Quantum Curves

Knot Theory

Loop Quantum Gravity

Chern-Simons Theory

Low-dimensional Topology

Supersymmetric Gauge Theory

String/M-Theory

Muxin Han

with Hal M. Haggard, Aldo Riello, Wojciech Kaminski, Roland van der Veen
Warm up: Harmonic Oscillator

\[ \omega = dp \wedge dq \]

\[ L_A : \frac{1}{2}(p^2 + q^2) - E = 0 \]

**Quantization**

\[ q \mapsto \hat{q} \]
\[ p \mapsto \hat{p} = -i\hbar \partial_q \]
\[ \hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2) \]

\[ \hat{H}\psi(q) = E\psi(q) \]

\[ \left[ \frac{1}{2}(\hat{p}^2 + \hat{q}^2) - E \right] \psi(q) = 0 \]

**Quantized Circle**

WKB solution:

\[ \psi^{(\pm)}(q) = \exp \left[ \frac{i}{\hbar} \int_{q_0}^{q} p(q') dq' + o(\log \hbar) \right] \]

\[ = \exp \left[ \frac{i}{\hbar} \int_{q_0}^{q} \pm \sqrt{2E - q'^2} \ dq' + o(\log \hbar) \right] \]
Quantum Curve

Complex symplectic manifold: \((M, \omega, J)\) \hspace{1cm} \dim_{\mathbb{C}} M = 2N

holomorphic symplectic structure \hspace{1cm} complex structure

Holomorphic coordinates:
\[
\omega = \sum_{m=1}^{2N} dv_m \wedge du_m
\]

Holomorphic Lagrangian submanifold (Classical curve): \(L_A\)

\[
\omega|_{L_A} = 0 \hspace{1cm} \dim_{\mathbb{C}} L_A = N
\]

Holomorphic: polynomial eqns \(A_m(e^u, e^v) = 0, \ m = 1, \cdots, N\)

Quantization:  
\[
[\hat{u}_m, \hat{v}_n] = i\hbar \delta_{m,n}, \quad \hat{u}_m f(u) = u f(u), \quad \hat{v}_m f(u) = -i\hbar \partial_{u_m} f(u)
\]

Quantization of \(L_A\):  
\[
\hat{A}_m(e^{\hat{u}}, e^{\hat{v}}, \hbar) Z(u) = 0, \ m = 1, \cdots, N
\]

The goal: find the holomorphic solution \(Z(u)\).
Quantum Curves

- Loop Quantum Gravity
- Chern-Simons Theory
- Supersymmetric Gauge Theory
- String/M-Theory
- Low-dimensional Topology
- Knot Theory
Moduli space of flat connections on 2-surface

Closed 2-surface \( \Sigma_g \)
e.g.

with a set of closed curves \( \{ c \} \)

s.t. \( \Sigma_g \setminus \{ c \} = \) a set of \( n \)-holed spheres

Complex symplectic manifold: \( \mathcal{M} = \mathcal{M}_{flat}(\Sigma_g, SL(2, \mathbb{C})) \) \( (A \text{ s.t. } F_A = 0) \)

- Hyper-Kahler: complex structures \( i, j, k = ij \)

from 2-surface from complex group

- Symplectic structure: \( \omega \sim \int_{\Sigma_g} \text{tr} [\delta A \wedge \delta A] \) (Atiyah-Bott-Goldman)
Holomorphic coordinates (w.r.t $j$)

Closed 2-surface $\sum_g$ with the set of closed curves $\{ c \}$

Complex Fenchel-Nielsen (FN) coordinates: $c \mapsto (x_c, y_c) \in (\mathbb{C}^*)^2$

FN length: $x_c$ holonomy eigenvalue along the curve $c$

FN twist: $y_c$ “conjugate momenta”

Holomorphic symplectic structure: $\omega = \sum_c d \ln y_c \wedge d \ln x_c + \omega_{n\text{-holed sphere}}$

$\dim_{\mathbb{C}} \mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C})) = 6g - 6$

e.g.

