Dynamical Chaos and the Volume Gap

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International Loop Quantum Gravity Seminar
relativity.phys.lsu.edu/ilqgs/

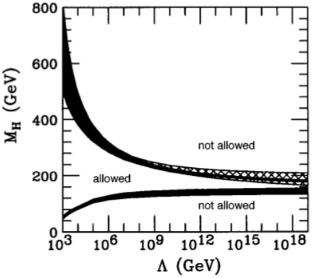
[PRD **87**, 044020], gr-qc/1211.7311

Divergences

"...no approach to quantum gravity can claim complete success that does not explain in full and convincing detail the ultimate fate of the divergences of perturbative quantum gravity." H. Nicolai

gr-qc/1301.5481

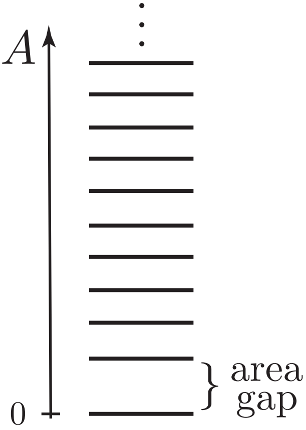
The mass of the LHC scalar boson (125 GeV) narrowly avoids vacuum instability and the Landau pole of self-couplings:



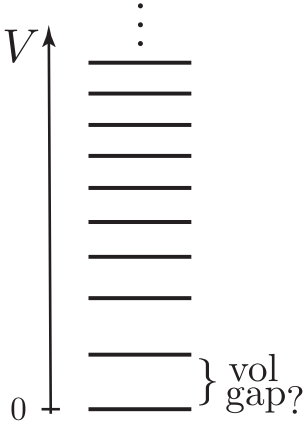
Currently understood physics may hold up to the Planck scale!

Volume gap

Well known area gap:



Discrete volume spectrum:



Is there a volume gap?

Volume gap

Number of intertwiners:

$$\dim \mathsf{inv}\, \mathcal{H}^{(j_1,\cdots,j_N)} = \frac{1}{\pi}\int_0^{2\pi} d\theta \sin^2\frac{\theta}{2}\prod_{a=1}^{N}\frac{\sin\left\{(2j_a+1)\frac{\theta}{2}\right\}}{\sin\frac{\theta}{2}}$$

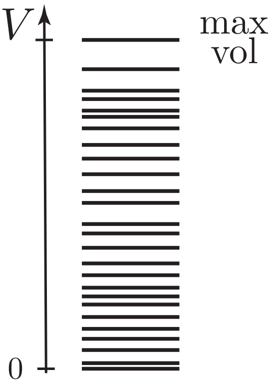
For 5 spins (areas), all equal:

$$\dim \mathsf{inv}\, \mathcal{H}^{(j,j,j,j,j)} = \frac{5}{2}j^2+\frac{5}{2}j+1.$$

Maximum classical volume:

$$V_{\mathsf{max}} \sim j^{3/2}$$

- Is a volume gap robust?



Thermalization

What does it mean to thermalize geometry (gravity)?

Bekenstein ('81): Information transfer rates fundamentally bound

$$\dot{I} < \frac{E}{\hbar},$$

E the total energy of the message. (Compare $\dot{I} \sim \Delta E/\hbar \sim kT/\hbar$.)

gr-qc/1302.0724

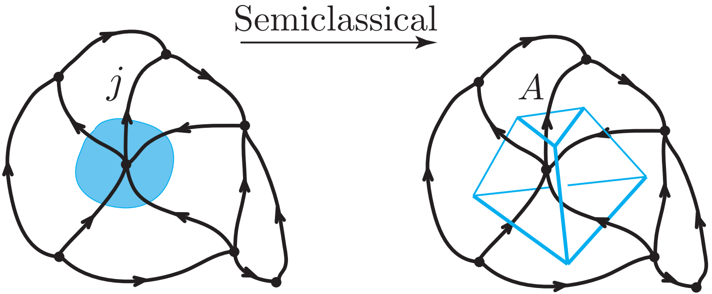
- Black holes saturate this bound.

