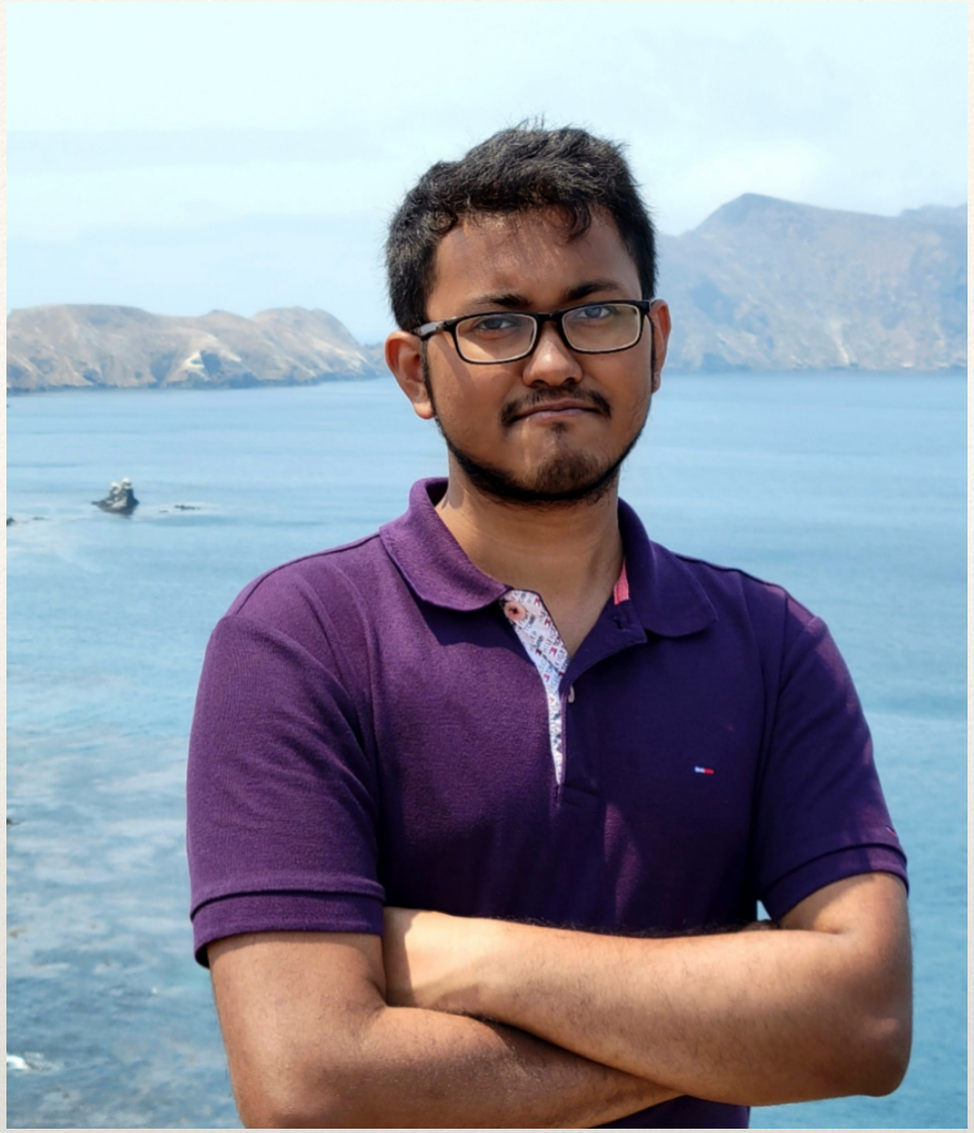
# Quantization of the Volume of the Simplest Grain of Space

Quantum Geometry,
Picard-Fuchs Equations, and
Perturbative/Non-Perturbative Relations

ILQGS
October 19th, 2021

Hal M. Haggard,
Bard College

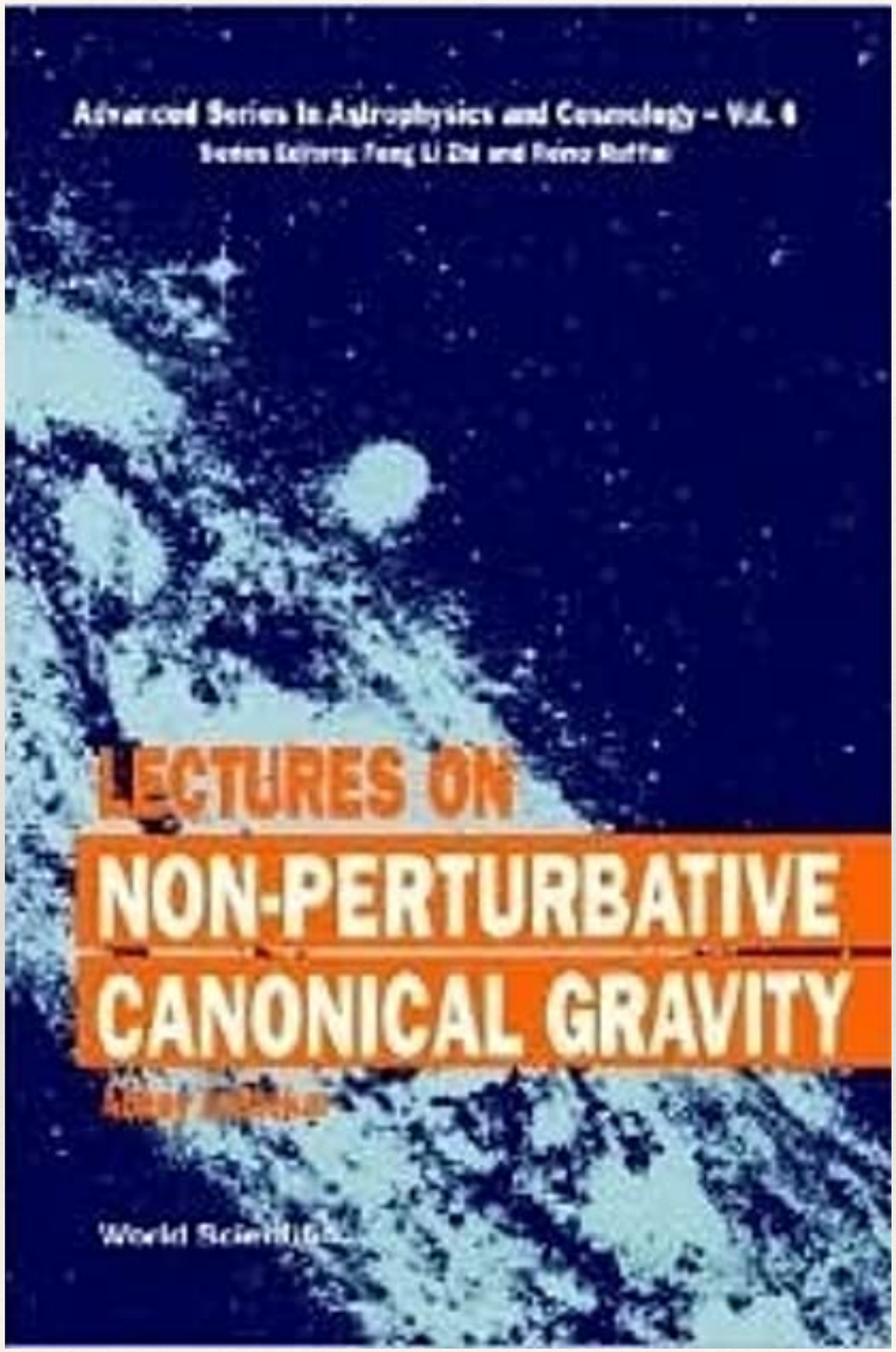


Antu Santanu

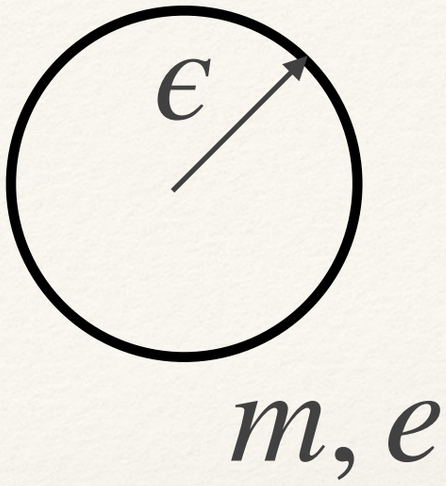
Today's talk is based on unpublished work done in collaboration with Antu Santanu

There is an argument that begins A. Ashtekar's book "Lectures on Non-Perturbative Canonical Gravity" that I've long found intriguing:

Are there features of classical GR that would indicate that non-perturbative quantum gravity is very different from perturbative quantum gravity? They proceed to a simple, but insightful computation:



Consider the self-energy of a shell of charge $e$ and uniform mass density as the radius, $\epsilon$, goes to zero.




Ignoring gravity,

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon}.$$

For a Newtonian self interaction

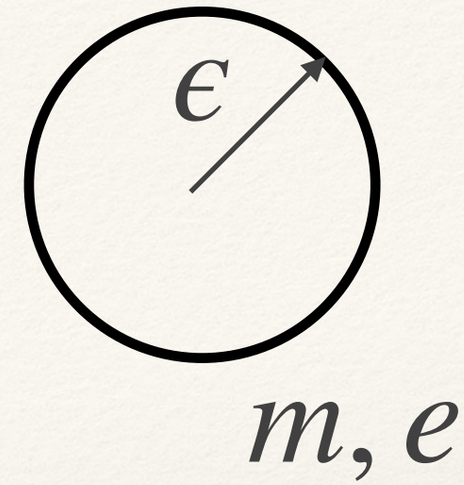
$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{G m_0^2}{\epsilon^2},$$

and in both cases the result diverges as $\epsilon \to 0$. In GR

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{G\, m^2(\epsilon)}{\epsilon^2} \quad \Longrightarrow \quad m(\epsilon) = \frac{-\epsilon}{2G} + \sqrt{\frac{e^2}{G} + \frac{m_0}{G}\epsilon + \frac{1}{4G^2}\epsilon^2}$$

This has a finite limit as $\epsilon \to 0$, $m \to e/\sqrt{G}$ !

Consider the self-energy of a shell of charge $e$ and uniform mass density as the radius, $\epsilon$, goes to zero.




But, if we expand around small $G$

$$m(\epsilon) = \frac{-\epsilon}{2G} + \frac{\epsilon}{2G}\sqrt{1 + \frac{4G}{\epsilon}\left(m_0 + \frac{e^2}{\epsilon}\right)}$$

$$= \left(m_0 + \frac{e^2}{\epsilon}\right) - \left(m_0 + \frac{e^2}{\epsilon}\right)^2 \frac{G}{\epsilon} + 2\left(m_0 + \frac{e^2}{\epsilon}\right)^3 \left(\frac{G}{\epsilon}\right)^2 + \cdots$$

Every term is divergent in the $\epsilon \to 0$ limit.

I take this cautionary tale seriously; beware of over interpreting perturbative divergences! Today I want to take up, what is for me, a new theme:

Perturbative divergences carry interesting information & structure $\rightsquigarrow$ known as resurgence

## *This Talk*

Asymptotic resurgence is impressively broad and impactful. To the best of my knowledge the perturbative/non-perturbative relations found in resurgence have yet to be applied to quantum gravity.

I show that these tools shed light on the quantization of the simplest grain of space, a quantum tetrahedron. The talk has 3 parts:

1. Resurgence and Perturbative/Non-Perturbative Relations
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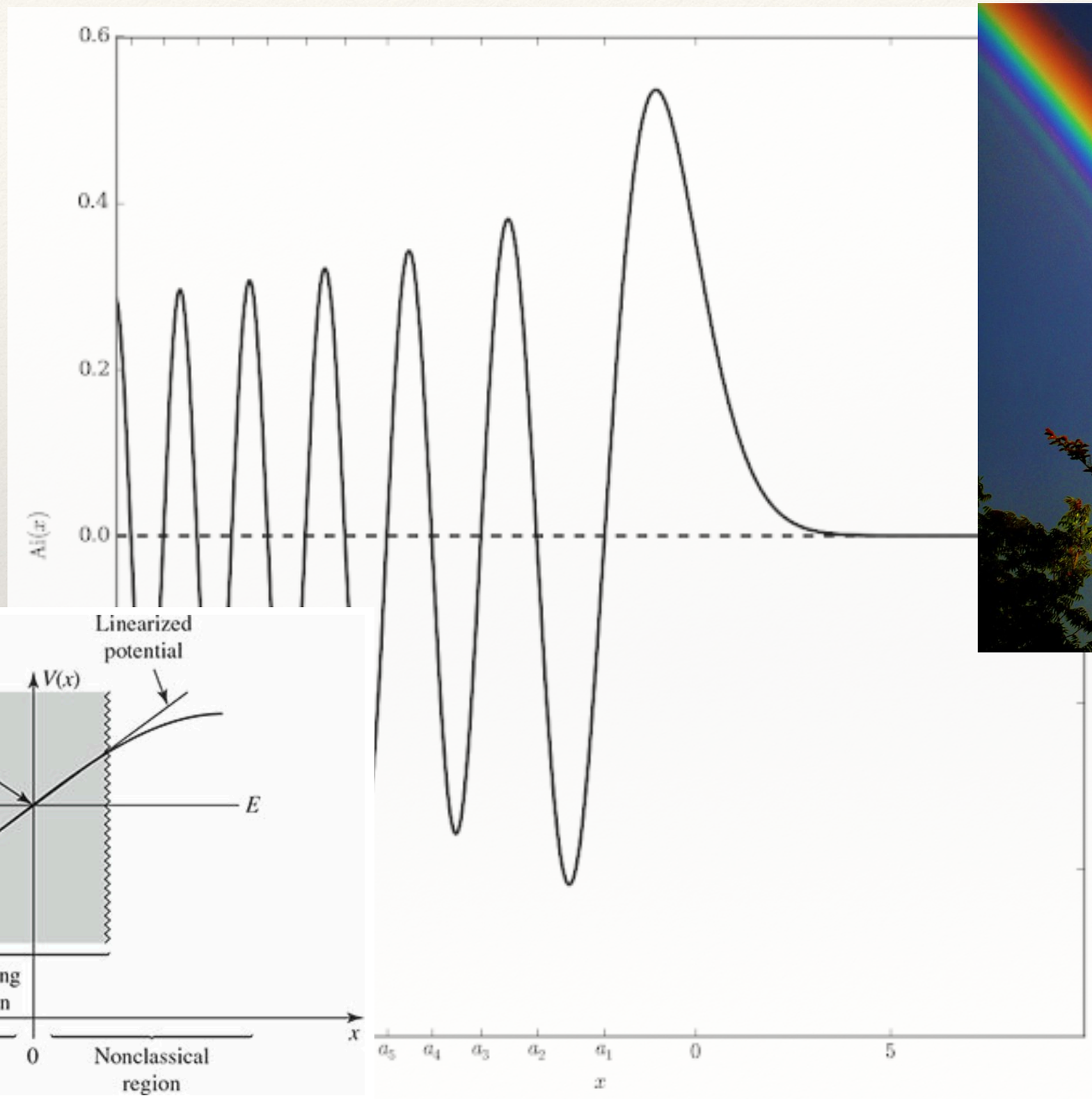
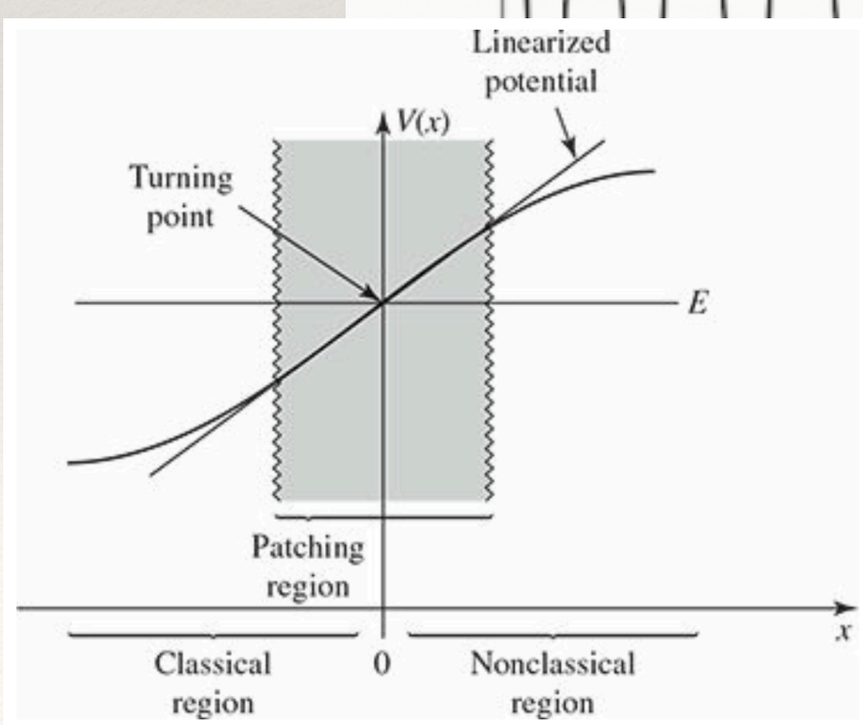
# Two physical examples where Airy functions arise are quantum mechanics of turning points and supernumerary rainbows







The Schrödinger equation at the turning point is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E + V'(0)x)\psi = E\psi \quad \text{or} \quad \frac{d^2\psi}{dx^2} = \alpha^3 x\psi$$

with $\alpha = \hbar^{-2/3}[2mV'(0)]^{1/3}$.