$g = 6$

\[ \dim = 30 = 2 \times 10 + 2 \times 5 \]

$\dim_{\text{2-holed sphere}} = 2$
Holomorphic Lagrangian submanifold

For a 3-manifold $M_3$ s.t. $\partial M_3 = \Sigma_g$

$\mathcal{M}_{flat}(M_3, \text{SL}(2, \mathbb{C})) \simeq \mathcal{L}_A \hookrightarrow \mathcal{M}_{flat}(\Sigma_g, \text{SL}(2, \mathbb{C}))$

Holomorphic polynomial eqns $A_m(x_c, y_c; \cdots) = 0, \ m = 1, \cdots, 3g - 3$

We focus on graph complement 3-manifold in 3-sphere: removing a tubular open neighborhood of a graph embedded in 3-sphere.

$M_3 = S^3 \setminus N(\Gamma) \equiv S^3 \setminus \Gamma$

$\partial M_3 = \Sigma_g = 6$
Quantization of flat connections on 3-manifold

\[ \mathcal{M} = \mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C})) \quad \omega = \sum_c d \ln y_c \wedge d \ln x_c + \omega_{n\text{-holed sphere}} \]

Holomorphic symplectic coordinates

\[ u_c = \ln x_c, \quad v_c = \ln y_c, \quad \cdots \]

\[ \hat{u}_c f(u, \cdots) = u_c f(u, \cdots), \quad \hat{v}_c f(u, \cdots) = -i\hbar \partial_{u_c} f(u, \cdots) \]

Quantization of

\[ \mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C})) \cong \mathcal{L}_A \hookrightarrow \mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C})) \]

\[ \hat{A}_m(e^{\hat{u}}, e^{\hat{v}}, \cdots, \hbar) Z(u, \cdots) = 0, \quad m = 1, \cdots, 3g - 3 \]

The holomorphic solutions \( Z(u, \cdots) \) are the physical states for quantum flat connections on 3-manifold, which quantizes SL(2,C) Chern-Simons theory on 3-manifold.

Dimofte, Gukov, Lenells, Zagier 2009
Dimofte 2011
Gukov, Sułkowski 2011
Gukov, Saberi 2012
Flat Connections in 3d v.s. Simplicial Geometry in 4d

A class of $\text{SL}(2, \mathbb{C})$ flat connection on $S^3 \setminus \Gamma_5$

\[ \partial \mathcal{M}_3 = \sum_{g=6} \mathcal{G} \]

= Lorentzian 4-simplex geometries with constant curvature $\Lambda$

The class of flat connections is specified by the boundary condition

Haggard, MH, Kamiński, Riello 2014
Boundary condition on $\mathcal{M}_{flat} \left( \sum_{g=6}, \text{SL}(2, \mathbb{C}) \right)$

- We consider the SL(2, C) flat connections on $\Sigma_{g=6}$ that reduce to SU(2) on each 4-holed sphere

- We associate each 4-holed sphere with an SU(2) subgroup of SL(2, C)

- We consider the SL(2, C) flat connections on $\Sigma_{g=6}$ that reduce to SU(2) on each 4-holed sphere

- Relates to the simplicity constraint in LQG
Flat Connections in d-1 v.s. Discrete Geometry in d

\[ \pi_1(M_{d-1}) \]
Fundamental group of a \((d-1)\)-manifold \(M_{d-1}\)
(defected \(S^{d-1}\))

\[ \pi_1(\text{sk}(M_d)) \]
Fundamental group of the 1-skeleton of a \(d\)-dim polyhedron \(M_d\)

\[ S \]

\[ \omega_{flat} \]

\[ \omega_{spin} \]

\[ \text{SO}(d) \text{ or } \text{SO}(d - 1, 1) \]

modulo gauge

\[ \omega_{spin} = \omega_{flat} \circ S \]
4-holed sphere v.s. tetrahedron

\[ \pi_1(4\text{-}\text{holed sphere}) = \langle l_1, \cdots, l_4 \mid l_4 l_3 l_2 l_1 = e \rangle \]

\[ \pi_1(\text{sk(Tetra)}) \cong \langle p_1, \cdots, p_4 \mid p_4 p_3 p_2 p_1 = e \rangle \]

\[ \omega_{\text{flat}} \]

\[ \langle H_1, \cdots, H_4 \in \text{SO}(3) \mid H_4 H_3 H_2 H_1 = 1 \rangle / \text{conjugation} \]
\[ \Gamma_5 \text{ graph complement v.s. 4-simplex} \]

