Canonical Cosmological Perturbation Theory using Geometrical Clocks

joint work with Adrian Herzog, Param Singh
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Plan of the Talk

I. Motivation for canonical cosmological perturbation theory

II. Brief remarks on Lagrangian framework

III. Canonical formulation

- Framework of extended ADM phase space
- Gauge fixing
- Relation formalism and observable map

IV. Construction of common gauge invariant quantities

- Two examples: Longitudinal and spatially flat gauge

V. Summary and Outlook

I. Motivation

Why Cosmological Perturbation Theory (CPT) ?

Interesting for us: Possibility to learn something about the early universe: Test (quantum) cosmological models

[Wilson-Ewing, Ashtekar, Agullo, Bojowald, Singh, Mena-Marugan, ...]

Gauge invariant perturbations: observations

Aim: Obtain gauge invariant equation of motion for perturbations

CPT: Common gauges and gauge invariant variables exist

I. Motivation

Why Cosmological Perturbation Theory (CPT) ?

Relational Formalism: Provides naturally a framework to construct gauge invariant quantities once reference fields (clocks) have been chosen.

In particular useful for higher order perturbations theory

Geometrical interpretation of gauges (dynamical observers)

However, often matter reference fields are chosen:

[(K.G. Thiemann),(Domagala, K.G., Kaminski, Lewandowski),(Husain,Pawlowski),(K.G., Oelmann)]

As far as reduced quantization is considered matter models have advantage of simple observable algebra

In LQC: scalar field and common gauge invariant quantities

I. Cosmological Perturbation theory

Phase space formulation:

Langlois: Only reduced ADM+ scalar field, only Mukhanov-Sasaki

Dittrich, Tambornino: Connection formulation, lapse and shift not dynamical but gauge fixed.

K.G., Hofmann, Thiemann, Winkler
Non-linear Brown-Kuchar-dust clocks, not all common gauges

Dapor, Lewandowski, Puchta: Different gauge invariant variables
Cosmological perturbations on quantum spacetimes

Elizaga Navascues, Martin de Blas, Mena Marugan: effects of quantum spacetime on perturbations

Indicates: Natural clocks for each common gauge choice?

Relevant for reduced quantized models in full and LQC?

II. Cosmological Perturbation theory

One considers perturbations around a flat FLRW spacetime

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j$$

$\longrightarrow$ Aim: Gauge invariant EOM for perturbations

$$\delta N,\ \delta N^a,\ \delta q_{ab}$$

Consider k=0 case only

Linearized perturbations theory: Scalar-vector-tensor decomposition

$$N = \overline{N}(1+\phi), N^a = B^{,a} + S^a$$
$$\delta q_{ab} = 2a^2(\psi \delta_{ab} + E_{,<a,b>} + F_{(a,b)} + \frac{1}{2} h^{TT}_{ab})$$

Restrict discussion to scalar sector here ϕ, ψ, E, B

At linear order perturbed Einstein equations decouple, however complicated Poisson algebra due to projections.

Lagrangian Formalism

One considers linear perturbations and linearized diffeom.:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} \qquad \xi^{\mu} = (\xi, \vec{\xi})^T = (\xi, \hat{\xi}^{,a} + \xi^a_{\perp})^T$$

Gravitational sector:

$$\begin{aligned} \phi' &= \phi + \frac{1}{\bar{N}}\xi_{,t} \quad B' = B - \frac{\bar{N}}{a^2}\xi + \hat{\xi}_{,t} \\ \psi' &= \psi + \frac{\tilde{\mathcal{H}}}{\bar{N}}\xi + \frac{1}{3}\Delta\hat{\xi} \quad E' = E + \hat{\xi} \end{aligned}$$

Matter sector:

$$\varphi' = \varphi + \frac{1}{\bar{N}}\bar{\varphi}_{,t}\xi$$ minimally coupled scalar field

Gauge invariance: Lagrangian Formalism

General idea: Choose among the gauge variant 5 scalar dof $\phi, B, \psi, E, \varphi$ two out of them to construct linearized gauge invariant extensions of the remaining three.