WKB theory suggests we introduce the action

$$a(x) = \pm\int\sqrt{\alpha^3 x}\,dx = \pm\frac{2}{3}\alpha^{3/2}x^{3/2},$$

and consider solutions

$$\psi(x) \approx \#\frac{e^{-a(x)}}{\sqrt{a'(x)}}(1+\cdots) \sim \frac{e^{\mp\frac{2}{3}(\alpha x)^{3/2}}}{(\alpha x)^{\frac{1}{4}}}\sum_{n=0}^{\infty}\frac{c_n}{[(\alpha x)^{3/2}]^n}.$$

A recursion relation determines the (factorially divergent) $c_n$:

$$c_n = (\mp 1)^n\frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{2\pi\, n!\left(\frac{4}{3}\right)^n}.$$

The $c_n$ are unexpectedly interesting

$$c_n^+ = \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{2\pi\, n!\left(\frac{4}{3}\right)^n} \rightsquigarrow \overset{\{c_0^+,\ c_1^+,\ c_2^+,\ c_3^+,\ \ldots\}}{\left\{1, \frac{5}{48}, \frac{385}{4608}, \frac{85085}{663552}, \ldots\right\}}.$$

But, now consider the large $n$ behavior of the $c_n^+$

$$c_n^+ \sim \frac{1}{2\pi}\frac{(n-1)!}{\left(\frac{4}{3}\right)^n}\left(1 - \frac{5}{36}\frac{1}{n} + \frac{25}{2592}\frac{1}{n^2} - \ldots\right)$$

If we organize this expansion in terms of factorial growth…

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…we see the same coefficients! The late orders of $c_n^+$ are the early orders of $c_n^-$.

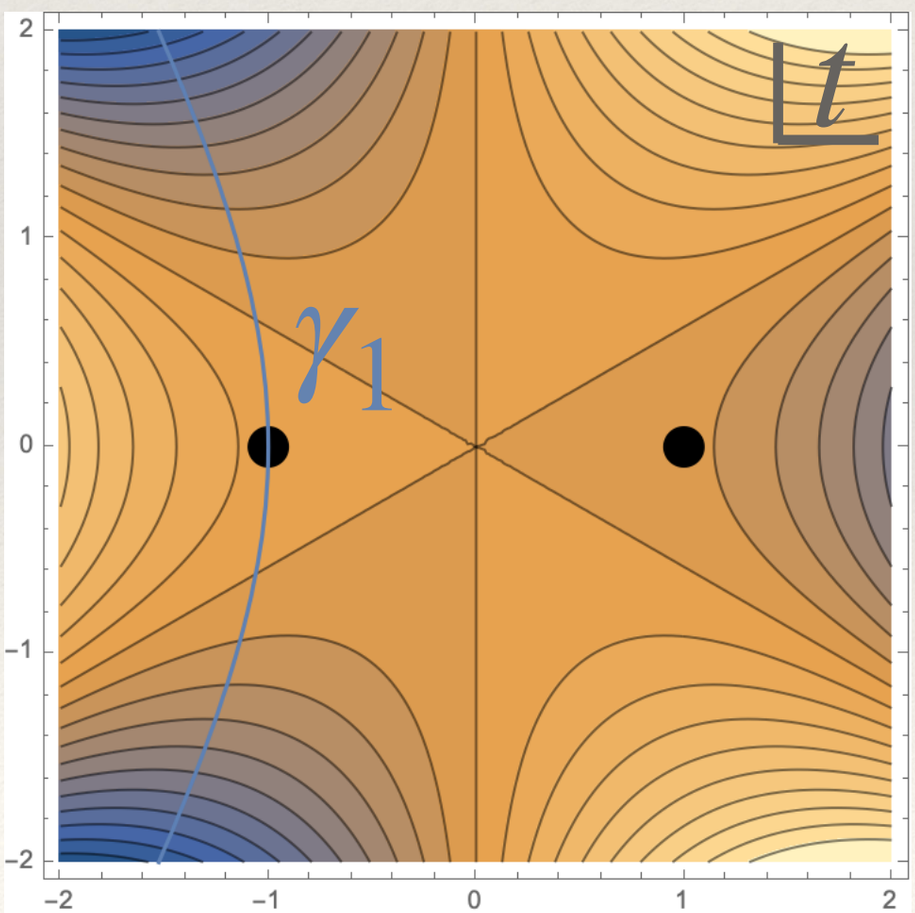
Why?!

The integral representation sheds light on this question:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\gamma_1} \exp\left(-zt + \frac{t^3}{3}\right) dt.$$

For $z = re^{i\theta}$, the critical points are at $t_c = \pm\sqrt{z} \sim \pm\, e^{i\theta/2}$.

For $z$ real and positive, i.e. $\theta = 0$, the integral is dominated by a single real critical point and exponentially decays.




Shading shows the value of the real part of the exponent:

Blue ⇝ decreasing real part

Yellow ⇝ increasing real part

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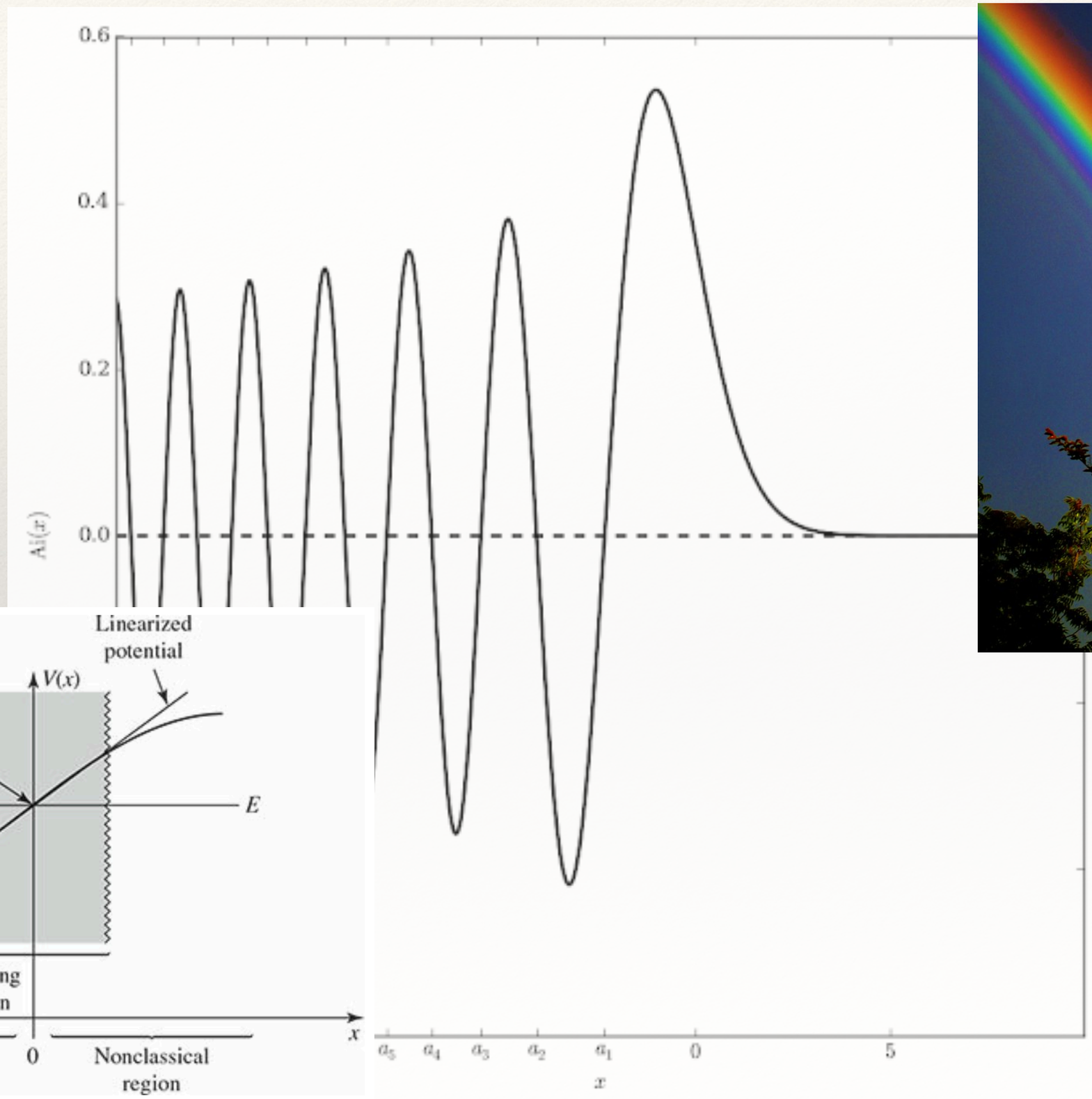



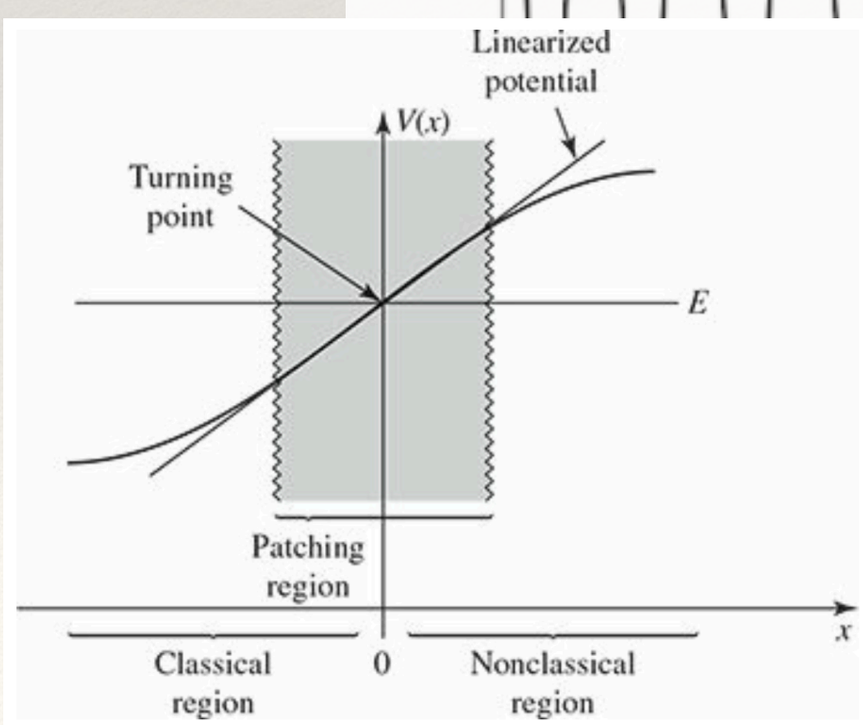

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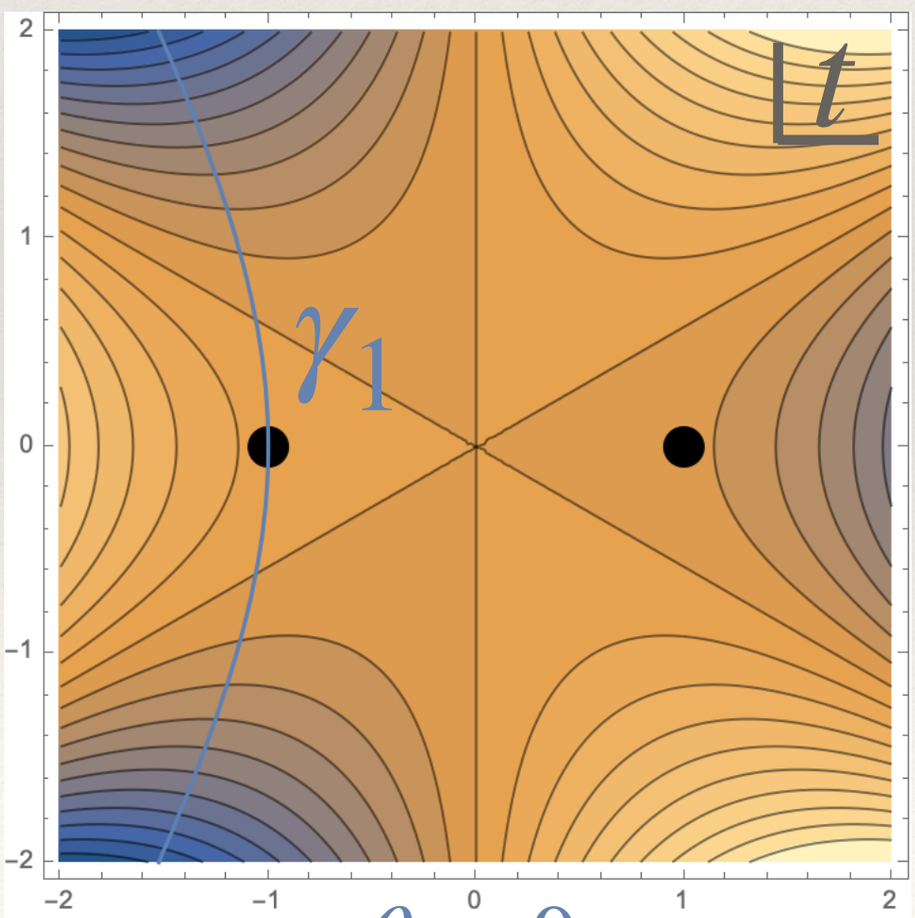
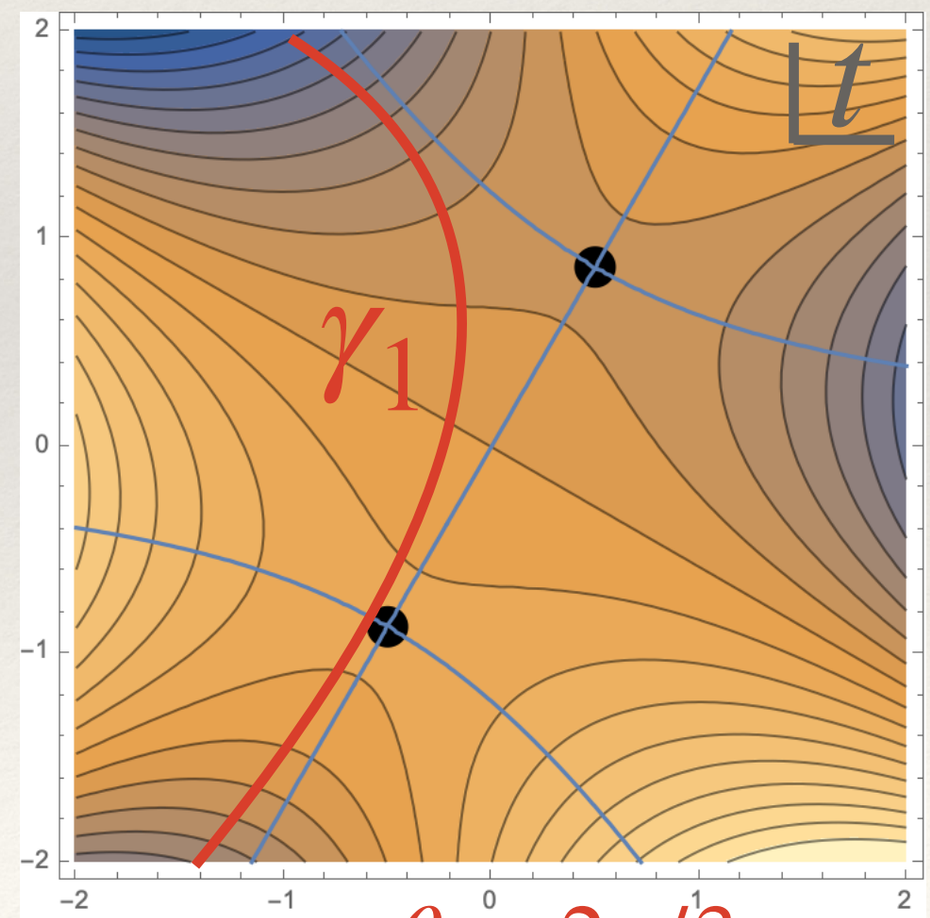
For $z = re^{i\theta}$, the normalized critical points are at $t_c = \pm\, e^{i\theta/2}$.