\[ S^3 \xrightarrow{\Gamma_5} S^3 \backslash \Gamma_5 \]

vertex 1: \[ I_{14} I_{13}^{(1)} I_{12} I_{15} = 1, \]
vertex 2: \[ I_{12} I_{24} I_{23} I_{25} = 1, \]
vertex 3: \[ I_{23}^{-1} (I_{13}^{(2)})^{-1} I_{34} I_{35} = 1, \]
vertex 4: \[ I_{34}^{-1} I_{24}^{-1} I_{14} I_{45} = 1, \]
vertex 5: \[ I_{25}^{-1} I_{35}^{-1} I_{45}^{-1} I_{15}^{-1} = 1, \]
crossing: \[ I_{13}^{(1)} = I_{24} I_{13}^{(2)} I_{24}^{-1}. \]

tetra 1: \[ p_{14} p_{13}^{(1)} p_{12} p_{15} = 1, \]
tetra 2: \[ p_{12}^{-1} p_{24} p_{23} p_{25} = 1, \]
tetra 3: \[ p_{23}^{-1} (p_{13}^{(2)})^{-1} p_{34} p_{35} = 1, \]
tetra 4: \[ p_{34}^{-1} p_{24}^{-1} p_{14}^{-1} p_{45} = 1, \]
tetra 5: \[ p_{25}^{-1} p_{35}^{-1} p_{45}^{-1} p_{15}^{-1} = 1, \]
“crossing”: \[ p_{13}^{(1)} = p_{24} p_{13}^{(2)} p_{24}^{-1}. \]

\[ \omega_{\text{flat}} \]
\[ \omega_{\text{spin}} \]

\[ \langle H_{ab} \in SO(3, 1) | \ldots \rangle / \text{conjugation} \]
\[ \omega_{spin} = \omega_{flat} \circ S \] are a set of holonomies along closed paths on 1-skeleton

How much do they know about the geometry?

In general they know very little.

But for constant curvature simplex, whose 2-faces are flatly embedded surfaces:

**Lemma**: Given 2-surface flatly embedded \((K=0)\) in constant curvature space, the holonomy of spin connection along the boundary of surface:

\[
h_{\partial f}(\omega_{spin}) = \exp \left[ -i \frac{\Lambda}{6} a_f \hat{n}_f \cdot \vec{\sigma} \right] \]

in 3d space

replaced by normal bivector in 4d spacetime

Area and normal data determine the simplex geometry
**Theorem:** There is 1-to-1 correspondence between

\[ A \in \mathcal{M}_{flat}(4\text{-holed sphere, PSU}(2)) \longleftrightarrow \text{A convex constant curvature tetrahedron geometry with } \Lambda > 0 \text{ or } \Lambda < 0 \]

**Remark:** The above statements hold as far as the geometry is nondegenerate.

**Remark:** Flat conn holonomy around defect = Spin conn holonomy around face.

**Theorem:** There is 1-to-1 correspondence between

\[ A \in \mathcal{M}_{flat}(S^3 \setminus \Gamma_5, PSL(2, \mathbb{C})) \text{ satisfying the boundary condition} \longleftrightarrow \text{A convex constant curvature 4-simplex geometry with } \Lambda > 0 \text{ or } \Lambda < 0 \text{ (Lorentzian)} \]

\[ S^3 \setminus \Gamma_5 \]

**Remark:** The above statements hold as far as the geometry is nondegenerate.
Dictionary between coordinates

Flat connection

$S^3 \setminus \Gamma_5$

4-simplex geometry

FN length: $x_{ab} = \pm \exp \left( -i \frac{\Lambda}{6} a_{ab} \right)$  \hspace{1cm} \text{triangle area}

FN twist: $y_{ab} = \pm \exp \left( -\frac{1}{2} \Theta^\Lambda_{ab} \right)$  \hspace{1cm} \text{4d dihedral angle}