Particular choice of such a pair: Common gauge in cosmology

Two examples: Bardeen potentials and Mukhanov-Sasaki variable

$$\Psi := \psi - \frac{1}{3}\Delta E + \frac{\mathcal{H}a^2}{\bar{N}^2}(B - \dot{E}) \qquad \Phi := \phi + \frac{1}{\bar{N}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^2}{\bar{N}}(B - \dot{E})\right)$$

gauge invariant extension of lapse and some spatial metric perturbations

(longitudinal gauge, B=0, E=0)

$$v := \delta\varphi - \frac{\dot{\bar{\varphi}}}{\tilde{\mathcal{H}}}\left(\psi - \frac{1}{3}\Delta E\right)$$

gauge invariant extension of scalar field (spatially flat gauge $\psi = 0, E = 0$)

III. Aim: Do the same on phase space

Here we follow the following strategy:

Step 1: Express Bardeen potentials, Mukhanov-Sasaki variable and their dynamics in ADM phase space + scalar field

Step 2: Use the relational formalism and choose appropriate geometrical clocks from which Bardeen potentials and Mukhanov-Sasaki variables are obtained naturally from the observable map, relation to gauge choice, physical dof.

Step 3: Analyze the dynamics of the observables and check that standard results can be reproduced

Step 4: Use results to go further: geometric interpretation of gauges, higher order perturbations, non-linear clocks (work in progress)

Why extended phase space?

Realize: Bardeen potential Φ is the gauge invariant extension of ϕ (lapse perturbation)

Explicit form of Φ, Ψ suggests that B is among the 'natural' clocks

In reduced ADM phase space we have:

$$\mathcal{M} \simeq \mathbb{R} \times \sigma \qquad g_{\mu\nu} \longrightarrow q_{ab}, N, N^a$$

Reduced ADM: $(q_{ab}, p^{ab}) \quad N, N^a \quad \Pi = 0, \Pi_a = 0$

Einstein's eqns: $\dot{q}_{ab} = \{q_{ab}, H_{\text{can}}\} \quad \dot{p}^{ab} = \{p^{ab}, H_{\text{can}}\}$

Constraints: $C(q,p) = 0 \quad C_a(q,p) = 0$

Can. Hamiltonian: $H_{\text{can}} := \int_\sigma d^3x (NC + N^a C_a)$

C, C_a acts trivially on lapse and shift, construction of Φ, B as clock?

Extension to full ADM Phase space

[Pons, Garcia]

CPT: Mostly discussed in Lagrangian framework Phase space:

$$\mathcal{M} \simeq \mathbb{R} \times \sigma \qquad g_{\mu\nu} \longrightarrow q_{ab}, N, N^a$$

Full ADM: $(q_{ab}, p^{ab}), (N, \Pi), (N^a, \Pi_a)$

Einstein's eqns:

$$\dot{q}_{ab} = \{q_{ab}, H_{\rm can}\} \quad \dot{p}^{ab} = \{p^{ab}, H_{\rm can}\}$$

$$\dot{N} = \{N, H_{\rm can}\} = \lambda \quad \dot{\Pi} = \{\Pi, H_{\rm can}\} \approx 0$$

$$\dot{N}^a = \{N^a, H_{\rm can}\} = \lambda^a \quad \dot{\Pi}_a = \{\Pi_a, H_{\rm can}\} \approx 0$$

Constraints: $C(q,p) = 0, C_a(q,p) = 0, \Pi = 0, \Pi_a = 0$

Can. Hamiltonian: $H_{\rm can} = \int d^3x (NC + N^a C_a + \lambda\Pi + \lambda^a \Pi_a)$

Extended Symmetry generator:

$$G_{b,\vec{b}} = C[b] + \vec{C}[b] + \Pi(\dot{b} + b^a N_{,a} - N^a b_{,a}) + \Pi_a(\dot{b}^a + q^{ab}(bN_{,b} - Nb_{,b}) - N^a b^b_{,a} + b^a N^b_{,a})$$

reduces to primary Hamiltonian for $b = N, b^a = N^a$

Remarks on Gauge Fixing

<u>Reduced ADM phase space:</u> Choose gauge fixing conditions:

$G_\mu \approx 0$ of the form $G_\mu = F_\mu(q,p) - f_\mu(x,t)$

Stability of the gauge fixing conditions:

$$\frac{dG_\mu}{dt} = \{G_\mu, H_{\text{can}}\} + \frac{\partial G_\mu}{\partial t} \overset{!}{\approx} 0$$

fixes lapse and shift if G_μ is appropriately chosen, no further cond.