However, as we vary $\theta$, the critical points and real-$t$ landscape vary. At $\theta = \frac{2\pi}{3}$ the second critical point begins to contribute.




$\theta = 0$

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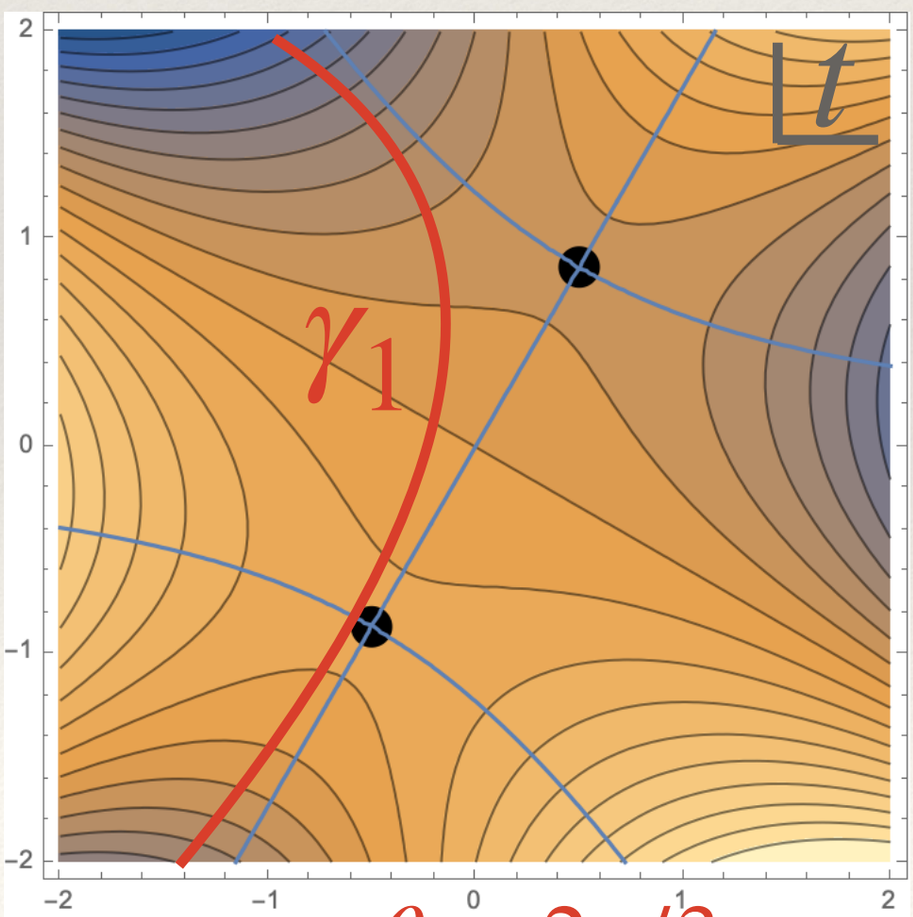
This change in dominance is quite clear in the Airy function:

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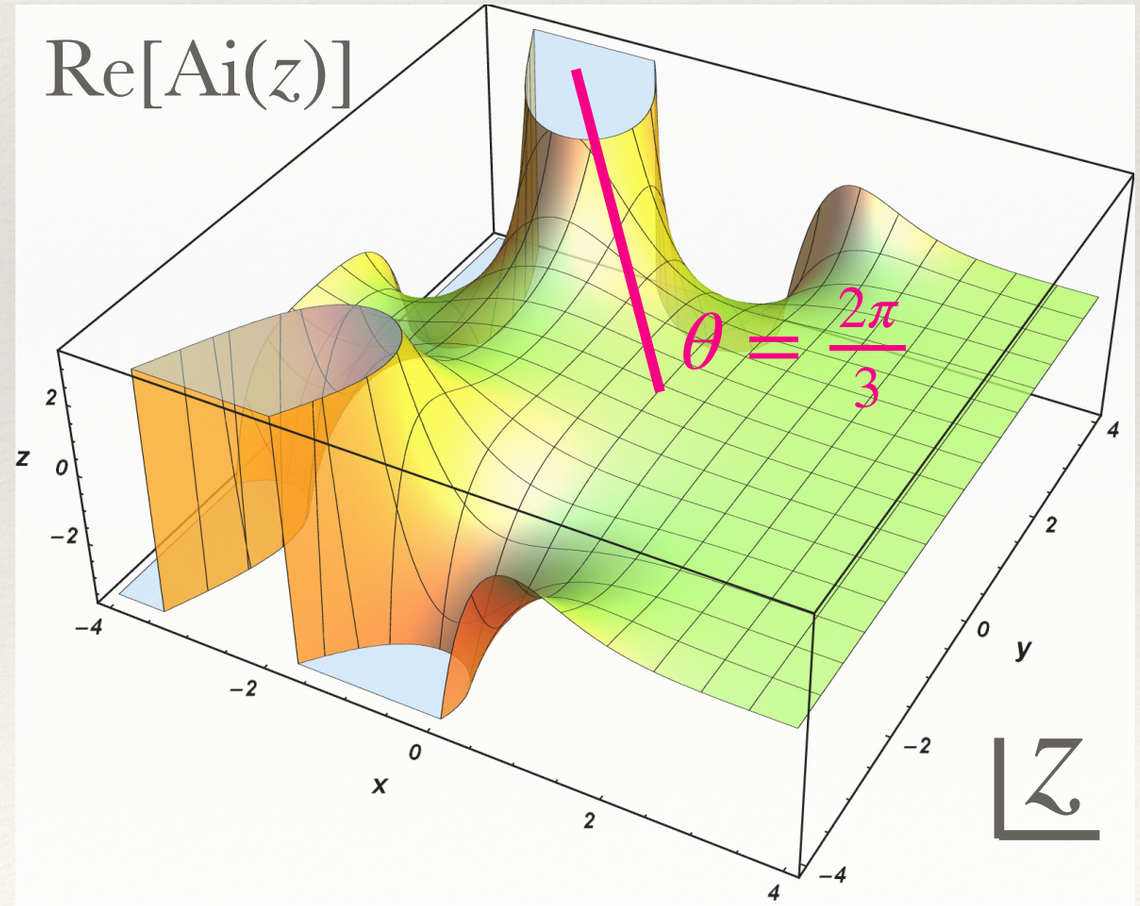

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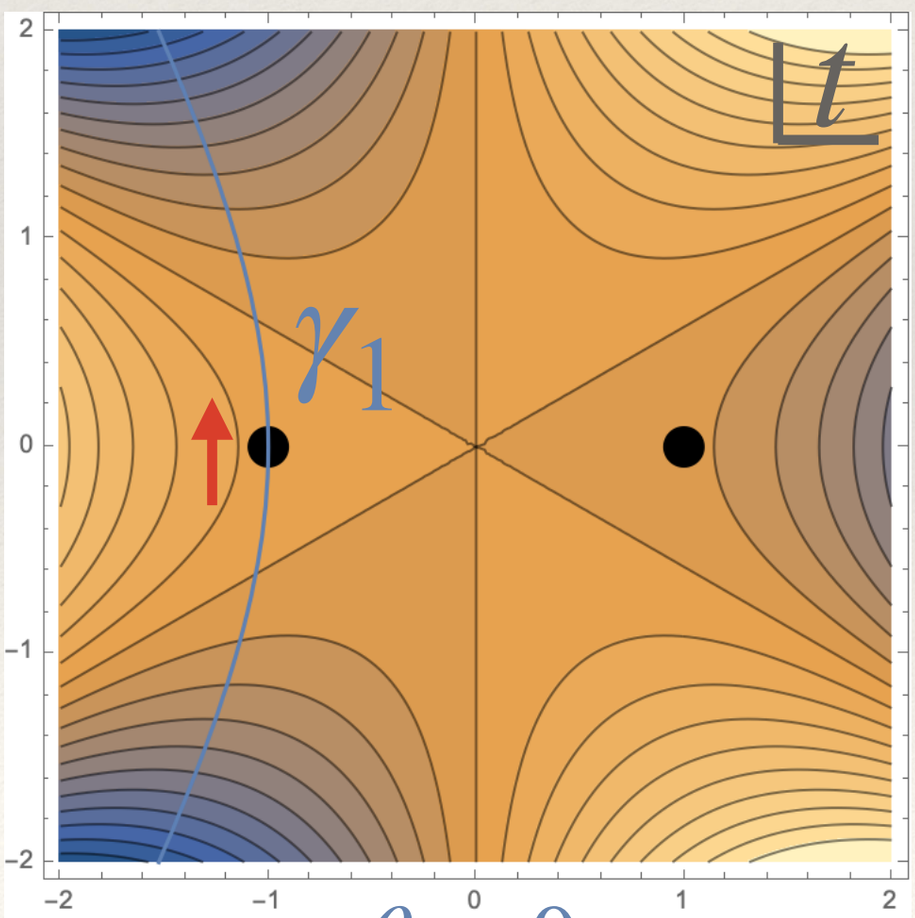
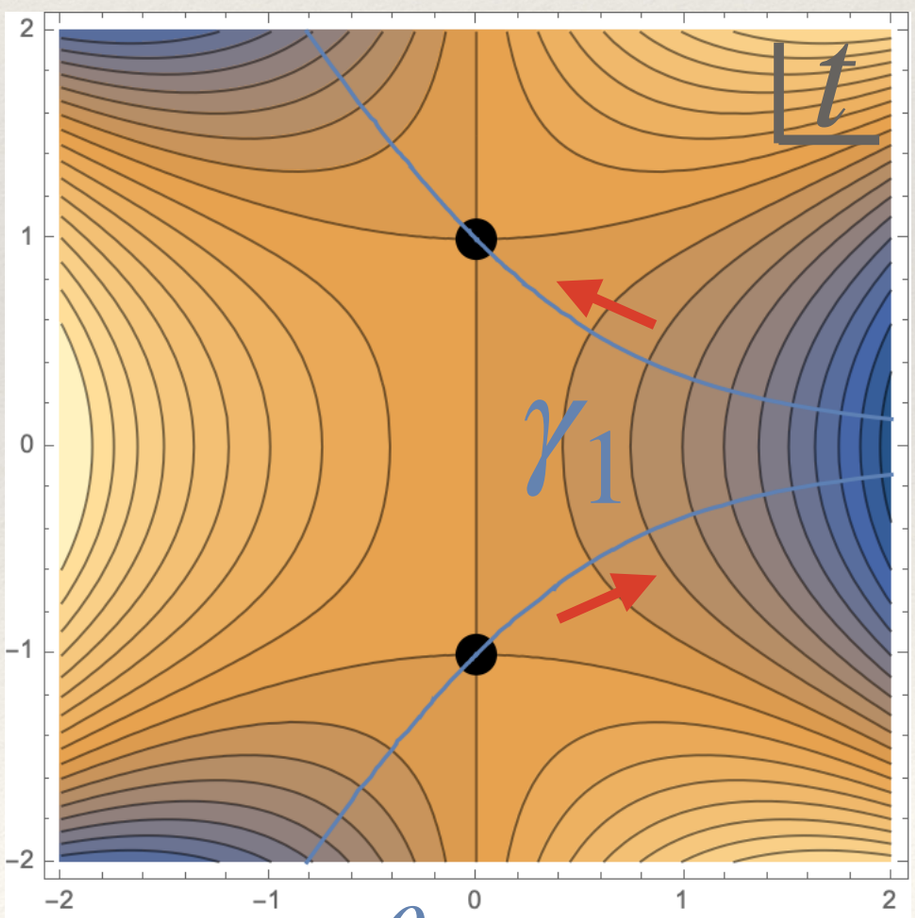
As we continue $\theta$ further to $\theta = \pi$, both critical points become purely imaginary and jointly lead to oscillatory behavior .

This change of 1 crit. point into 2 is the Stokes' phenomenon.




$\theta = 0$




$\theta = \pi$

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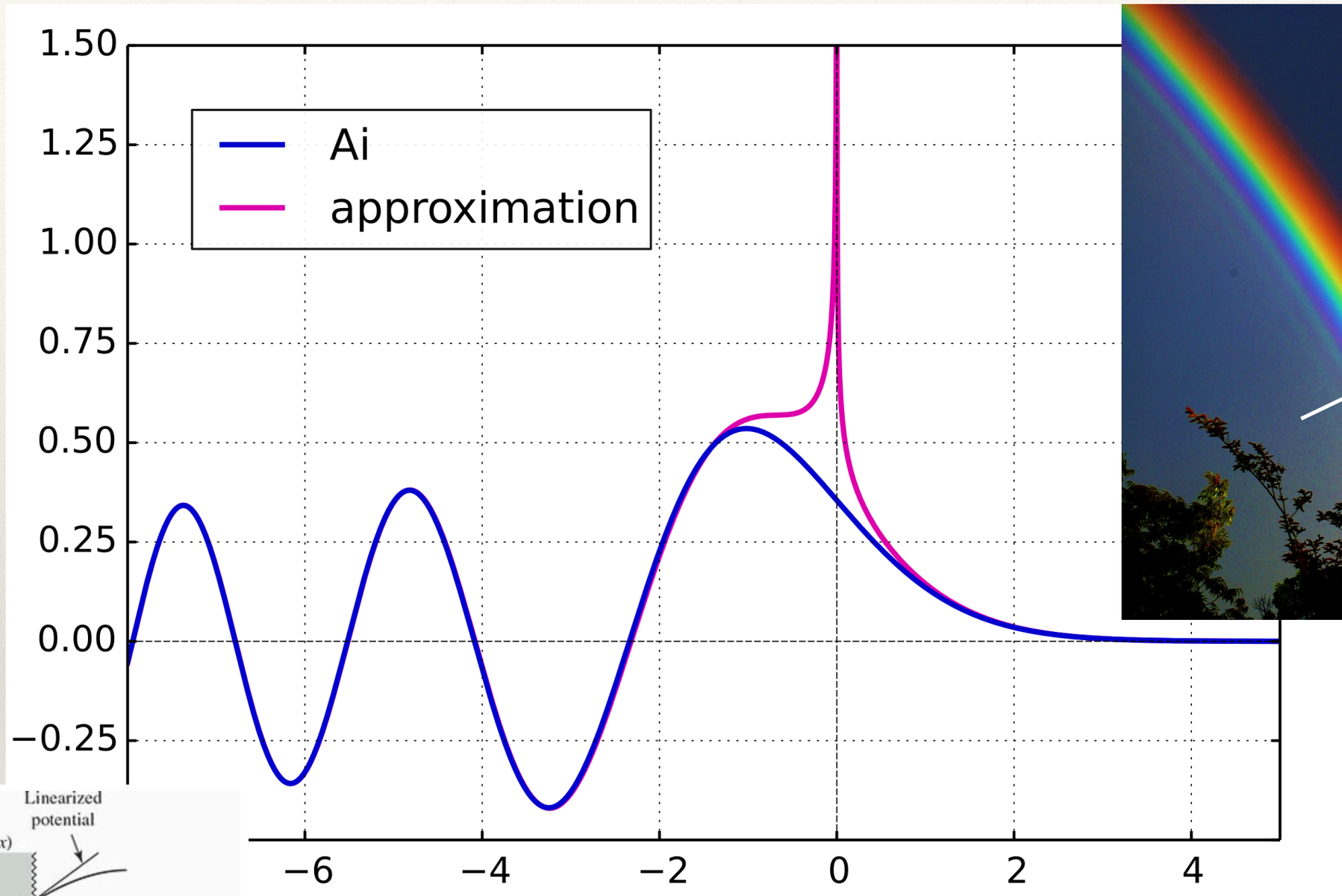
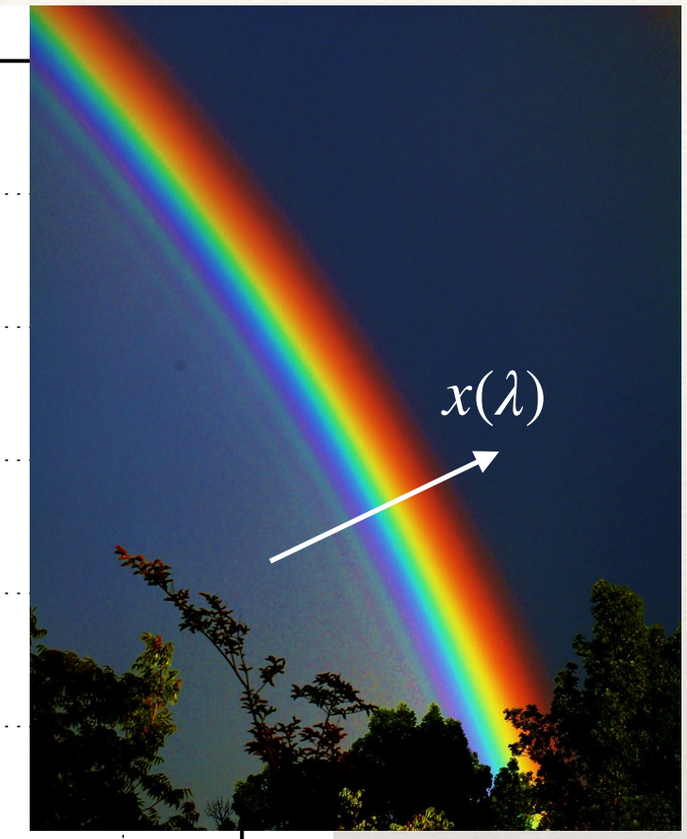
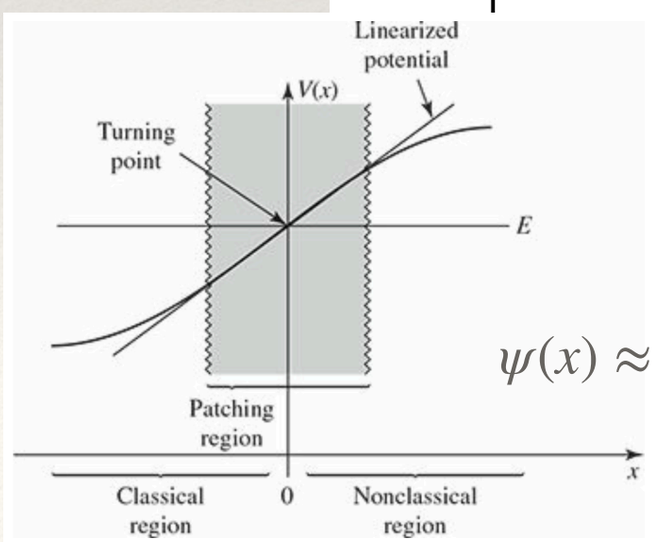



