Why study Bianchi models?

- Anisotropic spacetime introduces more degrees of freedom compared to isotropic spacetime.
- Much richer physics due to non-vanishing Weyl scalar.
- Classically the anisotropic shear scalar in Bianchi-I model varies as $\sigma^2 \propto a^{-6}$. Singularity can also take place due to diverging anisotropic shear as $a \rightarrow 0$.
- According to Belinskii-Khalatnikov-Lifshitz (BKL) behavior, during a generic approach to a spacelike singularity, each point transits from one Bianchi-I type universe to another Bianchi-I type (Kasner transition), giving rise to Mixmaster behavior.
Loop quantum cosmology of Bianchi-I spacetime

\[ ds^2 = -dt^2 + a_1^2 \, dx^2 + a_2^2 \, dy^2 + a_3^2 \, dz^2 \]

In the classical theory, approach to singularity can be classified as

(Doroshkevich, Ellis, Jacobs, MacCallum, Thorne ...)

- **Point or Isotropic singularity**: \( a_1, a_2, a_3 \to 0 \).
- **Barrel**: \( a_1 \to \text{const}, a_2, a_3 \to 0 \)
- **Pancake**: \( a_1 \to 0, a_2, a_3 \to \text{const} \)
- **Cigar**: \( a_1 \to \infty, a_2, a_3 \to 0 \)

Quantization performed by (Ashtekar, Wilson-Ewing(09)). Earlier approaches to quantization developed by Bojowald, Chiou, Date, Martin-Benito, Mena Marugan, Pawlowski, Szulc, Vandersloot

- Classical singularity resolved
- Resolution of all physical singularities studied in the effective dynamics (Singh (11))
- Physics of effective dynamics studied: big bang is replaced by bounce (Artymowski, Cailleteau, Chiou, Lalak, Maartens, Singh, Vandersloot)
Questions:

**Kasner Transitions:**
- What is the relation between the geometrical nature of spacetime in pre and post bounce regime?
- Are there transitions from one type to other?
- Are some transitions favored over others? If yes, depending on what?

**Inflation:**
- Does anisotropy prevent inflation?
- How does LQC modify the dynamics and the amount of inflation?
- How is the amount of inflation affected as compared to the isotropic spacetime?
Main results:

**Kasner Transitions:**
- Kasner transitions are seen in Bianchi-I spacetime for the first time, a feature not present in classical theory.
- Transitions are not random, there turns out to be a “selection rule”.
- Depending on the anisotropy and matter content some transitions are favored over others.

**Inflation:**
- Inflation takes place irrespective of the initial anisotropic shear.
- Non-trivial dependence of amount of inflation on the initial shear scalar.
- Modification in the initial value of inflaton field as compared to isotropic spacetime to generate a given amount of inflation.
Plan of the talk:

- Review of Kasner solution
- Review of effective dynamics of Bianchi-I
- Kasner transitions with perfect fluid with $P = \rho$
- Inflation in Bianchi-I spacetime with quadratic potential $V(\phi) = m^2 \phi^2 / 2$
Kasner solution: classical theory

**Vacuum:**

\[ a_i \propto t^{k_i} \text{ such that } k_1 + k_2 + k_3 = 1; \quad k_1^2 + k_2^2 + k_3^2 = 1 \] (1)

**Stiff matter,** \( w = P/\rho = 1: \)

\[ a_i \propto t^{k_i} \text{ such that } k_1 + k_2 + k_3 = 1; \quad k_1^2 + k_2^2 + k_3^2 = 1 - k^2 \] (2)

where \( k_i \) are Kasner exponents and \( k \) is a constant.

<table>
<thead>
<tr>
<th>Point</th>
<th>( k_1, k_2, k_3 &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barrel</td>
<td>( k_1 = 0, k_2, k_3 &gt; 0 )</td>
</tr>
<tr>
<td>Pancake</td>
<td>( k_1, k_2 = 0, k_3 &gt; 0 )</td>
</tr>
<tr>
<td>Cigar</td>
<td>( k_1 &lt; 0, k_2, k_3 &gt; 0 )</td>
</tr>
</tbody>
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\( 0 \leq w < 1: \)

