(A), X_e(A,E)\big) \end{array}$$

from continuous to discrete data

- ⋆ is a Poisson map,
- ⋆ is diffeomorphism-invariant (under suitable conditions),
- ⋆ can be inverted to reconstruct continuous geometries $\big[A(h_e), E(h_e, X_e)\big]$.

Let us write the symplectic potential of first order gravity on Δ, and evaluate it on the configurations in $\mathcal{C}$:

$$\begin{aligned}
\Theta &= \int_\Sigma \mathrm{Tr}(E \wedge \delta A) \\
&= \sum_v \int_{C_v} \mathrm{Tr}\big(E \wedge \delta(a_v \mathrm{d}a_v^{-1})\big) \\
&= \sum_v \int_{\partial C_v} \mathrm{Tr}\big(X_v \wedge \delta a_v a_v^{-1}\big) \\
&= \sum_e \int_{F_e} \Big[\mathrm{Tr}\left(X_{s(e)} \wedge \delta a_{s(e)}^{-1} a_{s(e)}\right) - \mathrm{Tr}\left(X_{t(e)} \wedge \delta a_{t(e)}^{-1} a_{t(e)}\right)\Big] \\
&= \sum_e \mathrm{Tr}\big(X_e \delta h_e h_e^{-1}\big) \\
&= \Theta_\Gamma
\end{aligned}$$

This shows that $\widetilde{\mathcal{P}}$ is finite-dimensional, and isomorphic to P_Γ.

Relating continuous and discrete geometries – Diff-invariance

Let us consider a diffeomorphism Φ_o connected to the identity and preserving Γ^* and the vertices of Γ.

The property $\mathcal{I} \circ \Phi_o = \mathcal{I}$ can be shown using the on-shell equivalence between diffeomorphisms and gauge transformations.

$$\mathcal{L}_\xi A = \iota_\xi F + \mathrm{d}_A(\iota_\xi A)$$

$$\mathcal{L}_\xi E = \iota_\xi \mathrm{d}_A E + \mathrm{d}_A(\iota_\xi E) + [E, \iota_\xi A]$$

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Since on $\mathcal{C}$ the curvature F vanishes outside of Γ^* and $\mathrm{d}_A E$ vanishes outside of V_Γ,

$$\mathcal{L}_\xi A = \iota_\xi F + \mathrm{d}_A(\iota_\xi A) = \mathrm{d}_A(\iota_\xi A)$$

$$\mathcal{L}_\xi E = \iota_\xi \mathrm{d}_A E + \mathrm{d}_A(\iota_\xi E) + [E, \iota_\xi A] = \mathrm{d}_A(\iota_\xi E) + [E, \iota_\xi A]$$

which shows that $\mathcal{L}_\xi = \delta^{\mathcal{F}}_{\iota_\xi E} + \delta^{\mathcal{G}}_{\iota_\xi A}$.

Gauge choices

At the continuous level, we have an equivalence class of gauge-related configurations:

$$\big(A, E\big) \sim \big(g \triangleright A, g(E + \mathrm{d}_A\phi)g^{-1}\big).$$

A choice of a representative in this equivalence class is a choice of gauge, i.e. a map

$$\begin{array}{cccccc} \mathcal{T} & : & P_\Gamma & \longrightarrow & \mathcal{C} \\ & & (h_e, X_e) & \longmapsto & (A, E) \end{array}$$

such that $\mathcal{I} \circ \mathcal{T} = \mathrm{Id}$.

There are lots of possibilities for the choice of $E(x)$.
Powerful criterion: Find gauge choices that are diffeomorphism-covariant.

- ⋆ Loop quantum gravity gauge $E|0\rangle = 0$, singular E outside of Γ.
- ⋆ Spin foam gauge $F|0\rangle_{\mathrm{F}} = 0$, flat E outside of Γ^*.