Sekino and Susskind ('08, '12): Call them fast scramblers.

hep-th/0808.2096

Can the polyhedral volume (Bianchi, Doná and Speziale) lend insight into these issues? gr-qc/1009.3402

$$\hat{V}_{\text{Pol}} = \text{The volume of a quantum polyhedron}$$



Outline

1. Pentahedral Volume

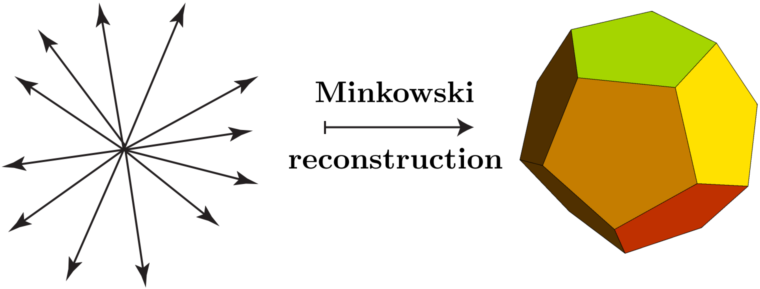
2. Chaos & Quantization

3. Volume Dynamics and Quantum Gravity

Minkowski's theorem: polyhedra

The area vectors of a convex polyhedron determine its shape:

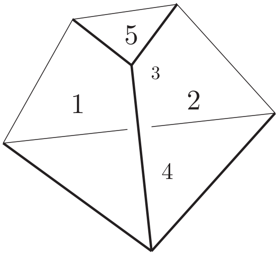
$$\vec{A}_1 + \cdots + \vec{A}_n = 0.$$

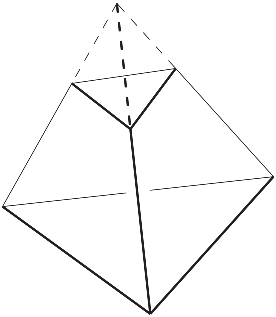


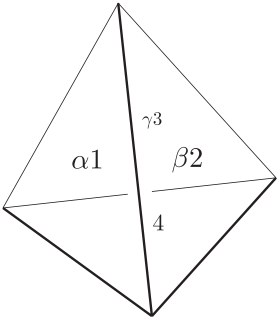
Only an existence and uniqueness theorem.

Volume of a pentahedron

A pentahedron can be completed to a tetrahedron

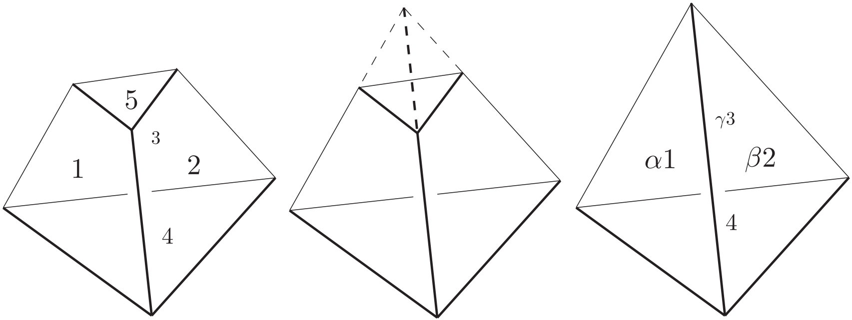






Volume of a pentahedron

A pentahedron can be completed to a tetrahedron



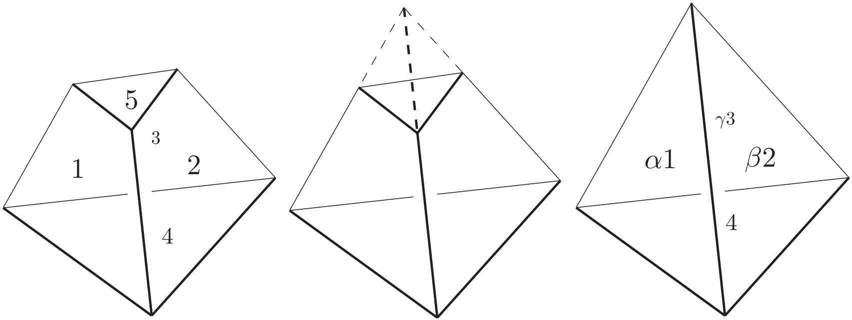
$\alpha, \beta, \gamma > 1$ found from,

$$\alpha \vec{A}_1 + \beta \vec{A}_2 + \gamma \vec{A}_3 + \vec{A}_4 = 0$$

e.g. $\implies \alpha = -\vec{A}_4 \cdot (\vec{A}_2 \times \vec{A}_3)/\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$