- Close to singularity, behaves like vacuum for all \( 0 \leq w < 1 \)
- In the future asymptotic limit
  - \( a_i \propto t^{2/3} \) for Dust (\( w = 0 \))
  - \( a_i \propto t^{1/2} \) for Radiation (\( w = 1/3 \))
Effective Hamiltonian: (Chiou, Vandersloot; Ashtekar, Wilson-Ewing)

$$H_{\text{eff}} = -\frac{1}{8\pi\gamma^2 V} \left( \frac{\sin(\bar{\mu}_1 c_1)}{\bar{\mu}_1} \frac{\sin(\bar{\mu}_2 c_2)}{\bar{\mu}_2} p_1 p_2 + \text{cyclic terms} \right) + H_{\text{mat}},$$  \hspace{1cm} (1)

where

$$\bar{\mu}_1 = \lambda \sqrt{\frac{p_2 p_3}{p_1}}, \quad \bar{\mu}_2 = \lambda \sqrt{\frac{p_1 p_3}{p_2}}, \quad \bar{\mu}_3 = \lambda \sqrt{\frac{p_2 p_1}{p_3}} \quad \text{and} \quad \lambda^2 = 4\sqrt{3}\pi\gamma\ell_{\text{Pl}}^2.$$

Triad and connection:  \hspace{1cm} p_1 = a_2 a_3 \quad \& \quad \text{classically, } c_1 = \gamma \dot{a}_1 \hspace{1cm} (2)

Hamilton’s equation of motion:  \hspace{1cm} \frac{\dot{p}_1}{p_1} = \frac{1}{\gamma\lambda} \left[ \sin(\bar{\mu}_2 c_2) + \sin(\bar{\mu}_3 c_3) \right] \cos(\bar{\mu}_1 c_1) \hspace{1cm} (3)

Energy Density: vanishing of the Hamiltonian constraint gives:

$$\rho = \frac{1}{8\pi G\gamma^2 \lambda^2} \left[ \sin(\bar{\mu}_1 c_1) \sin(\bar{\mu}_2 c_2) + \text{cyclic terms} \right] \leq \rho_{\text{crit}} = 0.41 \rho_{\text{Pl}} \hspace{1cm} (4)$$

Expansion scalar:

$$\theta = \frac{1}{2} \left( \frac{\dot{p}_1}{p_1} + \frac{\dot{p}_2}{p_2} + \frac{\dot{p}_3}{p_3} \right) \leq \theta_{\text{max}} = \frac{3}{2\gamma\lambda} \hspace{1cm} (5)$$

Shear scalar:

$$\sigma^2_{\text{max}} = \frac{10.125}{\gamma^2 \lambda^2} = \frac{11.57}{\ell_{\text{Pl}}^2} \hspace{1cm} (6)$$
Kasner transition: Stiff matter, $P = \rho$

• Classical trajectory undergoes singularity

• The mean scale factor $a = (a_1 a_2 a_3)^{1/3}$ in LQC bounces.

• The directional scale factors undergo Kasner transition across the bounce

• Transitions depend on anisotropy present in the spacetime
Stiff matter, \( w = 1 \)
### Kasner transition for $w = 1$

| $0 < |\delta| < \frac{1}{2}$ | $|\delta| = \frac{1}{2}$ | $\frac{1}{2} < |\delta| < \frac{1}{\sqrt{3}}$ | $|\delta| = \frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}} < |\delta| < 1$ |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| P ↔ P                      | P ↔ P                      | P ↔ P                      | P ↔ P                      | P ↔ P                      |
| B ↔ P                      | B ↔ P                      | B ↔ P                      | B ↔ P                      | B ↔ P                      |
| C ↔ P                      | C ↔ P                      | C ↔ P                      | C ↔ P                      | C ↔ P                      |
| B ↔ B                      | B ↔ B                      | B ↔ B                      | B ↔ B                      | B ↔ B                      |
| B ↔ C                      | B ↔ C                      | B ↔ C                      | B ↔ C                      | B ↔ C                      |
| C ↔ C                      | C ↔ C                      | C ↔ C                      | C ↔ C                      | C ↔ C                      |

where $|\delta| = \sqrt{\frac{3\sigma^2}{2\theta^2}}$.

- Depending on the value of $\delta$ some transitions are favored over others.
- In the low anisotropy regime only Point-Point transition takes place.
- Cigar-Cigar transition only happens in the large anisotropy regime.
Inflation: a phase of accelerated expansion in the early universe
Widely studied and explored in the classical theory. (Albrecht, Barrow, Guth, Linde, Rothman, Steinhardt, Steigman, Turner...)