<u>Full ADM phase space:</u> Choose gauge fixing conditions:

$G_\mu \approx 0$ of the form $G_\mu = F_\mu(q,p,N,\Pi,N^a,\Pi_a) - f_\mu(x,t)$

Stability here:

$$\frac{dG_\mu}{dt} = \{G_\mu, H_{\text{can}}\} + \frac{\partial G_\mu}{\partial t} \overset{!}{\approx} 0$$

since lapse and shift are no Lagrange multipliers, further stability fixes λ, λ^a

extended phase space allows to choose lapse & shift dof in G_μ!

Relational Formalism: Observables

[Rovelli, Dittrich]

Start with constrained theory $(q^A, P_A), \; \{C_I\}, \; I \text{ label set}$

Choose clocks T^I satisfying: (weak Abelianization, linear in clock momentum)

$$\{T^I\} \quad \text{s.t.} \quad \{T^I, C_J\} \approx \delta^I_J \quad \Leftrightarrow \quad \{G^I, C_J\} \approx \delta^I_J$$

Gauge invariant extensions (observables) associated with f:

$$O^C_{f,T}(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}(\tau - T^I)^n \{f, C_I\}_{(n)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(G^I)^n \{f, C_I\}_{(n)}$$

Dirac observables: $\{O_f, C_I\} \simeq 0$

Geometrical clocks

Choose clocks from metric dof

Algebra of observables: $\{O_f, O_g\} \simeq O_{\{f,g\}^*}$

Gauge invariant dynamics: $\frac{dO_f}{d\tau} = \{O_f, H_{\text{phys}}\} = O_{\{f,H\}^*}$

Observables map on full ADM Phase space

Extended observable map

$$G_{b,\vec{b}} = C[b] + \vec{C}[b] + \Pi(\dot{b} + b^a N_{,a} - N^a b_{,a}) + \Pi_a(\dot{b}^a + q^{ab}(bN_{,b} - Nb_{,b}) - N^a b^b_{,a} + b^a N^b_{,a})$$

One can show [Pons, Shepley, Sundermeyer]

$$G_{b,\vec{b}} \simeq C[b] + \vec{C}[b] + \Pi[\dot{b}] + \Pi_a[\dot{b}^a]$$

Weak abelianization has to be performed wrt primary and secondary

$$G^\mu = \tau^\mu - T^\mu, \;\; \tilde{G}^\mu := (G^\mu, \dot{G}^\mu), \;\; \tilde{C}_\mu := (C_\mu, \Pi_\mu)$$

Extended observable formula

$$O^G_{f,T} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x_1 \cdots d^3x_n \tilde{G}^\mu(x_1) \cdots \tilde{G}^\mu(x_n) \{f, \tilde{C}_\mu(x_1)\}, \cdots\}, \tilde{C}_\mu(x_n)\}$$

For (q_{ab}, p^{ab}) reduces to usual observable map $O^C_{f,T}$

IV. Use following Strategy:

<u>Step 1:</u> Express Bardeen potentials, Mukhanov-Sasaki variable and their dynamics in full ADM phase space + scalar field

Consider linearized phase space:

$$\delta N = \overline{N}\phi, \delta N^a = B^{,a} + S^a$$

$$\delta\Pi = \frac{1}{\overline{N}}p_\phi, \delta\Pi_a = \delta\Pi_{,a} + \delta\hat{\Pi}_a = (p_B)_{,a} + (p_S)_a$$

$$\delta q_{ab} = 2a^2(\psi\delta_{ab} + (E_{,<ab>} + F_{(a,b)} + \frac{1}{2}h^{TT}_{ab}))$$

$$\delta p^{ab} = 2\tilde{P}_{a^2}(p_\psi\delta^{ab} + (p_E)^{,<ab>} + (p_F)^{(a,b)} + \frac{1}{2}(p_h)^{TT}_{ab})$$

$$\delta\varphi, \delta\pi_\varphi$$

Derive Hamilton's equations for linearized perturbations and use them to construct analogues of Bardeen potentials and Mukhanov-Sasaki-variable