Volume of a pentahedron

A pentahedron can be completed to a tetrahedron

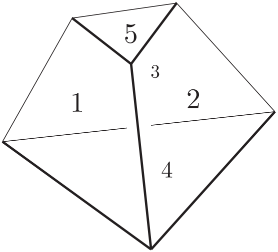


The volume of the pentahedron is then,

$$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right)\sqrt{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}$$

Volume spectrum

That does it:



A 54-pentahedron

$$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right)\sqrt{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}$$

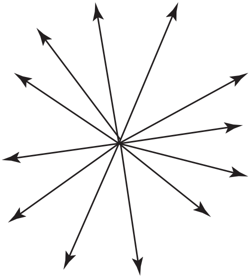
To study spectrum(V) semiclassically:

- set $H = V$
- use Bohr-Sommerfeld condition $\rightsquigarrow$ quantized values.

Won't work!

Adjacency and reconstruction

What's most difficult about Minkowski reconstruction? Adjacency.

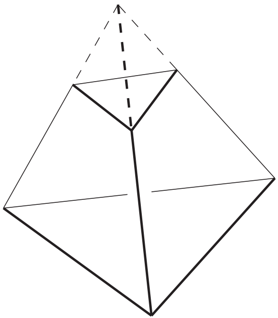


Remarkable side effect of introducing α, β and γ: they completely solve the adjacency problem!

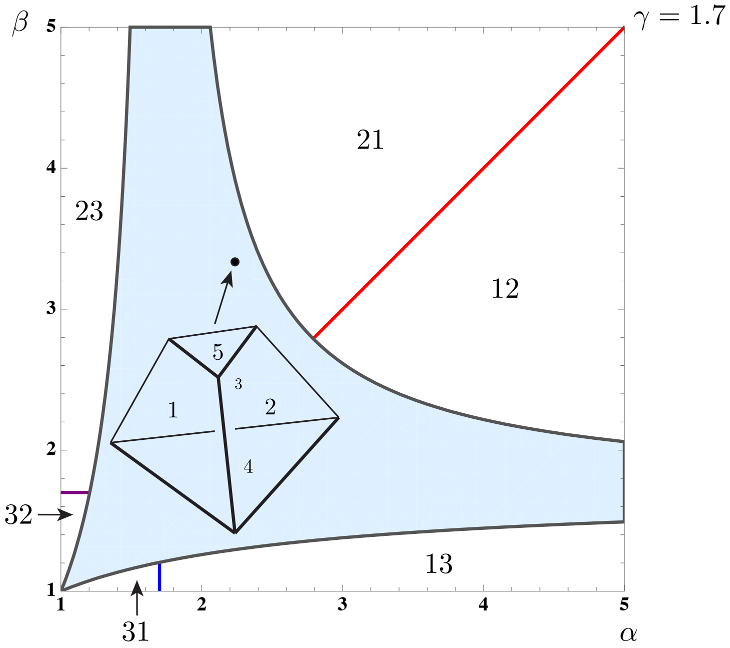
Determining the adjacency

Let $W_{ijk} = \vec{A}_i \cdot (\vec{A}_j \times \vec{A}_k)$. Different closures imply,

