The polyhedral picture is important for understanding loop gravity

- crucial to the dynamics of spin foams

- for fixed-graph Hilbert space the semiclassical limit is a collection of flat polyhedra

  &

  their extrinsic geometry is encoded in the Ashtekar-Barbero holonomy

- generalizes Regge calculus: tetrahedra $\rightsquigarrow$ polyhedra, & discontinuity is allowed; $\exists$ shape mismatch in the general case

Many interesting directions to explore...
Bianchi, Doná, and Speziale numerical reconstruction '11

Bianchi and HMH Bohr-Sommerfeld tetrahedral volume spectrum '11

HMH, Han, Kamiński, Riello curved tetrahedron reconstruction '16

HMH described full phase diagram of pentahedral adjacencies '13
Beautiful connection with quantum groups, Λ, and CS theory, maybe important for renormalization [Dittrich]

Projective perspective may be key to proving a generalized Minkowski theorem for curved polyhedra [Dupuis, Girelli, Livine, HHKR]

**Volumes from fluxes:**

Lasserre’s algorithm allows reconstruction of polyhedra (adjacency, edge lengths, volume) from the normals and heights wrt a ref pt;

Bianchi-Doná-Speziale: heights can be inverted for the areas numerically, to complete the reconstruction from the fluxes:

$$A_i(h, n) = \sum_{j,k=1}^{F} M_{ij}^{jk}(n_1, \ldots, n_F)h_j h_k$$

Can we do better, and have an analytic reconstruction procedure?

**Our central observation:** adjacency is projective
Outline

I. Classical Results in Projective Geometry

II. Analytic Polyhedral Adjacency

III. Projective Varieties and Spaces
Projective geometry uses only a straightedge in constructions.

Girard Desargues’ founding work will be important to us.

The heart of theorems is thus about incidence: points lying on lines and lines intersecting in points.
All pairs of lines in a projective plane meet in a point; sometimes a point at infinity

Harold and the Purple Crayon

And he set off on his walk, taking his big purple crayon with him.

The line of points at infinity we call the horizon

A special circumstance: three lines incident on a point
Two triangles in the plain are perspective from a point if...

...∃ a pairing of vertices s.t. lines through the pairs are incident.
Desargues’ Theorem: Two triangles perspective from a point are also perspective from a line.

(See animation)
The fundamental projective invariant is the cross ratio

\[ \rho = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_2 - x_3)(x_1 - x_4)} \]

Using triangle areas, can prove

\[ \rho = \frac{\sin(x_1Px_3)\sin(x_2Px_4)}{\sin(x_2Px_3)\sin(x_1Px_4)} \]

The latter implies immediately that \( \rho = \rho' \). Convenient to write

\[ \rho = (x_1, x_2; x_3, x_4) \overset{P}{=} (x'_1, x'_2; x'_3, x'_4) \]
Proof of Desargues’ theorem using cross ratios

Let \( U = BA \cap YX \) etc.

\[
(W, V; Q, BA \cap WV) \overset{B}{=} (W, C; N, A) \\
\overset{P}{=} (W, Z; M, X) \overset{Y}{=} (W, V; Q, YX \cap WV)
\]
[Brief aside...]

Some permutations of the four points change the cross ratio

\[
(x_1, x_2; x_3, x_4) = \rho \quad (x_1, x_2; x_4, x_3) = \frac{1}{\rho}
\]

\[
(x_1, x_3; x_4, x_2) = \frac{1}{1 - \rho} \quad (x_1, x_3; x_2, x_4) = 1 - \rho
\]

\[
(x_1, x_4; x_3, x_2) = \frac{\rho}{\rho - 1} \quad (x_1, x_4; x_2, x_3) = \frac{\rho - 1}{\rho}
\]

These permutations form a group called the **anharmonic group** — the other permutations leave it invariant...]
For some time no synthetic, constructive proof was known...