Does anisotropy prevent inflation? Barrow and Turner (81); Steigman and Turner (83)

Quantum theory of gravity is required (Rothman & Madsen (85), Rothman & Ellis(86)); Anisotropy helps attain more inflation (Maartens, Sahni and Saini (01))

Inflation in LQC in isotropic spacetime... (Ashtekar, Pawlowski, Singh to appear)

Isotropic inflation in effective theory studied in detail by Ashtekar & Sloan (09) establishing its inevitability (99.99%); other aspects also studied by Corichi & Karami (10)
Inflation in Bianchi-I spacetime

Generalized Friedmann equation: \[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{1}{6} \sigma^2 \] (1)

Raychaudhuri equation: \[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + 3P \right) - \frac{1}{3} \sigma^2 \] (2)

where \( a \) is the mean scale factor.

Conservation equation: \[ \ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \] (3)

\[ \rho = \frac{\dot{\phi}^2}{2} + V, \quad P = \frac{\dot{\phi}^2}{2} - V, \quad V = \frac{m^2 \phi^2}{2} \] (4)

\[ \frac{\ddot{a}}{a} = -\frac{8\pi G}{3} \left( \dot{\phi}^2 - V(\phi) \right) - \frac{1}{3} \sigma^2 \] (5)

Slow-roll: \( V(\phi) \gg \dot{\phi}^2 \)

\[ \ddot{a} > 0 \]

Anisotropy \( \Rightarrow \) Enhanced Hubble friction \( \Rightarrow \) Fast KE decay \( \Rightarrow \) Arrival of slow roll
Amount of inflation: Classical theory

Number of efoldings: \[ N = \ln \left( \frac{a_{\text{off}}}{a_{\text{on}}} \right) \propto \phi_{\text{on}}^2 \]

\[ |\dot{\phi}(0)| = (2 \cdot 10^{-5}, 2 \cdot 10^{-3}, 2 \cdot 10^{-2}, 0.1) \, m_{\text{Pl}}^2, \quad \phi(0) = 3.14 \, m_{\text{Pl}} \quad \text{and} \quad \epsilon^2 = \sigma^2 / 4\pi G \rho \]

\[ \begin{align*}
    \dot{\phi}(0) < 0 \\
    \dot{\phi}(0) > 0
\end{align*} \]

Figure: slow-roll starts at a higher value of \( \phi \) as \( \sigma^2(0) \) increases

Figure: slow-roll starts at a lower value of \( \phi \) as \( \sigma^2(0) \) increases

Depending on the initial condition on \( \dot{\phi} \) non-zero anisotropy, monotonically, either enhances or reduces \( N \) in the classical theory.
Bianchi-I inflation in LQC: amount of inflation

Hubble rate:

\[ \dot{\phi}_{\text{bounce}} < 0 \]

\[ \dot{\phi}_{\text{bounce}} > 0 \]

Number of e-foldings:

\[ \dot{\phi}_{\text{bounce}} < 0 \]

\[ \dot{\phi}_{\text{bounce}} > 0 \]
Maximum number of e-foldings \((N)\) for \(\phi(0) = 3.14 \, m_{Pl}\) various \(\dot{\phi}(0)\)

<table>
<thead>
<tr>
<th>(\dot{\phi}(0))</th>
<th>(-0.2)</th>
<th>(-0.02)</th>
<th>(-0.002)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>42.7503</td>
<td>60.3810</td>
<td>62.3939</td>
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</thead>
<tbody>
<tr>
<td>(N)</td>
<td>71.8433</td>
<td>65.0221</td>
<td>62.8843</td>
</tr>
</tbody>
</table>

Isotropic LQC: \(\rho_{\text{bounce}} = \rho_{\text{crit}} \Rightarrow \dot{\phi}(0) = -0.905 \, m_{Pl}^2\)

\[
N_{\text{iso}} = 3.12 \quad \text{for} \quad \phi(0) = 3.14 \, m_{Pl} \quad (1)
\]

In comparison to the isotropic spacetime, Bianchi-I spacetime generates more e-foldings for the same initial value of the inflaton at the bounce.

In the isotropic spacetime, for \(N \approx 60, \phi(0) \approx 5.50 \, m_{Pl}\) for an inflaton which is rolling down.

Bianchi-I spacetime widens the window of value of \(\phi(0)\) to generate a given number of e-foldings.
All trajectories meet the slow roll curve in their future evolution.

Slow-roll is an attractor for all these solutions in Bianchi-I spacetime.
There are Kasner transitions across the bounce in Bianchi-I spacetime.

These transitions follow a pattern and depending on anisotropy and matter content some of them are favored—"selection rule".

Inflation takes place irrespective of the initial anisotropic shear.

Anisotropy may enhance or reduce the amount of inflation depending on the initial conditions on the inflaton velocity.

Bianchi-I spacetime widens the window of the value of inflaton at the bounce, for a given number of e-foldings.