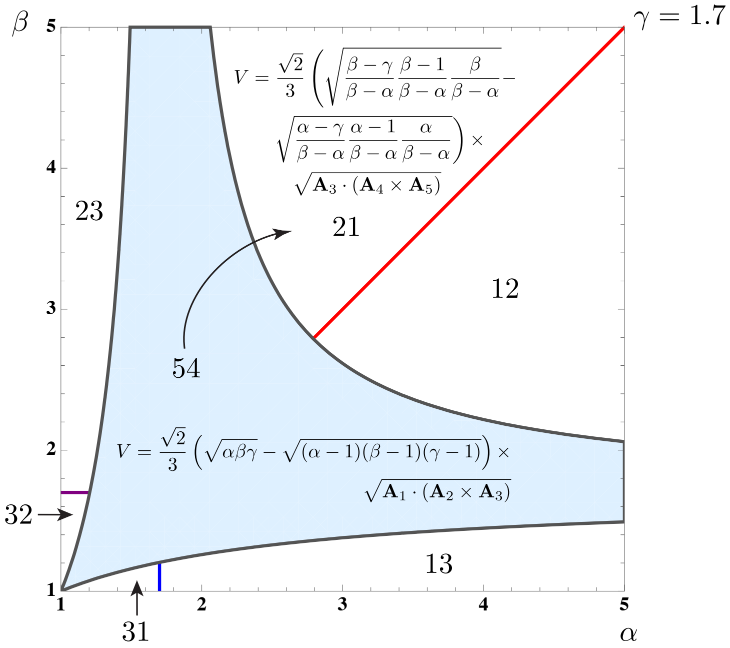
- Case 1: 54-pentahedron
  $\alpha_1 \vec{A}_1 + \beta_1 \vec{A}_2 + \gamma_1 \vec{A}_3 + \vec{A}_4 = 0,$
  $$\gamma_1 = -\frac{W_{124}}{W_{123}}$$
- Case 2: 53-pentahedron
  $\alpha_2 \vec{A}_1 + \beta_2 \vec{A}_2 + \vec{A}_3 + \gamma_2 \vec{A}_4 = 0,$
  $$\gamma_2 = -\frac{W_{123}}{W_{124}} = \frac{1}{\gamma_1}$$



Require $\alpha, \beta, \gamma > 1$: They are mutually incompatible!



β
5
4
3
2
1
1
2
3
4
5
α
$\gamma = 1.7$
21
12
23
13
32
31
1
2
3
4
5



$\gamma = 1.7$
β
α
1
2
3
4
5
$V = \frac{\sqrt{2}}{3}\left(\sqrt{\frac{\beta-\gamma}{\beta-\alpha}\frac{\beta-1}{\beta-\alpha}\frac{\beta}{\beta-\alpha}} - \sqrt{\frac{\alpha-\gamma}{\beta-\alpha}\frac{\alpha-1}{\beta-\alpha}\frac{\alpha}{\beta-\alpha}}\right) \times \sqrt{\mathbf{A}_3 \cdot (\mathbf{A}_4 \times \mathbf{A}_5)}$
$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right) \times \sqrt{\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3)}$
23
21
12
54
32
31
13

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3 Volume Dynamics and Quantum Gravity

EBK quantization

Sommerfeld and Epstein extended Bohr's condition, $L = n\hbar$,

$$S = \int_0^T p \frac{dq}{dt} dt = nh$$

and applied it to bounded, separable systems with f d.o.f,

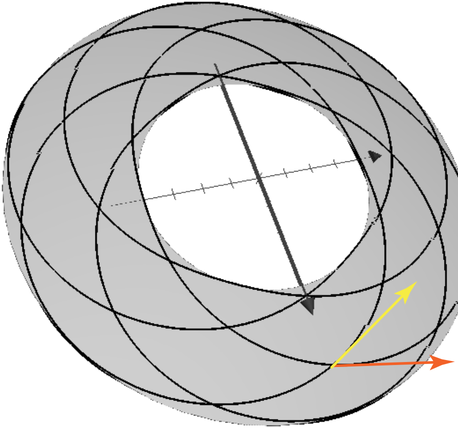
$$\int_0^{T_i} p_i \frac{dq_i}{dt} dt = n_i h, \qquad i = 1, \ldots, f$$

Here the T_i are the periods of each of the coordinates.

Einstein(!) was not satisfied. These conditions are not invariant under phase space changes of coordinates.