$$\psi(x) \approx \frac{e^{\mp\frac{2}{3}(\alpha x)^{3/2}}}{(\alpha x)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{c_n}{[(\alpha x)^{3/2}]^n} \sim \begin{cases} \dfrac{1}{2\sqrt{\pi}\,|x|^{1/4}} e^{-\frac{2}{3}x^{3/2}} (1+\cdots) & \text{for } x > 0 \\ \dfrac{1}{\sqrt{\pi}\,|x|^{1/4}} \sin\left(\dfrac{2}{3}|x|^{2/3} + \dfrac{\pi}{4}\right)(1+\cdots) & \text{for } x < 0 \end{cases}$$

The integral representation sheds light on this question:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\gamma_1} \exp\left(-zt + \frac{t^3}{3}\right) dt.$$

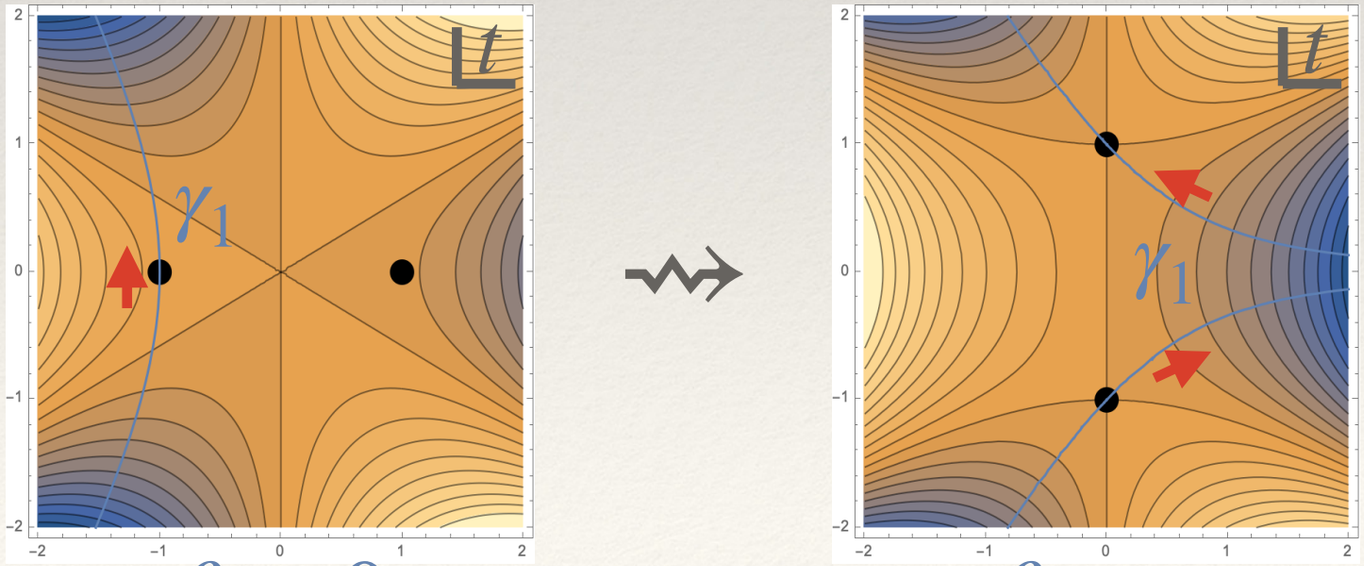
But still the question remains: why do both critical points contribute? The answer is that both saddles were always contributing, it's just that one was exponentially sub-dominant

$$\mathrm{Ai}(e^{\mp\frac{2\pi i}{3}}z) = \frac{1}{2}e^{\pm\frac{\pi i}{6}}\mathrm{Bi}(z) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Ai}(z).$$

We can understand that such connection formulae must exist from the differential equation:

$$\frac{d^2\psi}{dx^2} = \alpha^3 x\psi,$$

which only has two independent solutions.




$\theta = 0$ $\theta = \pi$

Gerald Dunne offers an elegant mnemonic image for these ideas: the wavelets surrounding droplets of water

The expansion of a quantity of physical interest around one of its critical points has late term contributions arising from the near term contributions of neighboring saddles.

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**Classical Area Geometry** — Areas, instead of lengths, can be used to describe all sorts of familiar geometry.

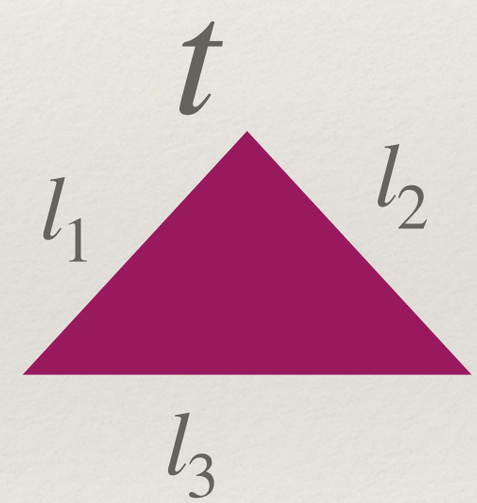
Heron's beautiful formula gives the area of a triangle as a function of its edge lengths

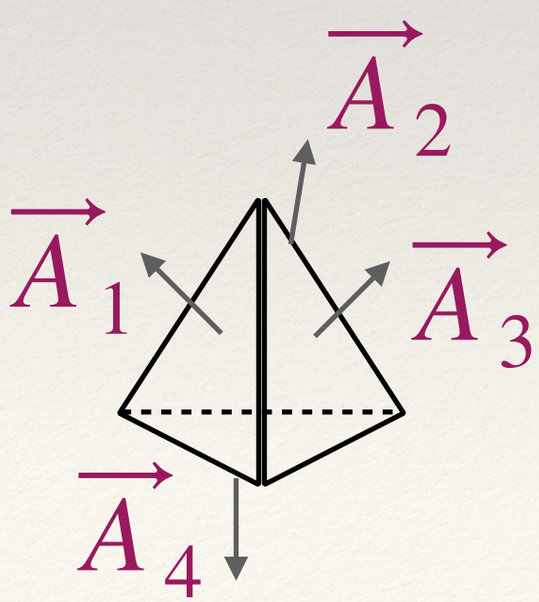
$$A_t^2(l_1, l_2, l_3) = \frac{1}{16}[l_1^2 - (l_2 - l_3)^2][(l_2 + l_3)^2 - l_1^2].$$

A classical theorem due to H. Minkowski gives a tetrahedron's geometry via its area vectors:

$$\vec{A}_t = A_t \hat{n}_t,$$

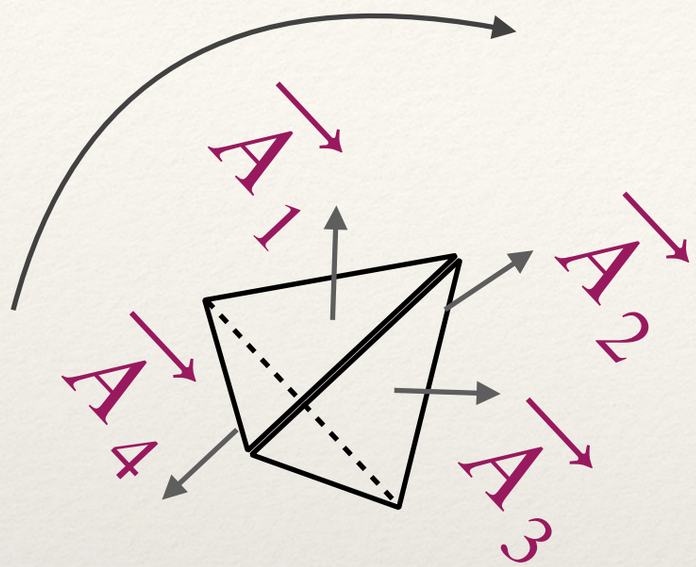
where $\hat{n}_t$ is the unit normal to triangle $t$.




$$\longleftrightarrow \quad \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0$$

Indeed, we can make a rather profound change in perspective, and view $\vec{A}_t$ as elements of the $\mathfrak{so}(3) \cong \mathbb{R}^3$ Lie algebra




$$R(\theta, \hat{u}) = e^{\theta(\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4)\cdot\hat{u}}$$

With this choice, not only does the closure

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0$$

describe the tetrahedral geometry, but is also the diagonal generator of overall rotations and expresses the invariance of the geometry under these rotations; this is the Gauss constraint of discrete geometry.

Counting edge lengths, we know that a tetrahedron has 6 independent parameters. The 12 components $\{\vec{A}_t\}_{t=1}^4$ are clearly overkill, while the 4 magnitudes $\{A_t\}_{t=1}^4$ are insufficient.

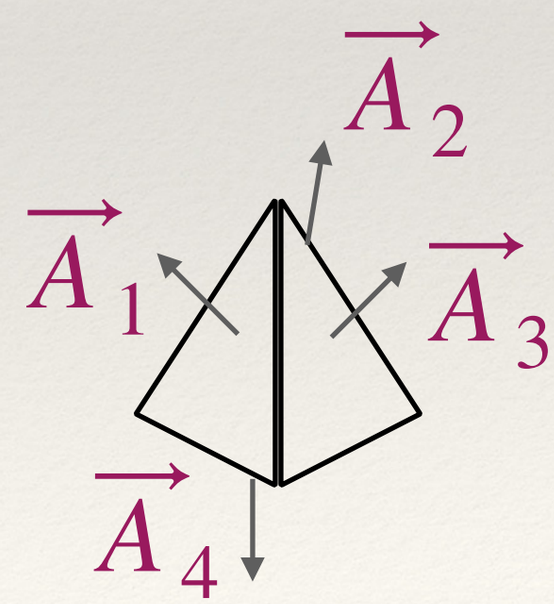
Closure provides a way out of this quandary. We seek rotational invariants to complete the characterization of the tetrahedral geometry. We can work with any two of:

the dot products

$\vec{A}_t \cdot \vec{A}_{t'}$, or the same $A_{tt'}^2 = (\vec{A}_t + \vec{A}_{t'})^2 = A_t^2 + A_{t'}^2 + 2\vec{A}_t \cdot \vec{A}_{t'}$

or the rotationally invariant triple products

$$\vec{A}_t \cdot (\vec{A}_{t'} \times \vec{A}_{t''}).$$

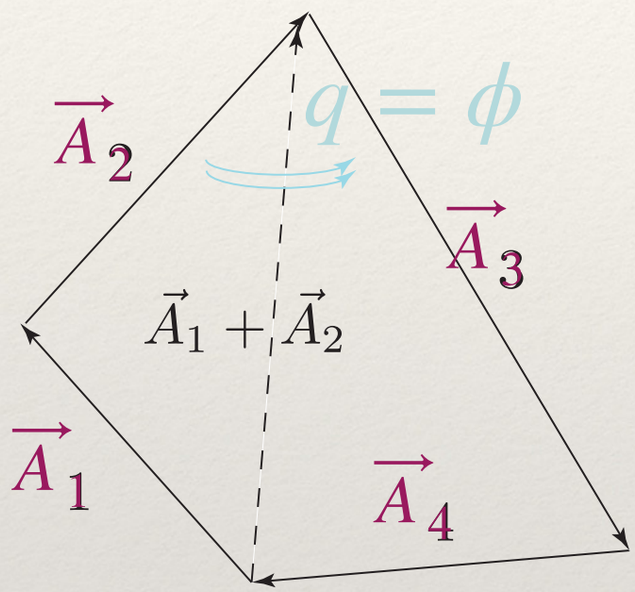



It's striking that the latter gives the volume squared of the tetrahedron

$$Q \equiv V_\tau^2 = \frac{2}{9}\vec{A}_t \cdot (\vec{A}_{t'} \times \vec{A}_{t''}).$$

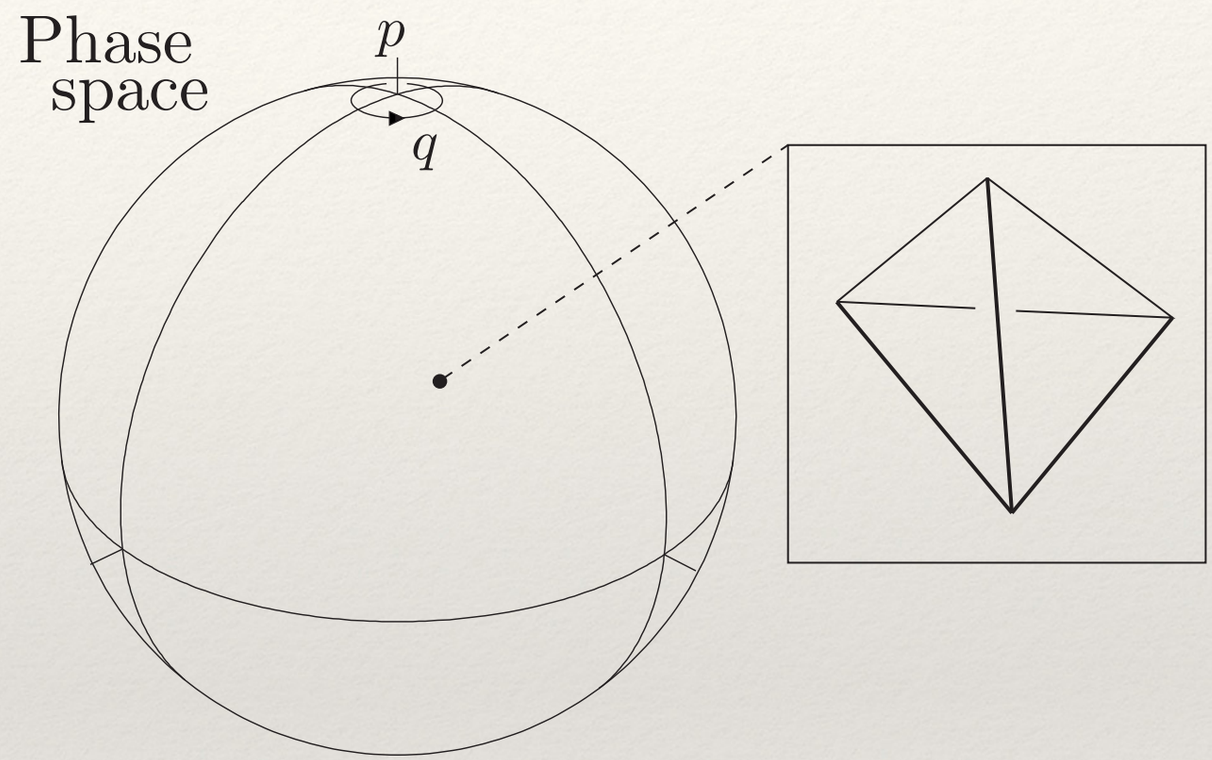
In sum, we have a classical phase space for tetrahedra!

