

# The $1/N$ Expansion in Colored Tensor Models

Răzvan Gurău

ILQGS, 2011

## Introduction

### Colored Tensor Models

- Colored Graphs

- Jackets and the  $1/N$  expansion

- Topology

- Leading order graphs are spheres

### Conclusion

# Space-time and Scales

Space-time is one of the most fundamental notions in physics. In many theories (e.g. quantum mechanics) it appears as a fixed **background**. The distances and lapses of time are measured **with respect** to this fixed background.

Scales encode causality: **effective** physics at large distance is determined by **fundamental** physics at short distance.

General relativity promotes the metric to a **dynamical** variable, and the length scales become dynamical!

- ▶ How to define **background independent scales** separating fundamental and effective physics?
- ▶ How to obtain the usual **space time as an effective phenomenon**?

# Matrix Models

A success story: Matrix Models in **two** dimensions

- ▶ An **ab initio** combinatorial statistical theory.
- ▶ Have **built in** scales  $N$ .
- ▶ Generate **ribbon graphs**  $\leftrightarrow$  discretized surfaces.
- ▶ They undergo a **phase transition** (“condensation”) to a continuum theory of **large** surfaces.

Physics: **quantum gravity in  $D = 2$** , **critical phenomena**, **conformal field theory**, the theory of **strong interactions**, **string theory**, etc.

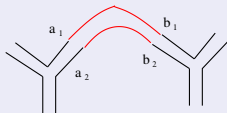
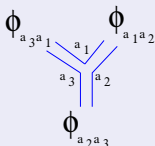
Mathematics: **knot theory**, **number theory** and the **Riemann hypothesis**, **invariants of algebraic curves**, **enumeration** problems, etc.

**All these applications rely crucially on the “ $1/N$ ” expansion!**

# Ribbon Graphs as Feynman Graphs

Consider the partition function.

$$Z(Q) = \int [d\phi] e^{-N \left( \frac{1}{2} \sum \phi_{a_1 a_2} \delta_{a_1 b_1} \delta_{a_2 b_2} \phi_{b_1 b_2}^* + \lambda \sum \phi_{a_1 a_2} \phi_{a_2 a_3} \phi_{a_3 a_1} \right)}$$



**Ribbon vertex** because the field  $\phi$  has two arguments.

The **lines** conserve the two arguments (thus having two **strands**).

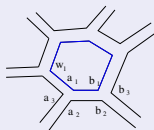
Strands close into **faces**.

$Z(Q)$  is a sum over **ribbon Feynman graphs**.

# Amplitude of Ribbon Graphs

The Amplitude of a graph with  $\mathcal{N}$  vertices is

$$A = \lambda^{\mathcal{N}} N^{-\mathcal{L} + \mathcal{N}} \sum \prod_{\text{lines}} \delta_{a_1 b_1} \delta_{a_2 b_2}$$

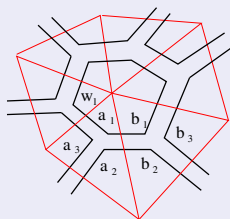


$$\sum \delta_{a_1 b_1} \delta_{b_1 c_1} \dots \delta_{w_1 a_1} = \sum \delta_{a_1 a_1} = N$$

$$A = \lambda^{\mathcal{N}} N^{\mathcal{N} - \mathcal{L} + \mathcal{F}} = \lambda^{\mathcal{N}} N^{2 - 2g(\mathcal{G})}$$

with  $g_{\mathcal{G}}$  is the **genus** of the graph.  $1/N$  expansion in the genus. **Planar graphs** ( $g_{\mathcal{G}} = 0$ ) dominate in the large  $N$  limit.

# Ribbon Graphs are Dual to Discrete Surfaces



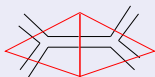
Place a **point** in the middle of each face. Draw a **line** crossing each ribbon line. The ribbon vertices correspond to **triangles**.

A ribbon graph encodes **unambiguously** a gluing of triangles.

Matrix models sum over all graphs (i.e. surfaces) with **canonical** weights (Feynman rules). The dominant **planar graphs** represent **spheres**.