EBK quantization II

Motivating example: central force problems

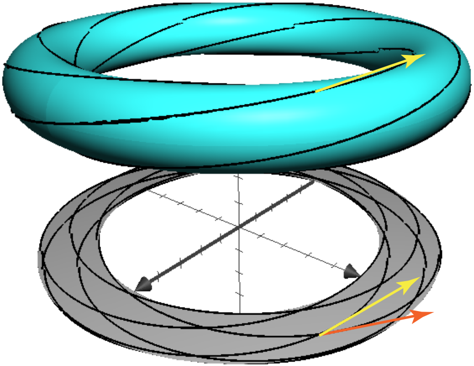


In configuration space trajectories cross;

momenta are distinct at such crossings

EBK quantization II

Motivating example: central force problems



In phase space the distinct momenta lift to the two sheets of a torus

EBK quantization III

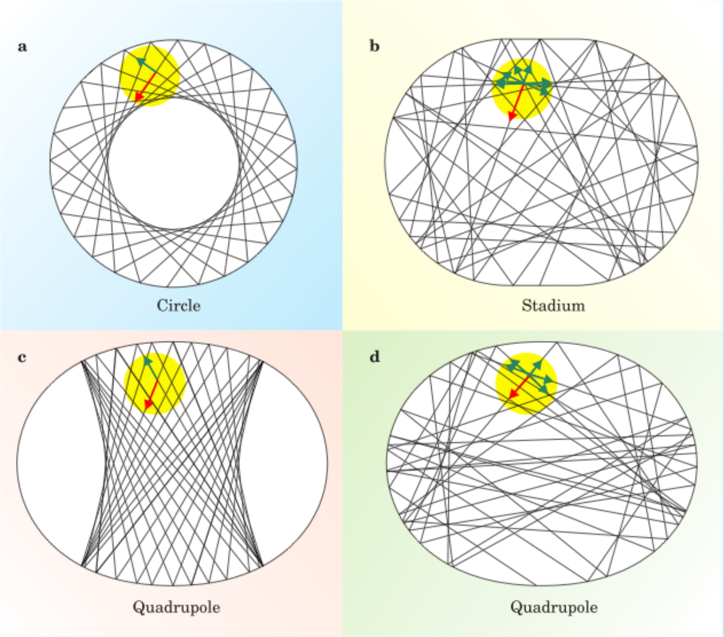
Following Poincaré, Einstein suggested that we use the invariant

$$\sum_{i=1}^{d} p_i dq_i$$

to perform the quantization.

The topology of the torus remains under coordinate changes, and so the quantization condition should be,

$$S_i = \oint_{C_i} \vec{p} \cdot d\vec{q} = n_i h.$$

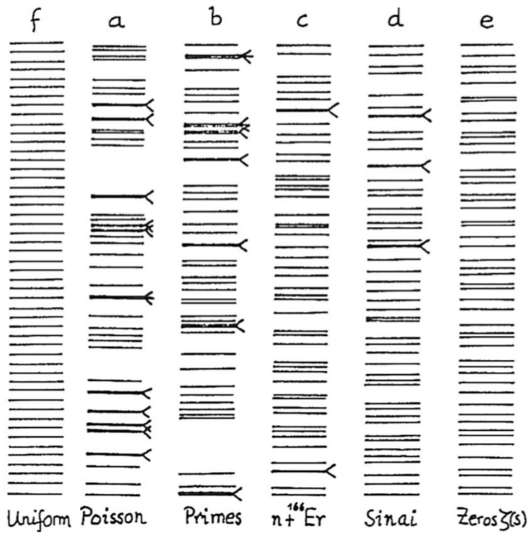

a
Circle
b
Stadium
c
Quadrupole
d
Quadrupole

Quantizing chaos

Einstein was completely correct. Quantizing (in the spectral sense) classically chaotic systems is difficult.

That was how things remained until Wigner introduced Random Matrix Theory (RMT) in the 60's:

Idea — For complex interactions we should treat the Hamiltonian as random and only subject to certain symmetries (Hermitian, perhaps time reversal or parity invariant).