4-holed sphere: $(x_a, y_a) = \text{shape of tetrahedron}$
Parity Pair

Given a flat connection \( A \in \mathcal{M}_\text{flat}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C})) \) satisfying boundary condition, it associates a unique \( \tilde{A} \in \mathcal{M}_\text{flat}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C})) \) satisfying boundary condition, with the same boundary data: \( (x_{ab}; x_a, y_a) \)

but with different twist variable:

\[
y_{ab} = \pm \exp \left[ -\frac{1}{2} \Theta^\Lambda_{ab} \right], \quad \tilde{y}_{ab} = \pm \exp \left[ \frac{1}{2} \Theta^\Lambda_{ab} \right]
\]

\( \mathcal{M}_\text{flat}(\Sigma_{g=6}, \text{SL}(2, \mathbb{C})) \)

2 constant curvature 4-simplex with the same geometry but with opposite 4d orientations
Quantum Theory

Flat connection on $S^3 \setminus \Gamma_5 = \text{4-simplex geometry}$

Quantum flat connection on $S^3 \setminus \Gamma_5 = \text{Quantum 4-simplex geometry}$

Quantization of 4d geometry

Quantization of flat connection on 3-manifold

(Quantization of holomorphic Lagrangian submanifold)
Quantization of flat connections on 3-manifold

\[ \mathcal{M} = \mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C})) \quad \omega = \sum_c \text{d} \ln y_c \wedge \text{d} \ln x_c + \omega_{n\text{-holed sphere}} \]

Holomorphic symplectic coordinates

\[ u_c = \ln x_c, \quad v_c = \ln y_c, \quad \cdots \]

\[ \hat{u}_c f(u, \cdots) = u_c f(u, \cdots), \quad \hat{v}_c f(u, \cdots) = -i\hbar \partial_{u_c} f(u, \cdots) \]

Quantization of \( \mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C})) \cong \mathcal{L}_A \rightarrow \mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C})) \)

\[ \hat{A}_m(e^{\hat{u}}, e^{\hat{v}}, \cdots, \hbar) Z(u, \cdots) = 0, \quad m = 1, \cdots, 3g - 3 \]

The holomorphic solutions \( Z(u, \cdots) \) are the physical states for quantum flat connections on 3-manifold, which quantizes SL(2,\mathbb{C}) Chern-Simons theory on 3-manifold.
\[ A_m(x_c, y_c; \cdots) = 0, \quad m = 1, \cdots, 3g - 3 \]

\[ \hat{A}_m(e^{\hat{u}}, e^{\hat{v}}, \cdots, \hat{h}) Z(u, \cdots) = 0, \quad m = 1, \cdots, 3g - 3 \]

WKB solutions: holomorphic 3d block

\[
Z^{(\alpha)}(M_3 | u) = \exp \left[ \frac{i}{\hbar} \int_{(u_0, v_0)}^{(u, v^{(\alpha)})} \vartheta + o(\log \hbar) \right]
\]

\( \alpha \) labels the branches of Lagrangian submanifold. Thus \((u, \alpha)\) corresponds to a unique SL(2,C) flat connection on \( M_3 \)

\[
Z^{(\alpha)}(M_3 | u) \text{ has ambiguities: (1) } Z^{(\alpha)}(M_3 | u) \mapsto Z^{(\alpha)}(M_3 | u) \exp \left( -\frac{2\pi i}{\hbar} u \right) \quad (v \sim v + 2\pi i)
\]

(2) starting point of contour \( \mapsto \) overall phase.
Wave function of 4-geometry

- Quantize $\mathcal{L}_A \approx \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$

- Impose the boundary condition

- Consider the branch $\alpha$ s.t. $(u, \alpha)$ corresponds to a constant curvature 4-simplex geometry

- $\alpha$ associates with its parity partner $\tilde{\alpha}$ s.t. $(u, \alpha)$ and $(u, \tilde{\alpha})$ are parity pair

Holomorphic 3d block defined at branch $\alpha$ with the reference at branch $\tilde{\alpha}$ is a state for quantum 4-simplex geometry

\[
Z^{(\alpha)}(S^3 \setminus \Gamma_5|u) = \exp \left[ \frac{i}{\hbar} \int (u, v^{(\alpha)}) \vartheta + o(\log \hbar) \right]
\]

Where $\mathcal{M}_{\text{flat}}(\Sigma_{g=6}, \text{SL}(2, \mathbb{C}))$
Quantum Geometry = Quantum Gravity