Gauge transformations on extended ADM

Behavior under linearized gauge transformations: (geometry)

$$\delta_{G'_{b,\vec{b}}}\phi = \frac{1}{\bar{N}}b_{,t} \qquad \delta_{G'_{b,\vec{b}}}B = -\frac{\bar{N}}{A}b + \hat{b}_{,t}$$

$$\delta_{G'_{b,\vec{b}}}\psi = \frac{\tilde{\mathcal{H}}}{\bar{N}}b + \frac{1}{3}\Delta\hat{b} \qquad \delta_{G'_{b,\vec{b}}}E = \hat{b}$$

Conjugate momenta:

$$\delta_{G'_{b,\vec{b}}}p_\psi = -\left(\frac{1}{4}\frac{\tilde{\mathcal{H}}}{\bar{N}} + \frac{\kappa}{8}\frac{\bar{N}}{\tilde{\mathcal{H}}}p\right)b + \frac{1}{6}\Delta\left(\frac{\bar{N}}{A\tilde{\mathcal{H}}}b + \hat{b}\right) \qquad \delta_{G'_{b,\vec{b}}}p_E = -\frac{\bar{N}}{4A\tilde{\mathcal{H}}}b - \hat{b}$$

$$\delta_{G'_{b,\vec{b}}}p_\phi = 0 \qquad \delta_{G'_{b,\vec{b}}}p_B = 0$$

expected transformation behavior similar to Lagrange framework

Step 1: Longitudinal gauge

Natural observables: Bardeen Potentials Φ, Ψ

$$\mathcal{LM}^*\Psi := \psi - \frac{1}{3}\Delta E + \frac{\mathcal{H}A}{\bar{N}^2}(B - \dot{E}).$$
$$B - \dot{E} \xrightarrow{\mathcal{LM}} 4\tilde{\mathcal{H}}(E + p_E).$$

used EOM for E to get result

Bardeen potential Ψ on phase and conjugate momentum:

$$\Psi \ = \psi - \frac{1}{3}\Delta E + \tilde{P}^2(E + p_E) \qquad \Upsilon := p_\psi - \frac{1}{6}\Delta E + \frac{2}{3}\Delta(E + p_E) - \left(\frac{1}{4}\tilde{P}^2 + \frac{\kappa}{2}Ap\right)(E + p_E)$$

Likewise we get:

$$\mathcal{LM}^*\Phi := \phi + \frac{1}{\bar{N}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A}{\bar{N}}(B - \dot{E})\right)$$

EOM yields directly to

$$\Phi = -\Psi$$

Phase space form of Bardeen potentials gives us hint what clocks to choose

Step 2: Choice of geometrical clocks

Realize: Natural choice of geometrical clock for given gauge in CPT

Longitudinal/Newtonian gauge: (E=0, B=0 + stability)

Observables via map $O^G_{f,T}$

Linear Perturbations: $$\delta O^G_{f,T} = O^G_{f,T} - O^G_{\overline{f,T}}$$

Explicitly on linearized phase space:

$$\begin{aligned}
\delta\mathcal{O}_{f,T}[\tau] &= \delta f + \int \mathrm{d}^3y \left[\delta\dot{G}^\mu(y)\overline{\{f,\tilde{\Pi}_\mu(y)\}} + \delta G^\mu(y)\overline{\{f,\tilde{\tilde{C}}_\mu(y)\}}\right] \\
&\approx \delta f + \int \mathrm{d}^3y \int \mathrm{d}^3z \bar{\mathcal{B}}^\nu_\mu(z,y)\left[\delta\dot{G}^\mu(y)\overline{\{f,\Pi_\nu(z)\}} - \delta G^\mu(y)\left(\overline{\{f,C_\nu(z)\}}\right.\right. \\
&\quad \left.\left. + \int \mathrm{d}^3w \int \mathrm{d}^3v\ \bar{\mathcal{B}}^\rho_\sigma(w,v)\overline{\{\dot{G}^\sigma(v),C_\nu(z)\}}\ \overline{\{f,\Pi_\rho(w)\}}\right)\right].
\end{aligned}$$

Question: What kind of gauge fixing and hence clocks needs to be chosen such that observables are the Bardeen potentials?