...brings us back to our story and loop gravity
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**Minkowski’s theorem:** The areas $A_l$ and the unit-normals $\vec{n}_l$ to the faces of a convex polyhedron fully characterize its shape. Let $\vec{A}_l = A_l \vec{n}_l$, then the *space of shapes of polyhedra* with $F$ faces of given areas $A_l$ is

$$
S_{KM} = \{ \vec{A}_l, l = 1 \ldots F | \sum_l \vec{A}_l = 0, \| \vec{A}_l \| = A_l \}/ISO(3)
$$

$$
\vec{A}_1 + \cdots + \vec{A}_n = 0 \iff
$$

Existence and uniqueness thm $\sim$ nothing to say about construction
A pentahedron can be completed to a tetrahedron

Define $\alpha$ as the ratio of the tetrahedron’s first face area, $A_{1\text{tet}}$, to the pentahedron’s first face area $A_1$, i.e.

$$A_{1\text{tet}} = \alpha A_1$$

and similarly for $\beta$ and $\gamma$
A pentahedron can be completed to a tetrahedron

\[ \alpha \vec{A}_1 + \beta \vec{A}_2 + \gamma \vec{A}_3 + \vec{A}_4 = 0 \]

\[ \text{e.g. } \Rightarrow \alpha = -\frac{\vec{A}_4 \cdot (\vec{A}_2 \times \vec{A}_3)}{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)} \]
Let \( W_{ijk} = \vec{A}_i \cdot (\vec{A}_j \times \vec{A}_k) \). Different closures imply,

- **Case 1: 54-pentahedron**
  \[
  \alpha_1 \vec{A}_1 + \beta_1 \vec{A}_2 + \gamma_1 \vec{A}_3 + \vec{A}_4 = 0,
  \]
  \[
  \gamma_1 = -\frac{W_{124}}{W_{123}}
  \]

- **Case 2: 53-pentahedron**
  \[
  \alpha_2 \vec{A}_1 + \beta_2 \vec{A}_2 + \vec{A}_3 + \gamma_2 \vec{A}_4 = 0,
  \]
  \[
  \gamma_2 = -\frac{W_{123}}{W_{124}} = \frac{1}{\gamma_1}
  \]

**Require \( \alpha, \beta, \gamma > 1 \): These cases are mutually incompatible!**
A representative sample of the 20 cases:

1. \( \alpha_1 \equiv \alpha \quad \beta_1 \equiv \beta \quad \gamma_1 \equiv \gamma \)  
2. \( \alpha_2 = \frac{\alpha}{\gamma} \quad \beta_2 = \frac{\beta}{\gamma} \quad \gamma_2 = \frac{1}{\gamma} \)  
3. \( \alpha_3 = \frac{\alpha}{\beta} \quad \beta_3 = \frac{\gamma}{\beta} \quad \gamma_3 = \frac{1}{\beta} \)  
4. \( \alpha_4 = \frac{\beta}{\alpha} \quad \beta_4 = \frac{\gamma}{\alpha} \quad \gamma_4 = \frac{1}{\alpha} \)  
5. \( \alpha_5 = 1 - \alpha \quad \beta_5 = 1 - \beta \quad \gamma_5 = 1 - \gamma \)  
6. \( \alpha_6 = \frac{1-\alpha}{1-\gamma} \quad \beta_6 = \frac{1-\beta}{1-\gamma} \quad \gamma_6 = \frac{1}{1-\gamma} \)  
7. \( \alpha_7 = \frac{1-\alpha}{1-\beta} \quad \beta_7 = \frac{1-\gamma}{1-\beta} \quad \gamma_7 = \frac{1}{1-\beta} \)  
8. \( \alpha_8 = \frac{1-\beta}{1-\alpha} \quad \beta_8 = \frac{1-\gamma}{1-\alpha} \quad \gamma_8 = \frac{1}{1-\alpha} \)  
9. \( \alpha_9 = \frac{\gamma-\alpha}{\gamma} \quad \beta_9 = \frac{\gamma-\beta}{\gamma} \quad \gamma_9 = \frac{\gamma-1}{\gamma} \)  
10. \( \alpha_{10} = \frac{\gamma-\alpha}{\gamma-1} \quad \beta_{10} = \frac{\gamma-\beta}{\gamma-1} \quad \gamma_{10} = \frac{\gamma}{\gamma-1} \)  
11. ... 20.