# From Matrix to **COLORED** Tensor Models

surfaces  $\leftrightarrow$  ribbon graphs



$$\text{Matrix } M_{ab},$$

$$S = N \left( M_{ab} \bar{M}_{ab} + \lambda M_{ab} M_{bc} M_{ca} \right)$$

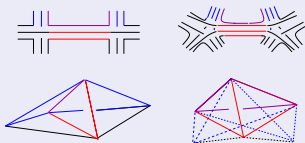
$g(\mathcal{G}) \geq 0$  genus

$1/N$  expansion in the genus

$$A(\mathcal{G}) = N^{2-2g(\mathcal{G})}$$

leading order:  $g(\mathcal{G}) = 0$ , **spheres**.

$D$  dimensional spaces  $\leftrightarrow$  colored  
stranded graphs



$$\text{Tensors } T^i_{a_1 \dots a_D} \text{ with color } i$$

$$S = N^{D/2} \left( T^i \bar{T}^i + \lambda T^0 T^1 \dots T^D \right)$$

$\omega(\mathcal{G}) \geq 0$  degree

$1/N$  expansion in the degree

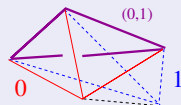
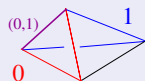
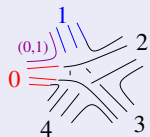
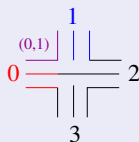
$$A(\mathcal{G}) = N^{D - \frac{2}{(D-1)!} \omega(\mathcal{G})}$$

leading order:  $\omega(\mathcal{G}) = 0$ , **spheres**.

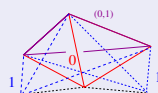
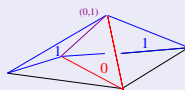
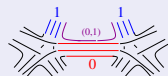
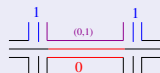


# Colored Stranded Graphs

Clockwise and anticlockwise turning colored **vertices** (positive and negative oriented  $D$  simplices).



**Lines** have a well defined color and  $D$  parallel strands ( $D - 1$  simplices).



**Strands** are identified by a couple of colors ( $D - 2$  simplices).

## Action

Let  $T_{a_1 \dots a_D}^i, \bar{T}_{a_1 \dots a_D}^i$  tensor fields with color  $i = 0 \dots D$ .

$$S = N^{D/2} \left( \sum_i \bar{T}_{a_1 \dots a_D}^i T_{a_1 \dots a_D}^i + \lambda \prod_i T_{a_{i-1} \dots a_{i0} a_{iD} \dots a_{i+1}}^i + \bar{\lambda} \prod_i \bar{T}_{a_{i-1} \dots a_{i0} a_{iD} \dots a_{i+1}}^i \right)$$

## Topology of the Colored Graphs

Amplitude of the graphs:

- ▶ the  $\mathcal{N} = 2p$  vertices of a graph bring each  $N^{D/2}$
- ▶ the  $\mathcal{L}$  lines of a graphs bring each  $N^{-D/2}$
- ▶ the  $\mathcal{F}$  faces of a graph bring each  $N$

$$A^G = (\lambda \bar{\lambda})^p N^{-\mathcal{L} \frac{D}{2} + \mathcal{N} \frac{D}{2} + \mathcal{F}} = (\lambda \bar{\lambda})^p N^{-p \frac{D(D-1)}{2} + \mathcal{F}}$$

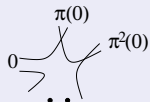
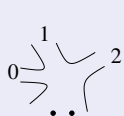
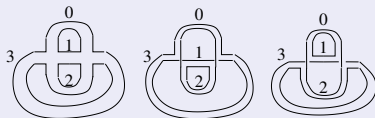
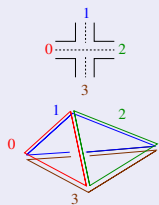
But  $\mathcal{N}(D+1) = 2\mathcal{L} \Rightarrow \mathcal{L} = (D+1)p$

Compute  $\mathcal{F}$  !

# Jackets 1

Define **simpler** graphs. **Idea**: forget the interior strands! Leads to a **ribbon graph**.

02 and 13: opposing edges of the tetrahedron. **But** 01, 23 and 12, 03 are perfectly equivalent. **Three jacket (ribbon) graphs.**



$\frac{1}{2}D!$  **jackets**. Contain **all** the vertices and **all** the lines of  $\mathcal{G}$ . A face belongs to  $(D-1)!$  jackets.

$0, 1, 2, \dots$

$0, \pi(0), \pi^2(0), \dots$

The **degree** of  $\mathcal{G}$  is  $\omega(\mathcal{G}) = \sum_{\mathcal{J}} g_{\mathcal{J}}$ .