Choice of geometrical clocks I

Realize: Natural choice of geometrical clock for given gauge in CPT

Longitudinal/Newtonian gauge: (E=0, B=0 + stability)

$$\delta T^0 = 2\tilde{P}a(E + P_E) \qquad \delta T^a = \delta^{ab}(E_{,b} + F_{,b}) \longrightarrow \delta\hat{T} = E$$

Stability of the clocks:

$$\delta T^0 \approx 0, \qquad \delta\hat{T} \approx 0, \qquad \delta\dot{T}^0 \approx 0, \qquad \delta\dot{\hat{T}} \approx 0,$$ is equivalent to

$$B \approx 0, \qquad E \approx 0, \qquad p_E \approx 0, \qquad \phi \approx -\psi\ .$$

Results for observables:

$$O^G_{\psi,T} = \Psi,\ O^G_{\psi,T} = \Upsilon,\ O^G_{\phi,T} = -\Psi,\ O^G_{p_\phi,T} = \frac{1}{N}\delta\Pi$$

correct form of Bardeen potentials, others vanish

$$O^G_{\delta\varphi,T} = \delta\varphi^{(gi)}, \qquad O^G_{\delta\pi_\varphi,T} = \delta\pi_\varphi^{(gi)}$$

correct form of gauge invariant matter dof

Step 3: Dynamics of observables

We are interested in EOMs of physical degrees of freedom.

Two possibilities to compute EOM for observables: (more in outlook)

1. Derive Poisson algebra for observables: $\{O_f, O_g\} \simeq O_{\{f,g\}^*}$

Dirac bracket is quite complicated due to non-commuting clocks

$$\{\delta T^0(x), \delta T^0(y)\} = 0 \quad \{\delta T^0(x), \delta T^a(y)\} = -\frac{3}{4}\frac{\kappa}{\sqrt{A}} \int \mathrm{d}^3 z\, G(x,z) \frac{\partial G(z,y)}{\partial y_a}$$

2. Use that observables are combinations of gauge variant quantities

We derived already all EOMs for all gauge variant quantities, use these

Expected result for Bardeen potential:

$$\ddot{\Psi} + 3\mathcal{H}\dot{\Psi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\Psi = -4\pi G A \delta T^{(gi)}.$$

remaining 2 phys. dof in tensor sector.

Step1: Spatially flat gauge

Repeat the same for spatially flat gauge.

Mukhanov-Sasaki variable

$$\mathcal{L}\mathcal{M}^*(v) = \delta\varphi - \frac{\dot{\bar{\varphi}}}{\mathcal{H}}\left(\psi - \frac{1}{3}\Delta E\right)$$

Mukhanov-Sasaki variable on phase space:

$$v = \delta\varphi - \frac{\lambda_\varphi}{A^{3/2}\tilde{\mathcal{H}}}\bar{N}\bar{\pi}_\varphi\left(\psi - \frac{1}{3}\Delta E\right)$$

$$\pi_v = \delta\pi_\varphi - \bar{\pi}_\varphi\Delta E + \frac{1}{2}\frac{a^3}{\lambda_\varphi}\frac{\mathrm{d}V}{\mathrm{d}\varphi}(\bar{\varphi})\frac{\bar{N}}{\tilde{\mathcal{H}}}\left(\psi - \frac{1}{3}\Delta E\right)$$