We adopt canonical coordinates by rearranging closure as vector addition:




$$p = A = |\vec{A}_1 + \vec{A}_2|$$

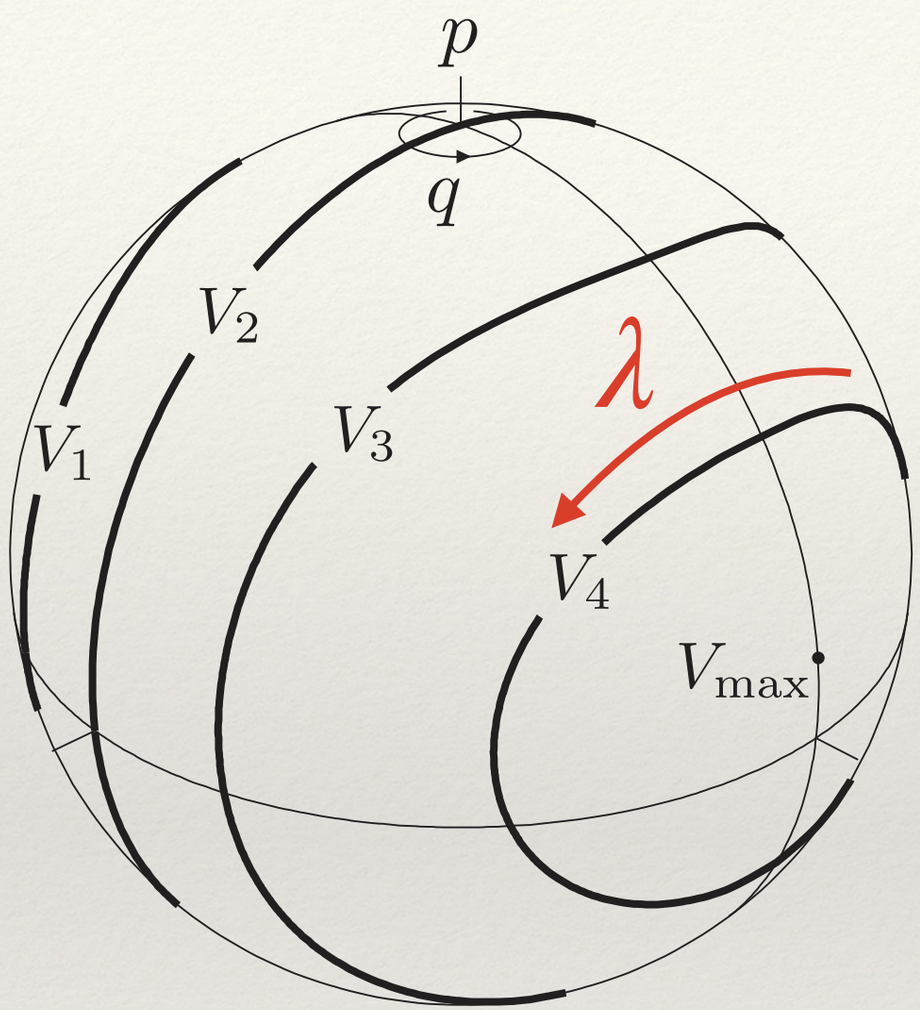
$\Leftrightarrow$



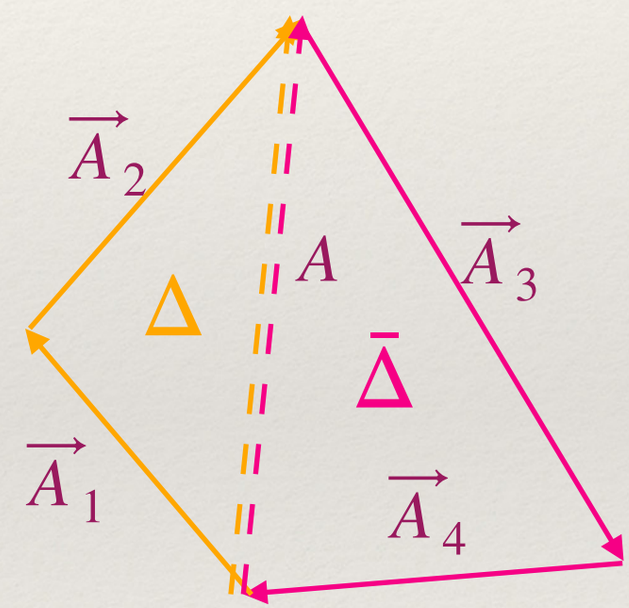

Because we understand the vectors as elements of $\mathfrak{so}(3)$, the coords are naturally equipped with a Poisson bracket and:

$$\{q, p\} = \{\phi, A\} = 1.$$

In 2011, Eugenio Bianchi and I considered the evolution generated by $V$, more precisely $Q = V^2 = \frac{2}{9}\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$, on this space with fixed $\{A_1, A_2, A_3, A_4\}$:

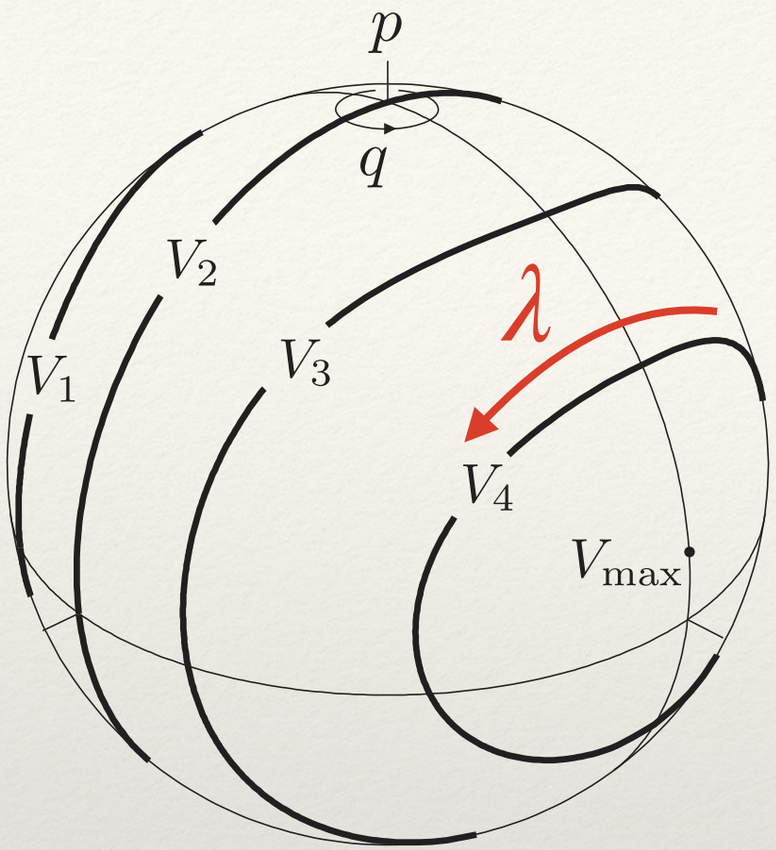



$$Q = V^2 = \frac{8}{9}\frac{\Delta\bar{\Delta}}{A}\sin\phi$$




$$\frac{d\phi}{d\lambda} = \{\phi, Q\} \qquad \text{and} \qquad \frac{dA}{d\lambda} = \{A, Q\}$$

In 2011, Eugenio Bianchi and I considered the evolution generated by the volume $V$; this evolution is integrable:

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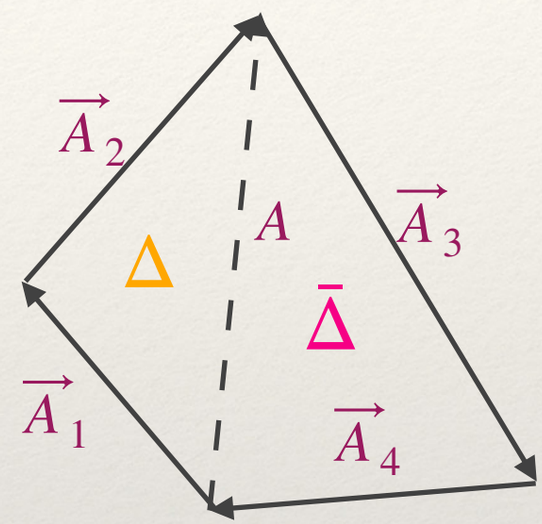
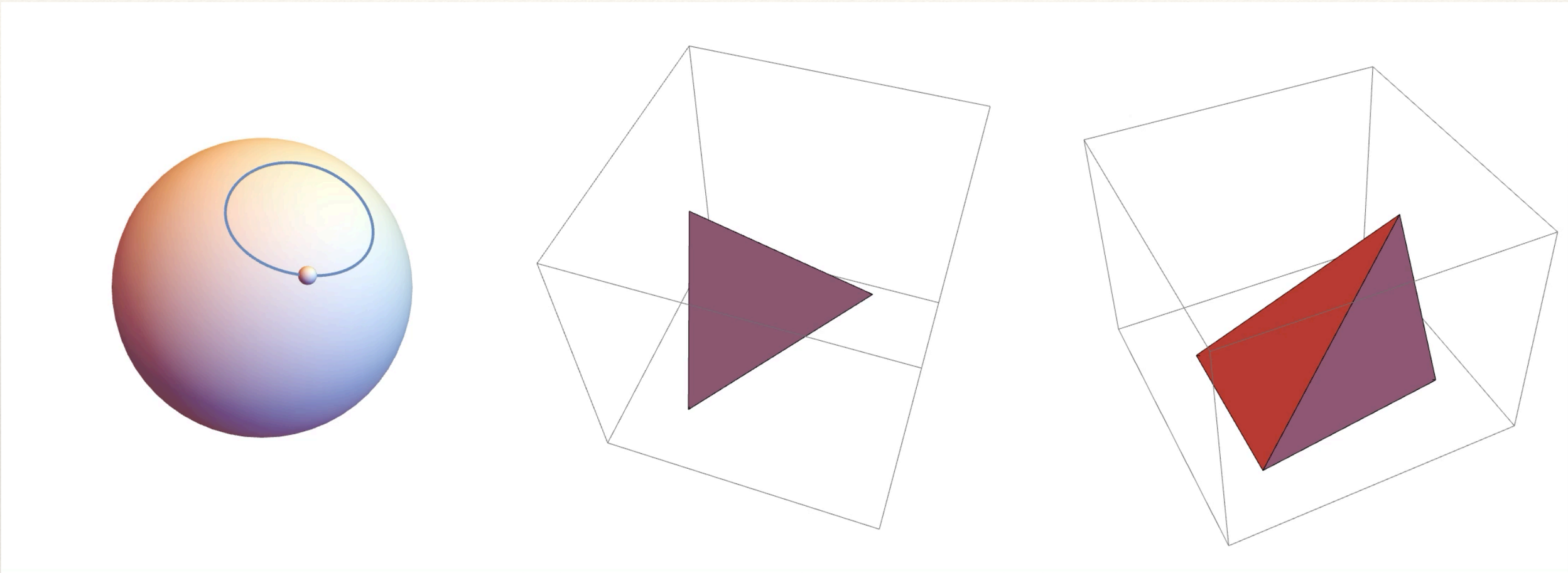



$$\bar{r}_1^2 = (A_1 - A_2)^2$$
$$\bar{r}_2^2 = (A_3 - A_4)^2$$
$$\bar{r}_3^2 = (A_1 + A_2)^2$$
$$\bar{r}_4^2 = (A_3 + A_4)^2$$

$$\frac{dA^2}{d\lambda} = \frac{1}{9}\sqrt{(4\Delta)^2(4\bar{\Delta})^2 - A^2(18Q)^2}$$
$$= \frac{1}{9}\sqrt{[A^2 - \bar{r}_1^2][A^2 - \bar{r}_2^2][\bar{r}_3^2 - A^2][\bar{r}_4^2 - A^2] - A^2(18Q)^2}$$

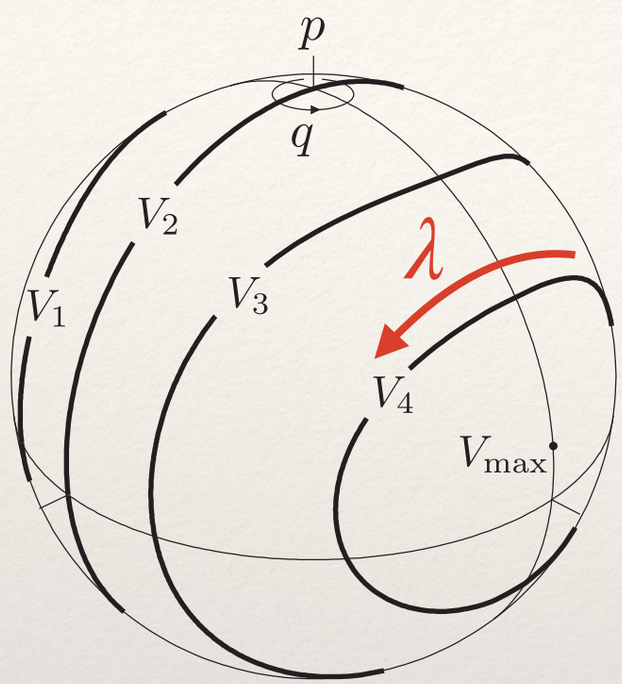
The solutions, $A^2(\lambda)$, are ratios of Jacobi elliptic functions.