N.B. anharmonic group appears for \( \gamma \) in cases 1, 2, 5, 6, 9, & 10

[For defs of cases see arXiv:1211.7311]
Requiring: \[ \beta > \alpha > \gamma > 1 \quad \text{and} \quad \gamma \geq \frac{\alpha \beta}{(\alpha + \beta - 1)} \]
guarantees that the 54 pentahedron is constructible.
Closely related to a synthetic proof of Desargues’ theorem

Corresponding to every pentahedron is a unique Desargues configuration up to projective transformations
Desargues configurations are quite symmetrical—every pt of the config is a pt of perspectivity for a pair of triangles in the config

Coxeter’s Theorem: The moduli space of Desargues configurations is captured by five parameters $\lambda_\alpha$ such that

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0, \quad \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \neq 0$$

with all 20 cross ratios of the configuration given by $\rho_{\alpha,\beta} = -\frac{\lambda_\alpha}{\lambda_\beta}$
A small gap in pentahedral adjacency $\iff$ Desargues remains

Certainly we can take

$$\lambda_1 = W_{234}, \quad \lambda_2 = -W_{134}, \quad \lambda_3 = W_{124}, \quad \lambda_4 = W_{123},$$

and $\lambda_5 = -\sum_{\alpha=1}^{4} \lambda_\alpha$

and we will have a Desargues configuration with the same cross ratios as in the 54-pentahedral construction...

...but, we have not yet succeeded in proving that this is the Desargues config resulting upon projection of the pentahedron

We expect a clear correspondence and are working to close this gap
What about polyhedra with more faces?

Two dominant hexahedral classes: cuboids and pentagonal wedges

Can draw several lessons from the pentahedral case:
Cuboids occupy isolated islands surrounded by pentagonal wedges

cuboid adj = 5!! = 15  and  pent wedge adj = \( \binom{6}{2} \binom{4}{2} \cdot 2 = 15 \cdot 12 \)

Scaling procedures from the pentahedral work of ’13 apply, but also there is a projective configuration, the Steiner-Plücker config
But how general is this really? We do not know yet, but...

Collect the $F$ area vectors $\vec{A}_i$ into a $3 \times F$ matrix

$$A = \begin{pmatrix} \vec{A}_1 & \cdots & \vec{A}_F \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \vdots \end{pmatrix}$$

**Weyl’s Theorem:** Every projectively invariant polynomial of the components of $A$ can be expressed as a polynomial in the $3 \times 3$ sub-determinants of $A$.

Lasserre’s algorithm is not polynomial in the components of the area vectors—our approach is in the case of pentahedra

Motivates looking for a general argument that adjacency is polynomial in the area vector components $\rightsquigarrow$ look closely at $W_{ijk}$
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Connections with projective geometry were already highlighted in nice papers by Freidel, Krasnov, & Livine; [e.g. 1005.2090]

Using spinors they showed explicitly the isomorphism:

\[ PC^n \times S_{KM} \cong Gr_{\mathbb{C}}(2, n) \]

A projective variety: embedded in \( \mathbb{C}^n \wedge \mathbb{C}^n \) via spinor bilinears \( F_{ij} = z_i^0 z_j^1 - z_j^0 z_i^1 \) and Plücker relations \( F_{ij} F_{kl} - F_{ik} F_{jl} + F_{il} F_{jk} = 0 \)

Questions:

- What is the precise relation btwn the complex Grassmannian projectivity and the real projectivity of the adjacencies?
- Can we analytically reconstruct the adjacency using projective geometry, either the complex one of the \( F \)'s, or the real one of the \( W_{ijk} \)'s?
Can the $W_{ijk}$ be taken as coordinates for the space of shapes?