6 spectra normalized to same mean level spacing, "unfolded"

Spectral statistics

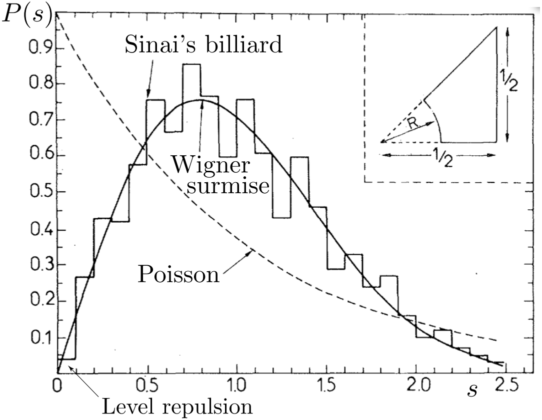
Shifts focus from detailed spectral info onto spectral statistics:

Wigner's surmise (chaos):

$$P(s) = \frac{\pi}{2} s \exp(-\pi s^2/4)$$

Semiclassics (Integrable):

$$P(s) = e^{-s}$$



Proof of BGS conjecture recently found semiclassically: nlin/0906.1960

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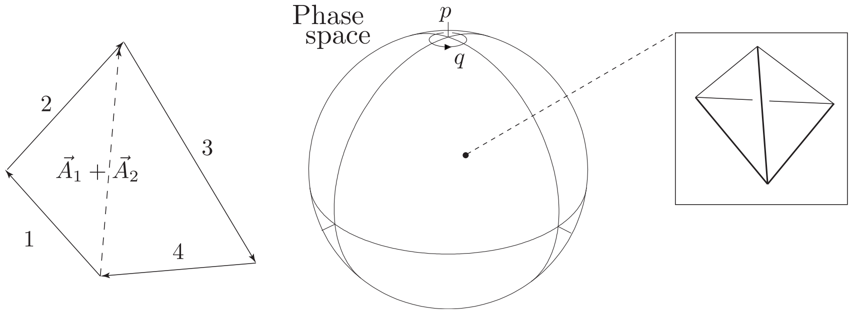
Minkowski's theorem: a tetrahdedron

Why discuss chaos? For fixed areas $A_1, \ldots, A_4$, tet has one d.o.f

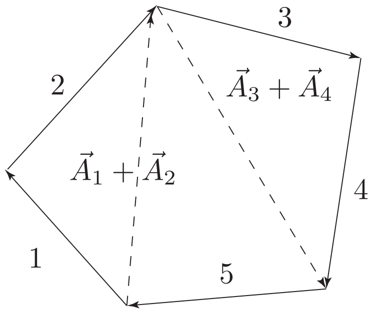


$p = |\vec{A}_1 + \vec{A}_2|$ $\qquad$ $q =$ Angle of rotation generated by p:

$$\{q, p\} = 1$$

Phase space of the pentahedron I

The pentahedron has two fundamental degrees of freedom,

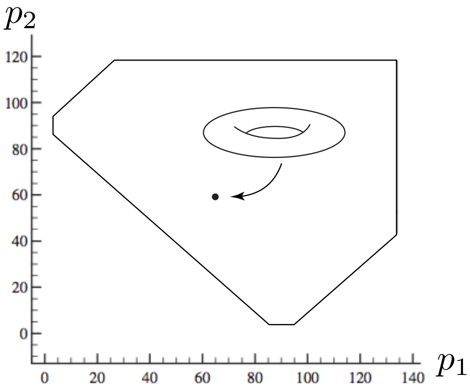


The angles generated by $p_1 = |\vec{A}_1 + \vec{A}_2|$ and $p_2 = |\vec{A}_3 + \vec{A}_4|$.