The flow along the curves of constant $V$ is integrable and describes a family of tetrahedra with different shapes, but equal volumes



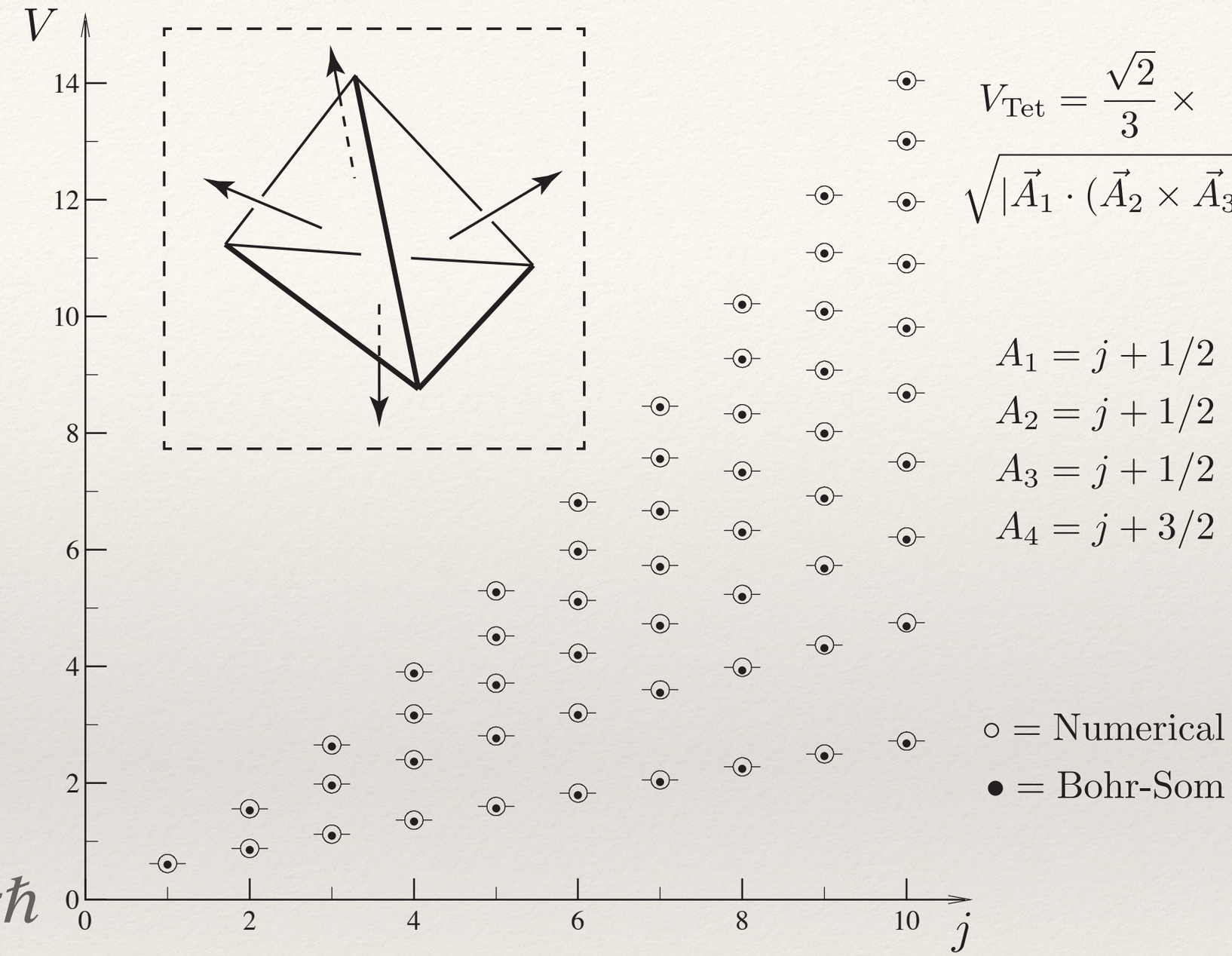
At left: the phase space of shapes with a constant volume contour; at right: two different views of the same tetrahedron as it undergoes the volume flow (click to play)

# Our first result was a semiclassical quantization of the volume eigenvalues:




$$a(V_n) = -\oint \phi dA$$

$$= \left(n + \frac{1}{2}\right) 2\pi\hbar$$

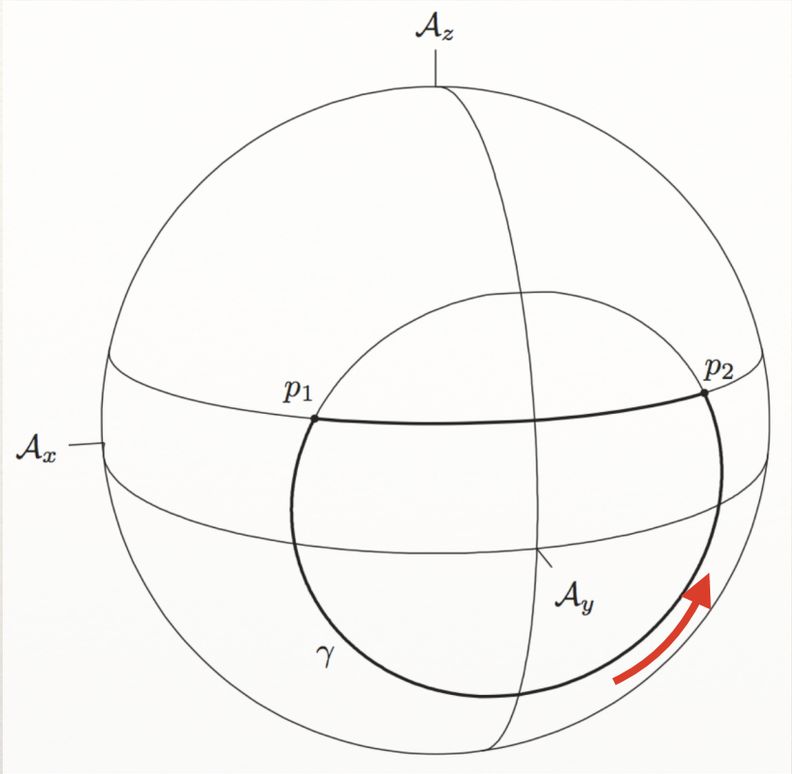


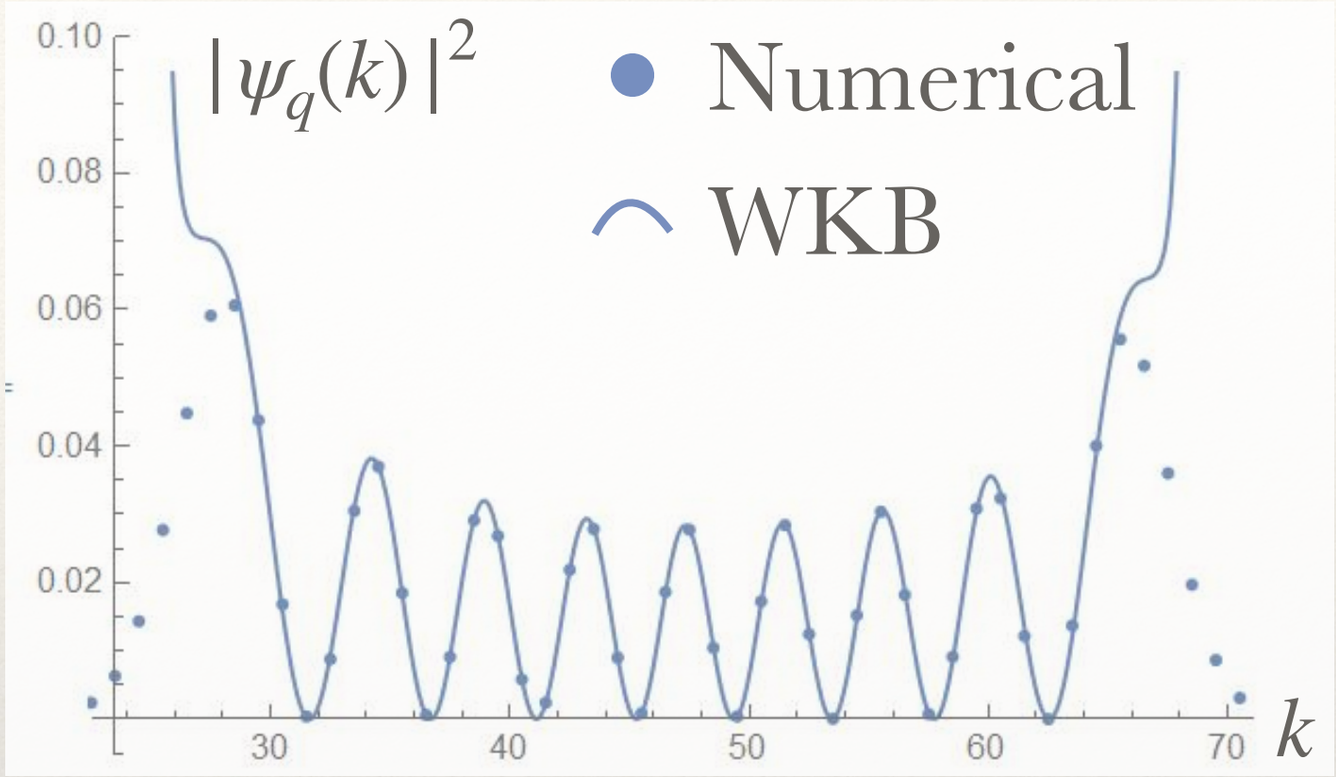

$$V_{\text{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$

Here the Bohr-Sommerfeld values (solid dots) are compared to the numerical eigenvalues from a full quantum treatment (open circles). More on the quantum treatment below…

A new result that I can share with you today is that Antu and I have been able to find the WKB wave functions too




$$\psi_q(k) = \sqrt{\frac{4A}{gK(m)}} \frac{\eta}{|[(16\Delta\bar{\Delta})^2 - (18AQ)^2]^{1/4}|} \cos\left(\frac{1}{2}(\Delta S_q - \Delta S_A) - \frac{\pi}{4}\right)$$

where

$$S_{q,m} = 18gq\left[\frac{\lambda_{p_m}}{9g} - \sum_i \left(\frac{r_1}{r_1 - \bar{r}_i}\lambda_{p_m} - \frac{\bar{r}_i(r_2 - r_1)}{(r_2 - \bar{r}_i)(r_1 - \bar{r}_i)}\Pi\left(\alpha_i^2, \mathrm{am}\left(\frac{\lambda_{p_m}}{9g}, m\right), m\right)\right)\right]$$

## *This Talk*

Asymptotic resurgence is impressively broad and impactful. To the best of my knowledge the perturbative/non-perturbative relations found in resurgence have yet to be applied to quantum gravity.

I show that these tools shed light on the quantization of the simplest grain of space, a quantum tetrahedron. The talk has 3 parts:

1. Resurgence and Perturbative/Non-Perturbative Relations
   (I draw heavily on the outstanding introductions by C. Howls, A.O. Daalhuis, and G. Dunne from the ARA School)
2. Grains of Quantum Space: the area geometry of tetrahedra, their classical phase space, and semiclassical quantization
3. Perturbative/Non-Perturbative Relations in the Quantization of the Volume of a Grain of Space

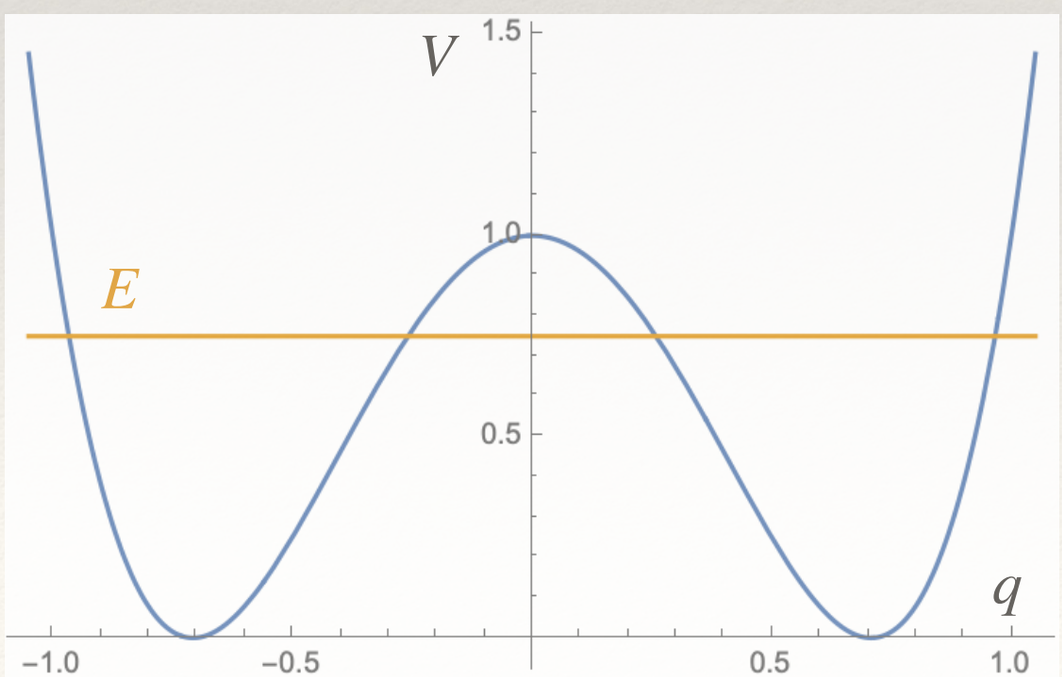
**Constructive Resurgence** — There is a remarkable second version of resurgence, often called constructive resurgence

Results are detailed and explicit, but largely restricted to phase spaces representable by (genus 1) Riemann surfaces

## Canonical Example

$V(q) = (2q^2 - 1)^2$, $p = \sqrt{2m[E - V]}$

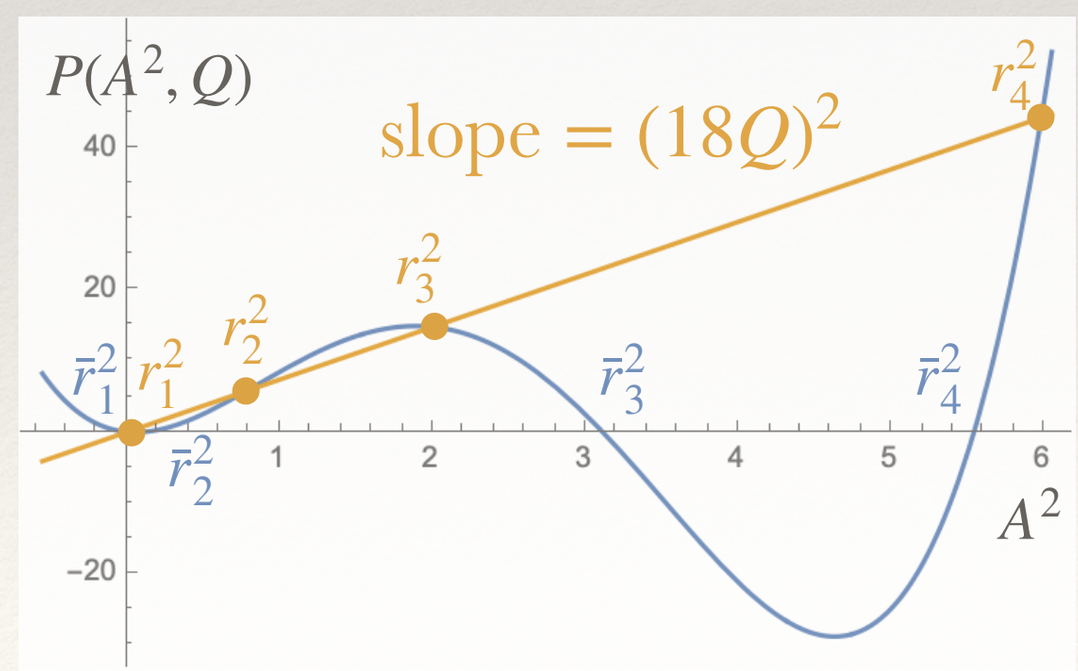
$$p = \sqrt{2m[E - (2q^2 - 1)^2]}$$




## Volume Evolution Example

$$\frac{dA^2}{d\lambda} = \frac{1}{9}\sqrt{(4\Delta)^2(4\bar{\Delta})^2 - A^2(18Q)^2}$$

$$= \frac{1}{9}\sqrt{[A^2 - \bar{r}_1^2][A^2 - \bar{r}_2^2][\bar{r}_3^2 - A^2][\bar{r}_4^2 - A^2] - A^2(18Q)^2}$$



# In both examples we can view the phase space as a genus 1 torus by complexifying variables and gluing along branch cuts

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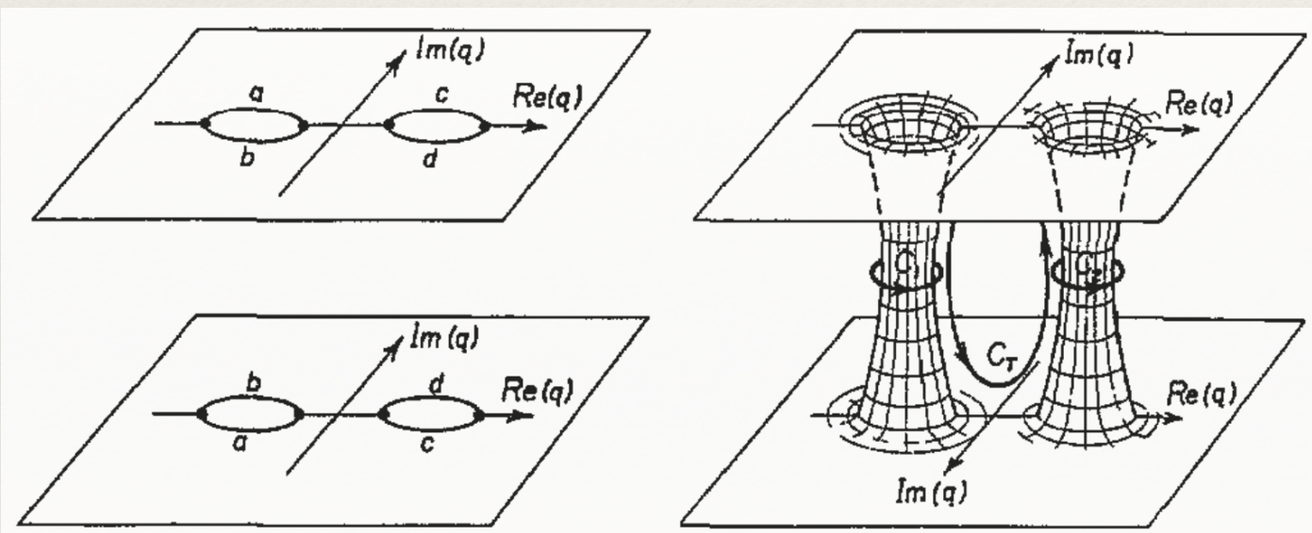