## Jackets 2: Jackets and Amplitude

### Theorem

$\mathcal{F}$  and  $\omega(\mathcal{G})$  are related by

$$\mathcal{F} = \frac{1}{2}D(D-1)p + D - \frac{2}{(D-1)!}\omega(\mathcal{G})$$

**Proof:**  $\mathcal{N} = 2p$ ,  $\mathcal{L} = (D+1)p$

For each jacket  $\mathcal{J}$ ,  $2p - (D+1)p + \mathcal{F}_{\mathcal{J}} = 2 - 2g_{\mathcal{J}}$ .

Sum over the jackets:  $(D-1)!\mathcal{F} = \sum_{\mathcal{J}} \mathcal{F}_{\mathcal{J}} = \frac{1}{2}D!(D-1)p + D! - 2\sum_{\mathcal{J}} g_{\mathcal{J}}$  □

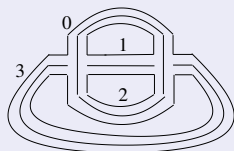
The amplitude of a graph is given by its degree

$$A^{\mathcal{G}} = (\lambda\bar{\lambda})^p N^{-p\frac{D(D-1)}{2} + \mathcal{F}} = (\lambda\bar{\lambda})^p N^{D - \frac{2}{(D-1)!}\omega(\mathcal{G})}$$

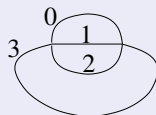
# Topology 1: Colored vs. Stranded Graphs

**THEOREM:** [M. Ferri and C. Gagliardi, '82] Any  $D$ -dimensional piecewise linear orientable manifold admits a colored triangulation.

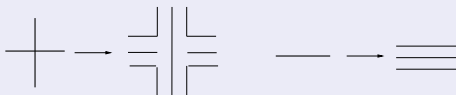
We have **clockwise** and **anticlockwise** turning vertices. Lines connect opposing vertices and have a **color** index. All the information is encoded in the **colors**



represented as



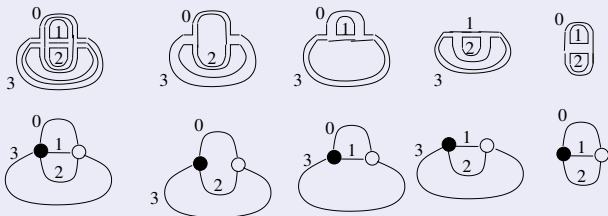
Conversely: expand the vertices into stranded vertices and the lines into stranded lines with parallel strands



## Topology 2: Bubbles

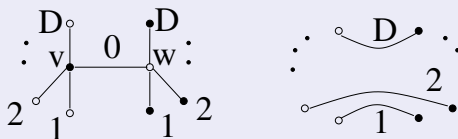
The **vertices** of  $\mathcal{G}$  are subgraphs with **0** colors. The **lines** are subgraphs with exactly **1** color. The **faces** are subgraphs with exactly **2** colors.

The  **$n$ -bubbles** are the maximally connected subgraphs with  **$n$**  fixed colors (denoted  $\mathcal{B}_{(\sigma)}^{i_1 \dots i_n}$ , with  $i_1 < \dots < i_n$  the colors).



A colored graph  $\mathcal{G}$  is dual to an orientable, normal,  $D$  dimensional, simplicial pseudo manifold. Its  $n$ -bubbles are dual to the links of the  $D - n$  simplices of the pseudo manifold.

## Topology 3: Homeomorphisms and 1-Dipoles



A **1-dipole**: a line (say of color 0) connecting two vertices  $v \in \mathcal{B}_{(\alpha)}^{1\dots D}$  and  $w \in \mathcal{B}_{(\beta)}^{1\dots D}$  with  $\mathcal{B}_{(\alpha)}^{1\dots D} \neq \mathcal{B}_{(\beta)}^{1\dots D}$ .

A 1-Dipole can be **contracted**, that is the lines together with the vertices  $v$  and  $w$  can be deleted from  $\mathcal{G}$  and the remaining lines reconnected **respecting the coloring**. Call the graph after contraction  $\mathcal{G}/d$ .