- Generically systems of 2 or more d.o.f. are chaotic

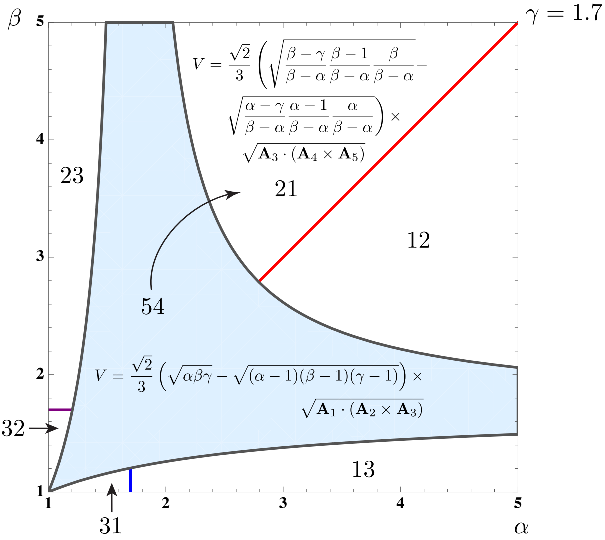
Phase space of the pentahedron II

For fixed p_1 and p_2 these angles sweep out a torus.



The phase space consists of tori over a convex region of the $p_1 p_2$-plane.

Volume is very nonlinear



Recall:

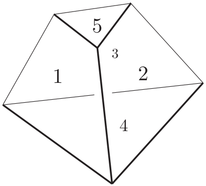
$$\alpha = -\frac{\vec{A}_4\cdot(\vec{A}_2\times\vec{A}_3)}{\vec{A}_1\cdot(\vec{A}_2\times\vec{A}_3)},$$

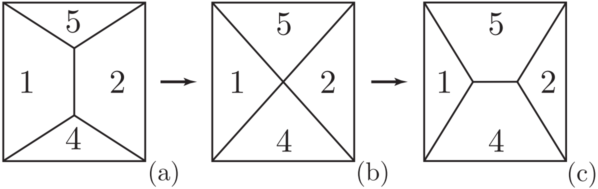
similarly for $\beta,\ \gamma$.

Forced to integrate it numerically.

Volume dynamics: first results

A Schlegel diagram projects a 3D polyhedron into one of its faces:



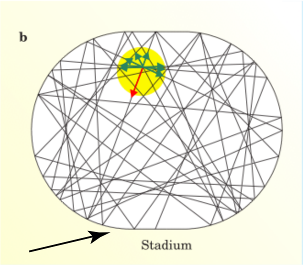


Pachner moves generated by volume evolution connect adjacencies.

Conjectures

Analytic and numerical findings led to conjectures [HMH PIRSA talk]:

- Phase space is mixed containing both stability and instability
- In analogy with billiards, adjacency changes should lead to instability
- Thermalization effectively erases adjacency information

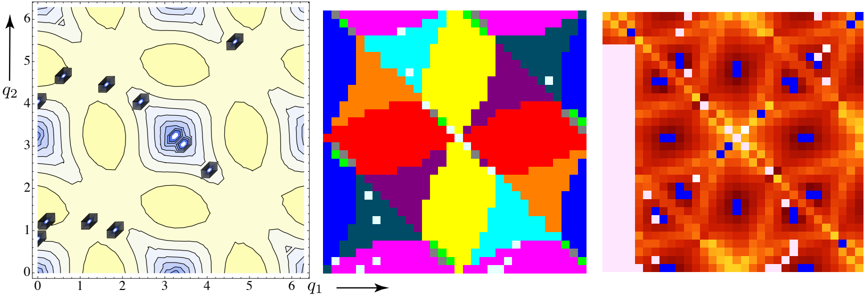


B. Müller's Group at Duke simultaneously work on pentahedron:

- Coleman-Smith and Müller numerically study equi-area case
- Trouble w/ long-time trajectories $\rightsquigarrow$ brilliant idea to focus on local stability

Equal Area Pentahedron

Coleman-Smith and Müller (CSM) [PRD, gr-qc/1212.1930]

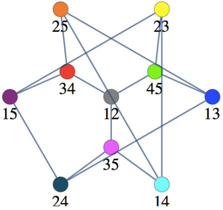


Volume contours:

■ =small volume

■ =large volume

Color code adjacency:

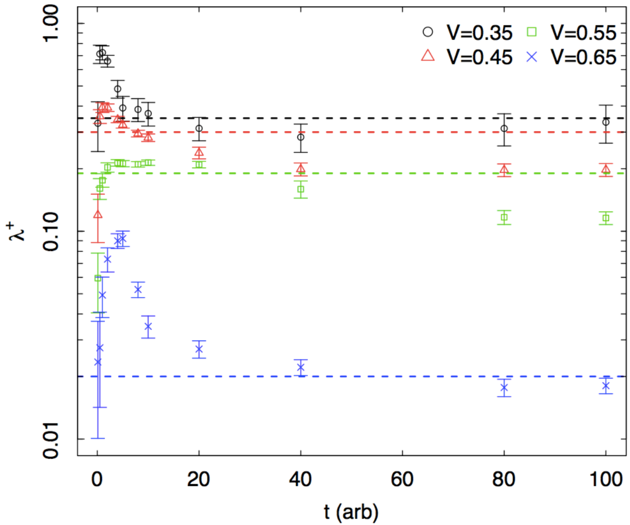


Local Lyapunov exp:

hot = unstable

cool = stable

Evidence for chaos



[CSM] Ensemble averaged intermediate Lyapunov exponents (ILE)

Is there a volume gap?

Chaos $\implies$ level repulsion; that tells about the unfolded spectrum.

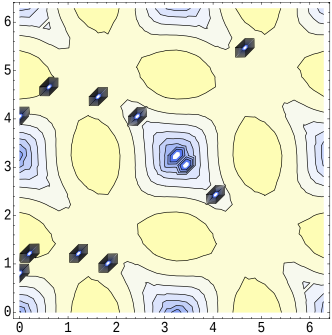
- Is there a volume gap? Does "refolding" mess up repulsion?

Unfolding: tightly packed states are spread out to the average spectral spacing and visa versa.

What is the density of states at zero volume?

- Need phase space (call it Φ) cells of volume $V_\Phi \sim \hbar^2$ to support quantum states.

At small (physical) volume, phase space volume $\to 0 \implies$ density of states goes to zero.



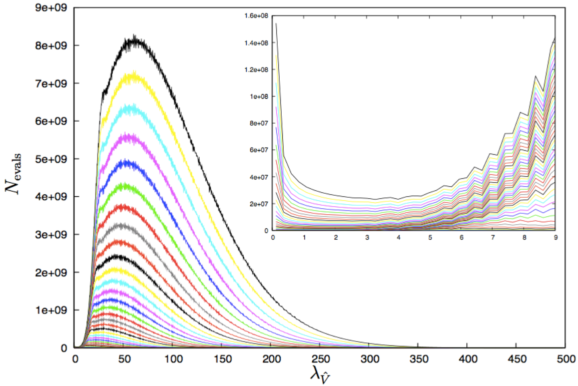
Chaotic volume dynamics leads robustly to a volume gap!

Volume gap previously

In a huge numerical effort Brunnemann and Rideout calculated $\sim 10^{12}$ e-values of the Ashtekar-Lewandowski volume for $j_{\text{max}} = 22$

gr-qc/0706.0469

$$V_{\text{Pent}} = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\bar{\alpha})(\bar{\beta})(\bar{\gamma})}\right) \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$



Why different results? Additivity of volume grains. Near integrability.

Gravitational Divergences

Loop gravity continues to indicate physical cutoffs at the Planck scale:

- Robust volume gap due to: chaos & low density of states at low volume

Meanwhile, Riello is finding divergences at large j (large distance) are tamer than first indicated, only logarithmic: gr-qc/1302.1781

Loop gravity has a coherent and, so far, consistent view of gravitational divergences.

Thermalization

A generic consequence of chaos is efficient thermalization.

Pentahedral volume dynamics $\implies$ eigenstates of quantum volume are spread out over different adjacencies.

- Coherent states peaked around a particular volume will rapidly transit through adjacency transition.
- Thus, thermalization of geometry, in this context is a rapid loss of adjacency information.

Can this window into the thermalization of geometry lend new insights into the extreme thermalization properties of black holes?

Happy Chinese New Year!

Thank you!