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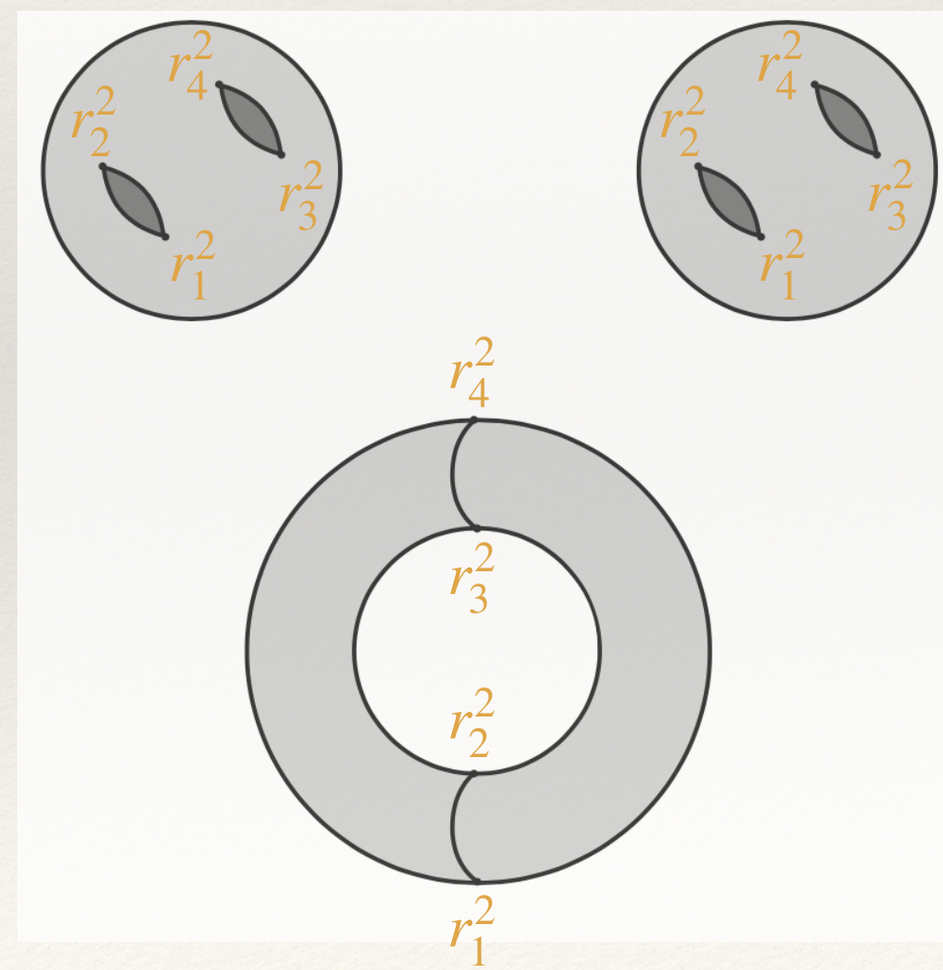

# In both examples we can view the phase space as a genus 1 torus by complexifying variables and gluing along branch cuts

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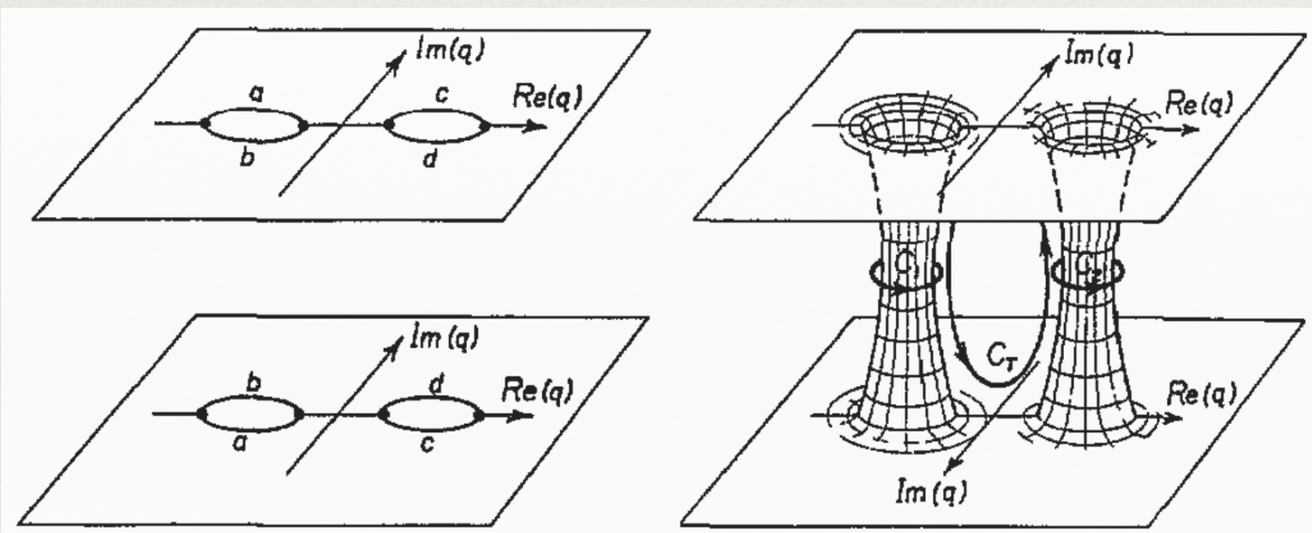


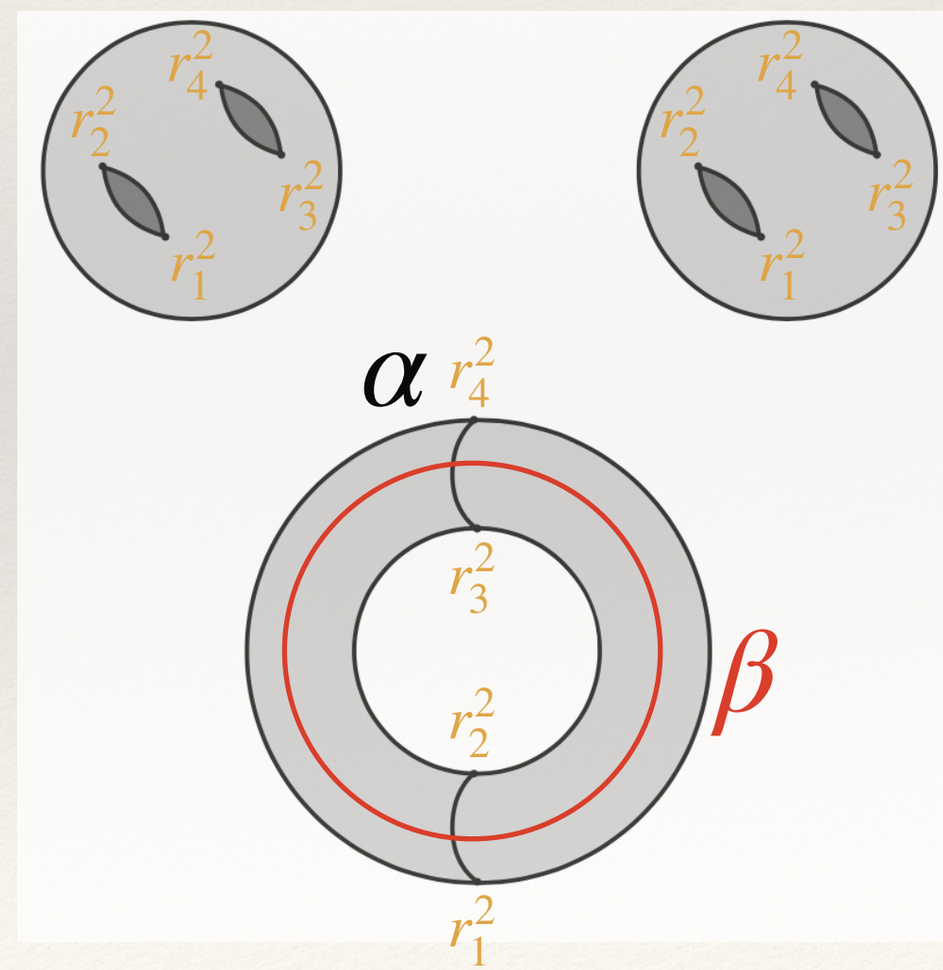

Bohr-Somm: $a = \oint_\alpha pdq = \left(n + \frac{1}{2}\right) 2\pi\hbar$

Tunneling-splitting: $a^D = \oint_\beta pdq$

## Volume Evolution Example

$$\frac{dA^2}{d\lambda} = \frac{1}{9}\sqrt{(4\Delta)^2(4\bar{\Delta})^2 - A^2(18Q)^2}$$

$$= \frac{1}{9}\sqrt{[A^2 - \bar{r}_1^2][A^2 - \bar{r}_2^2][\bar{r}_3^2 - A^2][\bar{r}_4^2 - A^2] - A^2(18Q)^2}$$



We can now borrow a nice result from algebraic topology: the # of independent cycles = # of independent 1-forms

A 1D complex manifold with genus $g$ and $\mathfrak{p}$ punctures, has

$$\#\text{ indep cycles} = 2g + 2\mathfrak{p} - 1.$$

Independence is up to an exact form. Taking $m = 1$, let

$$\theta(q, E) \equiv \; pdq = \sqrt{2[E - V(q)]},$$

then only $2g + 2p - 1$ of $\theta$ and

$$\theta_1 = \partial_E pdq = \frac{1}{\sqrt{2[E - V(q)]}} dq, \;\; \theta_2 \equiv \partial_E^2 pdq, \;\; \ldots \;\; , \theta_k \equiv \partial_E^k pdq$$

are independent. Express the dependency as $\sum_k c_k \theta_k(E) = df$,

$$\sum_k c_k \oint_\gamma \theta_k = \oint df = 0 \qquad \Longrightarrow \qquad c_k \partial_E^k a = 0 \quad \text{(Picard-Fuchs Eqn)}$$

For the symmetric quartic well this Picard-Fuchs equation is

$$E(1-E)\frac{d^2a}{dE^2} - \frac{3}{16}a = 0.$$

This equation captures the fact that as we vary $E$, the turning points, and hence the actions, vary in a predictable way.

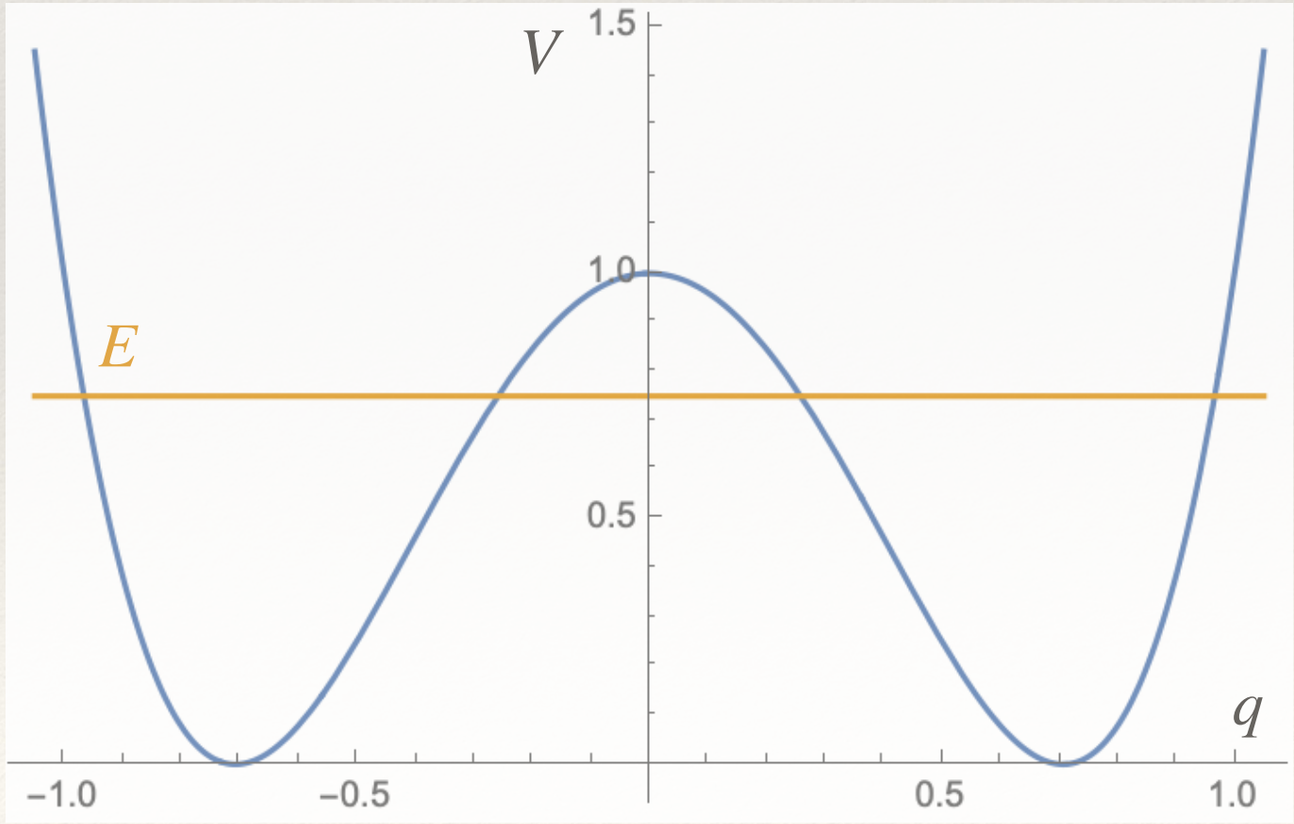
Both $a$ and $a^D$, which only differ by their integration cycles, satisfy this equation. Further, it implies a Wronskian relation:

$$a\partial_E a^D - a^D\partial_E a = \frac{4}{3}\pi i$$

or

$$a\,\omega^D - a^D\omega = \frac{4}{3}\pi i.$$

A connection formula between $a$ and $a^D$.

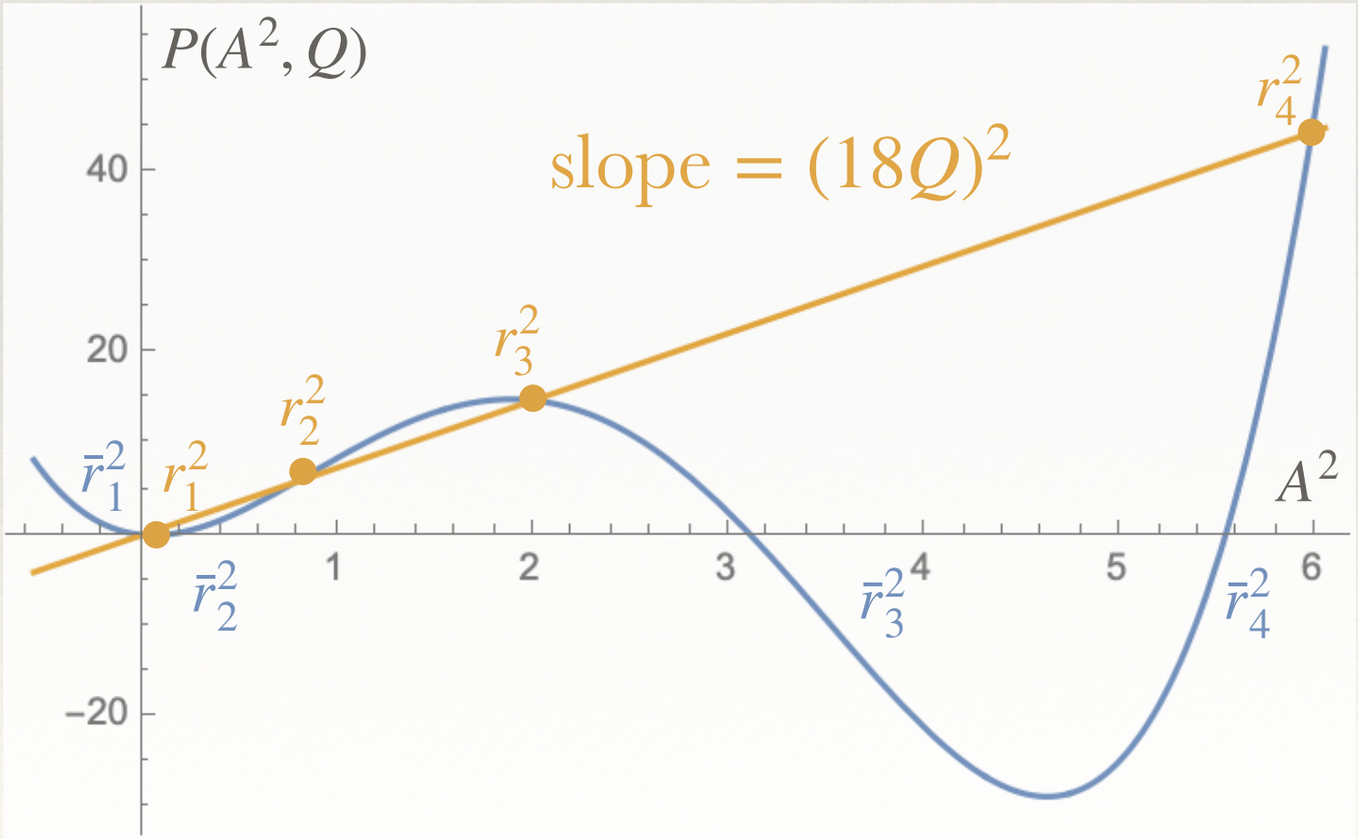


Antu and I have found an explicit 3rd order Picard-Fuchs equation for the volume evolution

$$D(r_i, Q)\frac{d^3a}{dQ^3} + c_2(r_i, Q)\frac{d^2a}{dQ^2} + c_1(r_i, Q)\frac{da}{dQ} = 0.$$

Here $D(r_i, Q)$ is the quartic discriminant, an 8th order polynomial in $Q$ that vanishes iff $r_i$ coalesce and $c_1$ and $c_2$ are respectively of 10th and 12th order in Q.