Consider pentahedral case for proof of principle test:

Some notation

\[
\vec{A}_{ij} = \vec{A}_{ji} = \vec{A}_i + \vec{A}_j, \\
W_{ijk} = \vec{A}_i \cdot [\vec{A}_j \times \vec{A}_k], \\
\vec{B}_{ij} = \vec{A}_i \times \vec{A}_j, \quad \vec{B}_{(kl)(ij)} = \vec{A}_{kl} \times \vec{A}_{ij}
\]

Then

\[
W_{ijk} = \vec{B}_{ij} \cdot \vec{B}_{(kl)(ij)} \frac{(\vec{A}_{kl} \cdot \vec{A}_k)}{A_{kl}^2} \sin \theta_{ij} + \frac{\vec{B}_{ij} \cdot \vec{B}_{kl}}{A_{kl}} [\cos \theta_{ij} \sin \theta_{kl} + \frac{\vec{A}_{ij} \cdot \vec{A}_k}{A_{ij} A_{kl}} \sin \theta_{ij} \cos \theta_{kl}]
\]
Could be just good luck; already for hexahedra there are 6 KM variables, but 10 $W$’s (after using closure)—seems hopeless...

But, in $\mathbb{R}^3$ there must be linear dependencies amongst 4 or more vectors, e.g. $\sum_{i=1}^{4} a_i \vec{A}_i = 0$, solve for $a_i$ to find

$$W_{234} \vec{A}_1 - W_{134} \vec{A}_2 + W_{124} \vec{A}_3 - W_{123} \vec{A}_4 = 0$$

and dot in $\{\vec{A}_1 \times \vec{A}_5, \vec{A}_2 \times \vec{A}_5, \vec{A}_3 \times \vec{A}_5\}$ to get

$$W_{134} W_{125} - W_{124} W_{135} + W_{123} W_{145} = 0$$
$$W_{234} W_{125} - W_{124} W_{235} + W_{123} W_{245} = 0$$
$$W_{234} W_{135} - W_{134} W_{235} + W_{123} W_{345} = 0$$

These are quadratic Plücker relations amongst the $W$’s and...
...now the counting works. Simpler to use \( w_{ijk} = W_{ijk} / A_i A_j A_k \)

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
A_i's & \phi_{ij} & \phi_{ij} \text{'s lin.dep.} & \phi_{ij} - \text{lin.dep.} & \phi_{ij} - \text{lin.dep. - clos} \equiv \text{KM} & w's & w's \text{'s lin.dep.} & w's - \text{lin.dep.} \\
\hline
\text{n} & \text{(n, 2)} & n(n-5)/2 + 3 & \equiv w - \text{lin.dep. - clos} & \equiv KM & \text{(n, 3)} & n(n^2 - 3n - 10)/6 + 3 & 4(+1 \text{ missing}) \\
\hline
4 & 6 & 1 & 5 & 2 & 4 & -1 & 4 \\
5 & 10 & 3 & 7 & 4 & 10 & 3 & 7 \\
6 & 15 & 6 & 9 & 6 & 20 & 11 & 9 \\
7 & 21 & 10 & 11 & 8 & 35 & 24 & 11 \\
\hline
\end{array}
\]

In fact, the Plückers of the \( W \)'s and the \( F' \)'s are exactly the same

\[ W_{ijk} = \frac{i}{4} F_{ij} E_{ik} \bar{F}_{jk} + c.c. \]

\[ F_{ij} = z_i^0 z_j^1 - z_j^0 z_i^1 = [z_i | z_j] \]

\[ E_{ij} = \bar{z}_i^0 z_j^0 + \bar{z}_i^1 z_j^1 = \langle z_i | z_j \rangle \]

\[ W_{ijk} W_{ilm} - W_{ilk} W_{ijm} + W_{ilj} W_{ikm} = 0 \leftrightarrow F_{ij} F_{kl} - F_{ik} F_{jl} + F_{il} F_{jk} = 0 \]

Plücker relations amongst the complex variables also follow directly from the spinorial identity
In conclusion

The face adjacency of a pentahedron is completely determined by cross ratios that can unusually be expressed as the ratio of just two numbers, e.g. \( \alpha = -W_{124}/W_{123} \)

Indeed pentahedral adjacency is likely completely determined by a Desargues’ configuration and projective configurations are also important to hexahedral geometry.

We have provided a new set of real projective tools with rich connections to the geometry of polyhedra and exposed a computational foundation for the motto *adjacency is projective*.