Again we use these to choose the natural clocks for this gauge

Step 2: Choice of geometrical clocks

Realize: Natural choice of geometrical clock for given gauge in CPT

Spatially flat gauge: ($\psi = 0, E = 0$ + stability)

$$\delta T^0 = -2\tilde{P}a(\psi - \frac{1}{3}\Delta E) \qquad \delta T^a = \delta^{ab}(E_{,b} + F_{,b}) \longrightarrow \delta\hat{T} = E$$

Stability of clocks:

$$\delta T^0 \approx 0, \ \delta\hat{T} \approx 0, \ \delta\dot{T}^0 \approx 0, \ \delta\dot{\hat{T}} \approx 0$$

is equivalent to

$$\psi \approx 0, \ E \approx 0, \ \phi \approx -2p_\psi - \frac{4}{3}\Delta p_E, \ B \approx 4\tilde{H}p_E,$$

Results for observables: $O^G_{\delta\varphi,T} = v, \ O^G_{\delta\pi_\varphi,T} = \pi_v$

$$O^G_{\phi,T} = -2\Upsilon - (\frac{1}{2} + \frac{\kappa}{\tilde{P}^2}Ap)\Psi, \ O^G_{p_\phi,T} = \frac{1}{\bar{N}}\delta\Pi$$

$$O^G_{B,T} = \frac{\bar{N}^2}{\tilde{H}a^2}\Psi, \ O^G_{p_B,T} = \delta\hat{\Pi}$$

$$O^G_{p_\psi,T} = \Upsilon + \alpha\Psi, \ O^G_{p_E,T} = \frac{1}{\tilde{P}^2}\Psi$$

Combinations of Bardeen potential and its momentum, others vanish

Step 3: Dynamics of observables

Again two possibilities to compute EOM for observables:

1. Derive Poisson algebra for observables: $\{O_f, O_g\} \simeq O_{\{f,g\}^*}$

$$\{\delta T^\mu(x), \delta T^\nu(y)\} = 0, \ \ \mu, \nu = 0, \cdots, 3$$

Reason: Projectors to define variables

2. Using again the EOMs of the gauge variant quantities we can derive the Mukhanov-Sasaki equation from the Hamiltonian EOM of $v, \ \pi_v$

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0$$

remaining 2 phys. dof in tensor sector.

here already formulated in terms of Fourier modes.

Further gauges:

In our papers we also considered 3 more common gauges:

1. **Uniform field gauge:** $\delta\varphi = E = 0$,

2. **Synchronous gauge:** $\phi = B = 0$,

3. **Comoving gauge:** $\delta\varphi = B = 0$.

We could also determine the natural clocks for these gauges and construct associated observables.

Outlook: Dynamics and physical Hamiltonian

Reconsider dynamics: Once observables constructed natural to compute EOM at the gauge invariant level

Might be complicated due to Dirac bracket involved, see also paper by Dittrich and Tambornino

However, here: All gauge fixing conditions are of the form

$$G_\mu = \bar{g}(t)F(\phi, p_\phi, B, p_B, E, p_E, \psi, p_\psi, \delta\varphi, \delta\pi_\varphi)$$

Work in progress: Candidate for a physical Hamiltonian s.t.

$$\frac{dO^G_{f,T}}{dt} = \{O^G_{f,T}, H_{\mathrm{phys}}\}$$

WIP with Singh & Winnekens

Could be framework to efficiently compute EOM

III. Summary and Conclusions

Idea: Use observable map in relational formalism and geometrical clocks to derive gauge invariant quantities relevant for CPT

First derived formulation of linearized perturbation theory in full ADM phase space

Bardeen potentials and Mukhanov-Sasaki variable in phase space formulation with associated natural clocks

Consistent with Lagrangian dynamics at linear order

Choice of natural geometrical clocks associated to common gauges in CPT, Analyzed dynamics

III. Open Questions and Outlook

Geometrical interpretation of clocks:

In the light of non-linear clocks one would like to choose consistent clocks in each order of perturbation theory

Not possible for all clocks that are used in CPT

Reduced quantization with geometric clocks?

Computed perturbed linearized observable algebra

Matter versus geometrical clocks?

Non-linear clocks:

Learn more about their particular geometric properties

Would also allow to perturb gauge invariant Einstein Eqn

now possible to discuss in full ADM phase space