Gauge choices – Singular gauge (LQG)

The singular electric field can be constructed explicitly. It is given by

$$
\begin{aligned}
E_{\mathrm{S}}(x) &\equiv \mathrm{d}_A\left(\sum_{e\in\Gamma} h_{\pi_e}(x)^{-1}X_e h_{\pi_e}(x)\int_{e(y)}\omega(x,y)\right)\\
&= \sum_e h_{\pi_e}(x)^{-1}X_e h_{\pi_e}(x)\delta_e(x),
\end{aligned}
$$

and satisfies

$$E_{\mathrm{S}}(x)=0,\quad \forall\, x\notin\Gamma \qquad \text{(singular condition)},$$

$$\mathrm{d}_A E_{\mathrm{S}}=0 \qquad \text{(Gauss law)},$$

$$\mathcal{I}\big(E_{\mathrm{S}}(h_e,X_e)\big)=X_e \qquad \text{(gauge choice)}.$$

Gauge choices – Flat gauge (spin foams)

A flat geometry within each cell of $\Sigma \setminus \Gamma^*$ is defined by

- ⋆ a flat connection A,
- ⋆ a frame field e satisfying $\mathrm{d}_A e = 0$.

Indeed, since $\mathrm{d}_A e = \gamma[K, e]$, the torsion-free condition and the invertibility of e imply $K = 0$, and therefore $A = \Gamma(e)$. If A is flat, the metric determined by e is therefore flat as well.

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It is always possible to use the gauge freedom

$$\mathcal{F} : E \longrightarrow E + \mathrm{d}_A \phi$$

to find a E such that $\mathrm{d}_A e = 0$.

P_Γ can be seen as the space of piecewise metric-flat geometries on $\Sigma \setminus \Gamma^*$.

In fact, $P_\Gamma = T^* \mathcal{M}_{\Gamma^*}$, where $\mathcal{M}_{\Gamma^*}$ is the moduli space of flat connections modulo gauge transformations (cf. Bianchi [0907.4388]).

Gauge choices – Flat gauge and Regge geometries

If we have a frame field such that $\mathrm{d}_A e = 0$, then

$$e_v(x) = a_v(x)\mathrm{d}x_v a_v(x)^{-1},$$

where $x_v(x)$ are flat coordinates. The flux elements become

$$X_e^i = \frac{1}{2}\varepsilon^i{}_{jk} \int_{F_e} h_{\pi_e}(x) \left(e_v^j \wedge e_v^k\right) h_{\pi_e}(x)^{-1} = \frac{1}{2}\varepsilon^i{}_{jk} \int_{F_e} \mathrm{d}x_v^j \wedge \mathrm{d}x_v^k.$$

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If the two-cells are forced to be flat, $X_e = j_e N_e$.

- ⋆ LQG = piecewise flat geometries.
- ⋆ Regge = piecewise-linear flat geometries:
  only the component of K parallel to e is non-vanishing.

Cylindrical consistency

What is the relationship between operators $\mathcal{O} \in \mathcal{P}$ and $\mathcal{O}_\Gamma \in P_\Gamma$?

The discrete operators $\mathcal{O}_\Gamma$ are called cylindrically consistent if

$$\mathcal{O}|_{\mathcal{C}}(A, E) = \mathcal{O}_\Gamma\big(h_e(A), X_e(A, E)\big).$$

Because of our construction,

$\mathcal{O}$ is a cylindrical operator iff $\mathcal{O}|_{\mathcal{C}}(A, E)$ is invariant under $\mathcal{F} \times \mathcal{G}$.

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<u>Example</u>

- ⋆ Continuous area operator: $\mathbf{A}(S) = \int_S \sqrt{E^i_a E^a_i}$
- ⋆ LQG area operator: $\mathbf{A}_{\text{LQG}}(S) = \sum_{e|e \cup S \neq 0} \sqrt{X^i_e X_{ei}}$

The continuous area operator is clearly not cylindrical, therefore $\mathbf{A}(S)|_{\mathcal{C}} \neq \mathbf{A}_{\text{LQG}}(S)$. In fact, the LQG operator is the continuous operator in the singular gauge: it can be written exclusively in terms of fluxes.

What about other operators and other gauges?

Conclusion

- ⋆ We have given a parametrization of the discrete LQG phase space P_Γ in terms of continuous geometries on $\widetilde{\mathcal{P}}$.
- ⋆ The data (h_e, X_e) labels an equivalence class of continuous geometries.
- ⋆ Clarifies the relation between the geometry of LQG, spin foams, and Regge.
- ⋆ The geometric operators in LQG are gauge-fixed operators.
- ? LQG as the quantization of a TQFT with defects?
- ? Formulation of classical general relativity in terms of (h_e, X_e)?
- ? Other gauge choices for the geometric operators.
- ? What happens in the case of a non-vanishing cosmological constant?

Merci