**THEOREM:** [M. Ferri and C. Gagliardi, '82] If either  $\mathcal{B}_{(\alpha)}^{1\dots D}$  or  $\mathcal{B}_{(\beta)}^{1\dots D}$  is dual to a sphere, then the two pseudo manifolds dual to  $\mathcal{G}$  and  $\mathcal{G}/d$  are homeomorphic.

It is in principle **very difficult** to check if a bubble is a sphere or not.

# Jackets, Bubbles, 1-Dipoles

The  $D$ -bubbles  $\widehat{\mathcal{B}}_{(\rho)}^i$  of  $\mathcal{G}$  are graphs with  $D$  colors, thus they admit jackets and have a **degree**. The degrees of  $\mathcal{G}$  and of its bubbles are not independent.

## Theorem

$$\omega(\mathcal{G}) = \frac{(D-1)!}{2} \left( p + D - \mathcal{B}^{[D]} \right) + \sum_{i,\rho} \omega(\widehat{\mathcal{B}}_{(\rho)}^i)$$

## Theorem

The degree of the graph is **invariant** under 1-Dipole moves,  $\omega(\mathcal{G}) = \omega(\mathcal{G}/d)$



## Degree 0 Graphs are Spheres

$$\omega(\mathcal{G}) = \frac{(D-1)!}{2} \left( p + D - \mathcal{B}^{[D]} \right) + \sum_{i,\rho} \omega(\widehat{\mathcal{B}}_{(\rho)}^i)$$

In a graph  $\mathcal{G}$  with  $2p$  vertices and  $\mathcal{B}^{[D]}$   $D$ -bubbles I contract a full set of 1-Dipoles and bring it to  $\mathcal{G}_f$  with  $2p_f$  vertices and exactly one  $D$ -bubble for each colors  $\widehat{i}$ .

Every contraction:  $p \rightarrow p - 1$ ,  $\mathcal{B}^{[D]} \rightarrow \mathcal{B}^{[D]} - 1$

$$p - p_f = \mathcal{B}^{[D]} - \mathcal{B}_f^{[D]} = \mathcal{B}^{[D]} - (D + 1) \Rightarrow p + D - \mathcal{B}^{[D]} = p_f - 1 \geq 0$$

Thus  $\omega(\mathcal{G}) = 0 \Rightarrow \omega(\widehat{\mathcal{B}}_{(\rho)}^i) = 0$ .

### Theorem

If  $\omega(\mathcal{G}) = 0$  then  $\mathcal{G}$  is dual to a  $D$ -dimensional sphere.

**Proof:** Induction on  $D$ .  $D = 2$ : the colored graphs are ribbon graphs and the degree is the genus. In  $D > 2$ ,  $\omega(\mathcal{G}) = 0 \Rightarrow \omega(\widehat{\mathcal{B}}_{(\rho)}^i) = 0$  and all  $\omega(\widehat{\mathcal{B}}_{(\rho)}^i)$  are a spheres by the induction hypothesis. 1-Dipole contractions do not change the degree and are homeomorphisms.  $\mathcal{G}_f$  is homeomorphic with  $\mathcal{G}$  and has  $p_f = 1$ . The only graph with  $p_f = 1$  is a sphere.

# From Matrix to **COLORED** Tensor Models

Tensors  $T^i_{a_1 \dots a_D}$  with color  $i$

$$S = N^{D/2} \left( T^i_{\dots} \bar{T}^i_{\dots} + \lambda T^0_{\dots} T^1_{\dots} \dots T^D_{\dots} + \bar{\lambda} \bar{T}^0_{\dots} \bar{T}^1_{\dots} \dots \bar{T}^D_{\dots} \right)$$

$$\omega(\mathcal{G}) = \sum_{\mathcal{J}} g_{\mathcal{J}} \geq 0 \text{ degree}$$

$$1/N \text{ expansion in the degree } A(\mathcal{G}) = N^{D - \frac{2}{(D-1)!} \omega(\mathcal{G})}$$

colored stranded graphs  $\leftrightarrow$   $D$  dimensional pseudo manifolds

leading order:  $\omega(\mathcal{G}) = 0$  are **spheres**

## Conclusion: A To Do List

- ▶ Is the dominant sector summable?
- ▶ Does it lead to a phase transition and a continuum theory?
- ▶ What are the critical exponents?
- ▶ Multi critical points?
- ▶ More complex models, driven to the phase transition by renormalization group flow.
  
- ▶ Generalize the results obtained using matrix models in higher dimensions.