The evident constant solution has a nice explanation: it is due to punctures in the Riemann surface.



# The Picard-Fuchs coefficients are below: here $Q_b = 18Q$, $s_0 = r_1 r_2 r_3 r_4$, … , $s_3 = r_1 + r_2 + r_3 + r_4$ are symm root functions

```
D = (4 (Qb^2 - s1) (12 s0 + s2^2) + (3 (Qb^2 - s1)^2 + 32 s0 s2) s3 + (-Qb^2 + s1) s2 s3^2 - 9 s0 s3^3)
   (27 Qb^8 - 256 s0^3 - 108 Qb^6 s1 + 162 Qb^4 s1^2 - 108 Qb^2 s1^3 + 27 s1^4 - 144 Qb^4 s0 s2 + 288 Qb^2 s0 s1 s2 - 144 s0 s1^2 s2 + 128 s0^2 s2^2 + 4 Qb^4 s2^3 -
     8 Qb^2 s1 s2^3 + 4 s1^2 s2^3 - 16 s0 s2^4 + 2 (Qb^2 - s1) (9 (Qb^2 - s1)^2 s2 - 8 s0 (12 s0 + 5 s2^2)) s3 +
     (6 s0 (Qb^2 - s1)^2 - 144 s0^2 s2 - (Qb^2 - s1)^2 s2^2 + 4 s0 s2^3) s3^2 - 2 (Qb^2 - s1) (2 (Qb^2 - s1)^2 - 9 s0 s2) s3^3 + 27 s0^2 s3^4)

c2 =
 (18 (243 Qb^12 s3 + 3 Qb^10 (180 (12 s0 + s2^2) - 378 s1 s3 - 27 s2 s3^2 - 4 s3^4) +
      3 Qb^8 (-684 s1 (12 s0 + s2^2) + (675 s1^2 + 3024 s0 s2 + 68 s2^3) s3 + 117 s1 s2 s3^2 - 3 (255 s0 + 11 s2^2) s3^3 + 12 s1 s3^4 + 4 s2 s3^5) +
      Qb^6 (-8 (12 s0 + s2^2) (-351 s1^2 + 72 s0 s2 - 2 s2^3) - 12 s1 (135 s1^2 + 2160 s0 s2 + 44 s2^3) s3 + 2 (1008 s0^2 - 297 s1^2 s2 + 1860 s0 s2^2 - 4 s2^4) s3^2 +
          4 s1 (1617 s0 + 65 s2^2) s3^3 + (-24 s1^2 - 1618 s0 s2 + s2^3) s3^4 - 32 s1 s2 s3^5 + 180 s0 s3^6) +
      Qb^4 (-8 s1 (12 s0 + s2^2) (189 s1^2 - 72 s0 s2 + 2 s2^3) + (405 s1^4 + 8 (1632 s0^3 + 2916 s0 s1^2 s2 - 1392 s0^2 s2^2 + 45 s1^2 s2^3 + 118 s0 s2^4)) s3 +
          2 s1 (243 s1^2 s2 + 4 (-540 s0^2 - 873 s0 s2^2 + s2^4)) s3^2 - 2 (2817 s0 s1^2 - 2784 s0^2 s2 + 93 s1^2 s2^2 + 208 s0 s2^3) s3^3 -
          s1 (24 s1^2 - 2994 s0 s2 + s2^3) s3^4 + 3 (8 s1^2 s2 + 3 s0 (-63 s0 + 5 s2^2)) s3^5 - 324 s0 s1 s3^6) -
     (s0 (48 s1 - 32 s2 s3 + 9 s3^3) - s1 (3 s1 s3 + s2 (-4 s2 + s3^2)))
      (256 s0^3 - s0^2 (128 s2^2 + 192 s1 s3 - 144 s2 s3^2 + 27 s3^4) + 2 s0 (s1^2 (72 s2 - 3 s3^2) + 2 s2^3 (4 s2 - s3^2) + s1 s2 s3 (-40 s2 + 9 s3^2)) +
          s1^2 (-27 s1^2 + s2^2 (-4 s2 + s3^2) + 2 s1 (9 s2 s3 - 2 s3^3))) +
     Qb^2 (36 864 s0^4 - 48 s0^3 (320 s2^2 + 480 s1 s3 - 288 s2 s3^2 + 45 s3^4) +
          s1^2 (162 s1^3 s3 + 12 s1 s2^2 s3 (4 s2 - s3^2) - s2^3 (-4 s2 + s3^2)^2 + 9 s1^2 (12 s2^2 - 21 s2 s3^2 + 4 s3^4)) +
          s0^2 (864 s1^2 (8 s2 + 3 s3^2) + 18 s1 (128 s2^2 s3 - 192 s2 s3^3 + 33 s3^5) + s2 (768 s2^3 - 1024 s2^2 s3^2 + 540 s2 s3^4 - 81 s3^6)) +
          2 s0 (648 s1^4 + 6 s2^4 (-4 s2 + s3^2)^2 + s1^3 (-2592 s2 s3 + 522 s3^3) + s1 s2^2 s3 (-592 s2^2 + 256 s2 s3^2 - 27 s3^4) +
              3 s1^2 (64 s2^3 + 468 s2^2 s3^2 - 189 s2 s3^4 + 18 s3^6)))))

c1 =
 (972 Qb^2 (80 (Qb^2 - s1)^3 (12 s0 + s2^2) + 8 (128 s0^3 + 3 (Qb^2 - s1)^4 + 168 s0 (Qb^2 - s1)^2 s2 - 64 s0^2 s2^2 + 2 (Qb^2 - s1)^2 s2^3 + 8 s0 s2^4) s3 -
      8 (Qb^2 - s1) (-24 s0^2 + 3 (Qb^2 - s1)^2 s2 - 50 s0 s2^2) s3^2 - 8 (42 s0 (Qb^2 - s1)^2 - 32 s0^2 s2 + (Qb^2 - s1)^2 s2^2 + 4 s0 s2^3) s3^3 +
      (Qb^2 - s1) ((Qb^2 - s1)^2 - 184 s0 s2) s3^4 + ((Qb^2 - s1)^2 s2 + 4 s0 (-6 s0 + s2^2)) s3^5 + 21 s0 (Qb^2 - s1) s3^6)) /
  ((4 (Qb^2 - s1) (12 s0 + s2^2) + (3 (Qb^2 - s1)^2 + 32 s0 s2) s3 + (-Qb^2 + s1) s2 s3^2 - 9 s0 s3^3)
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```

Why is, even such a messy, Picard-Fuchs equation of interest?

These equations allow us to express the all orders in $\hbar$ quantization conditions of a genus-1 phase space in terms of the lowest order actions, i.e solutions of the Picard-Fuchs eqn.

The proof is delicate, but quite elegant [Basar, Dunne, Ünsal]:

1. The classical actions $a_0$ and $a_0^D$ are solutions to the Picard-Fuchs equation. They are related via a Wronskian condition. Hence ∃ only 1 indep classical action function on the torus.

2. WKB analysis can be extended to all orders in $\hbar$:

$$a(E,\hbar) = \sqrt{2}\left(\oint_\alpha \sqrt{E-V}dq - \frac{\hbar^2}{2^6}\oint_\alpha \frac{(V')^2}{(E-V)^{5/2}}dq - \frac{\hbar^4}{2^{13}}\oint_\alpha \left(\frac{49(V')^4}{(E-V)^{11/2}} - \frac{16V'V''}{(E-V)^{7/2}}\right)\right) + \cdots$$

The integrands are rational functions of $q$ and $p = \sqrt{\text{quartic}}$.

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The integrands are rational functions of $q$ and $p = \sqrt{\text{quartic}}$.

For genus 1 systems, the classical action and all higher order actions can be expressed in terms of the complete elliptic integrals: $\mathbb{K}$, $\mathbb{E}$, $\Pi$, which are closed under differentiation.

Hence the higher order actions $a_\ell(E) = \mathscr{D}_E^\ell a_0(E)$, for some $\mathscr{D}_E^\ell$.

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Repeated use of the Picard-Fuchs eqn allows us to reduce the order of this differential operator until:

$$a_\ell(E) = f_\ell^{(2)}(E)\frac{d^2a_0}{dE^2} + f_\ell^{(1)}(E)\frac{da_0}{dE} + f_\ell^{(0)}(E)a_0$$

# We have found explicit modular relations between the actions

The closed action integrals in the classically allowed region and the classically forbidden regions ($A^2 > r_3$ and $A^2 < r_2$) are:

$$a_1 = 18gQ\left(\left[1 - \sum_{i=1}^{4}\frac{r_1}{r_1 - \bar{r}_i}\right]K(m) - \sum_{i=1}^{4}\left[\frac{r_2}{r_2 - \bar{r}_i} - \frac{r_1}{r_1 - \bar{r}_i}\right]\Pi(\alpha_i^2, m)\right)$$

$$a_2^D = \frac{18gQ}{i}\left(\left[1 - \sum_{i=1}^{4}\frac{r_2}{r_2 - \bar{r}_i}\right]K(m_f) - \sum_{i=1}^{4}\left[\frac{r_3}{r_3 - \bar{r}_i} - \frac{r_2}{r_2 - \bar{r}_i}\right]\Pi(\alpha_{if}^2, m_f)\right)$$

$$a_3^D = \frac{18gQ}{i}\left(\left[1 - \sum_{i=1}^{4}\frac{r_3}{r_3 - \bar{r}_i}\right]K(m_f) - \sum_{i=1}^{4}\left[\frac{r_2}{r_2 - \bar{r}_i} - \frac{r_3}{r_3 - \bar{r}_i}\right]\Pi(\alpha_{iF}^2, m_f)\right)$$

$$\alpha_i^2 = \frac{(r_3 - r_2)(r_1 - \bar{r}_i)}{(r_3 - r_1)(r_2 - \bar{r}_i)}; \qquad \alpha_{if}^2 = \frac{(r_4 - r_3)(r_2 - \bar{r}_i)}{(r_4 - r_2)(r_3 - \bar{r}_i)}; \qquad \alpha_{iF}^2 = \frac{(r_2 - r_1)(r_3 - \bar{r}_i)}{(r_3 - r_1)(r_2 - \bar{r}_i)}$$

$$m = \frac{(r_3 - r_2)(r_4 - r_1)}{(r_4 - r_2)(r_3 - r_1)}; \qquad m_f = \frac{(r4 - r3)(r2 - r1)}{(r4 - r2)(r3 - r1)}$$

Using $\alpha_{if}^2\alpha_{iF}^2 = m_f$ and $\Pi(\alpha^2, m) = K(m) - \Pi(m/\alpha^2, m)$, one can show $a_2^D = a_3^D$.

Much still remains to be understood: the quantum equation underlying this geometry is rich, and differs from the Schrödinger equation (c.f. Bianchi, HMH 2011 for derivation)

The matrix elements of the $\hat{Q}$ operator in a recoupling basis satisfy the recursion relation

$$a_{k+1}\langle k+1\,|\,q\rangle - a_k\langle k-1\,|\,q\rangle + iq\langle k\,|\,q\rangle = 0$$

with

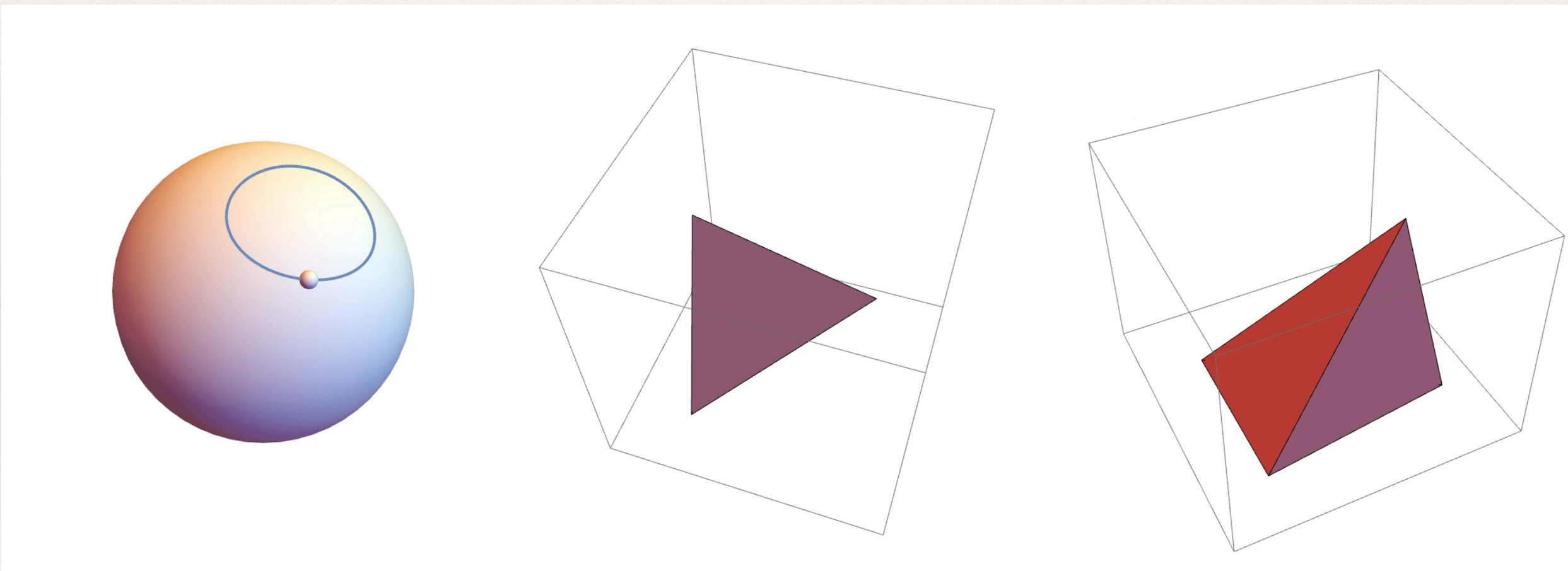
$$a_k = \frac{1}{8\sqrt{k-\frac{1}{2}}\sqrt{k+\frac{1}{2}}}\left(\sqrt{[(j_1+j_2+1)^2-k^2][k^2-(j_1-j_2)^2]}\sqrt{[(j_3+j_4+1)^2-k^2][k^2-(j_3-j_4)^2]}\right).$$

This can be viewed as a difference equation, which in the semiclassical limit gives a differential equation

$$\frac{\Delta^2\Phi_k^q}{\Delta k^2} + 2\frac{\Delta\Phi_k^q}{\Delta k} + \frac{iqK}{2\Delta(K,J_1,J_2)\Delta(K,J_3,J_4)}\Phi_k^q = 0.$$

At lowest order this is in excellent agreement with $a = -\oint \phi dA$, but both the coefficients and the wave function contribute at all orders in $\hbar$!

Quantization of geometry provides a remarkable laboratory for understanding resurgent perturbative/non-perturbative relations and, due to the richness of its underlying quantum structure, may even require extensions of this formalism.



Thank you!