Semiclassical limit of \[ Z^{(\alpha)} \left( S^3 \setminus \Gamma_5 \right| u \right) \longrightarrow \text{Discrete Einstein gravity in 4d} \]

Semiclassical limit \( \hbar \to 0 \)

\[
Z^{(\alpha)} \left( S^3 \setminus \Gamma_5 \right| u \right) \sim \exp \left[ \frac{i}{\hbar} S^\Lambda_{\text{Regge}} + o(\log \hbar) \right]
\]

Discrete 4d Einstein-Hilbert action on a constant curvature 4-simplex:

\[
S^\Lambda_{\text{Regge}} = \sum_{a < b} a_{ab} \Theta^\Lambda_{ab} - \Lambda \text{Vol}_4^\Lambda
\]
Variation of boundary data  

\[
[u_{ab}, u_a, v_a] \mapsto [u_{ab} + \delta u_{ab}, u_a + \delta u_a, v_a + \delta v_a]
\]

\[
\delta I^\alpha_{\tilde{\alpha}} = \int_{c \cup \tilde{c}} \sum_{a < b} v_{ab} du_{ab} \sim \text{“symplectic area of the square”}
\]

\[
= \delta u_{ab} [v_{ab} - \tilde{v}_{ab}] + o\left((\delta u)^2\right)
\]

\[
I^\alpha_{\tilde{\alpha}} = \int \sum_{a < b} v_{ab} du_{ab}
\]

\[
Z^{(\alpha)}(S^3 \setminus \Gamma_5|u) = \exp\left[\frac{i}{\hbar} I^\alpha_{\tilde{\alpha}} + o(\log \hbar)\right]
\]

Variation of boundary data
Dictionary between coordinates

Flat connection

\[ S^3 \setminus \Gamma_5 \]

4-simplex geometry

FN length: \[ x_{ab} = e^{u_{ab}} = \pm \exp \left[ -i \frac{\Lambda}{6} a_{ab} \right] \]

FN twist: \[ y_{ab} = e^{-2\pi i v_{ab}} = \pm \exp \left[ -\frac{1}{2} \Theta^\Lambda_{ab} \right] \]

\[ \tilde{y}_{ab} = e^{-2\pi i \tilde{v}_{ab}} = \pm \exp \left[ +\frac{1}{2} \Theta^\Lambda_{ab} \right] \]

\( t \) is CS coupling
Integrate by using Schafli identity
\[
\sum_{a<b} a_{ab} \delta \Theta_{ab} = \Lambda \delta \text{Vol}_4^A
\]

\[
\delta I_{\tilde{\alpha}} = \left( \frac{\Lambda t}{12\pi i} \right) \sum_{a<b} \delta a_{ab} \Theta_{ab} + \left( \frac{\Lambda t}{6} \right) \sum_{a<b} \delta a_{ab}
\]

Lorentzian Regge action in 4d

\[
Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \mid u\right) = \exp \left[ \frac{i}{\hbar} \left( \frac{\Lambda t}{12\pi i} \right) \left( \sum_{a<b} a_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^A + C_{\tilde{\alpha}}^{\alpha} + \frac{i}{\hbar} \left( \frac{\Lambda t}{6} \right) \sum_{a<b} a_{ab} + \cdots \right) \right]
\]

Integration const.

To obtain an oscillatory phase: consider full SL(2,C) Chern-Simons theory with both holomorphic and anti-holomorphic contribution

\[
Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \mid u\right) Z^{(\overline{\alpha})}\left(S^3 \setminus \Gamma_5 \mid \bar{u}\right) = \exp \left[ \frac{i}{\hbar} 2 \text{Re} \left( \frac{\Lambda t}{12\pi i} \right) \left( \sum_{a<b} a_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^A \right) + C_{\tilde{\alpha}}^{\alpha} + \frac{i}{\hbar} 2 \text{Re} \left( \frac{\Lambda t}{6} \right) \sum_{a<b} a_{ab} + \cdots \right]
\]

- **Gravitational coupling:** \( G_N = \left| \frac{3}{2 \text{Im}(i) \Lambda} \right| \) \( t \) is CS coupling
- **Independent of ambiguity:** \( 2 \text{Re} \left( \frac{\Lambda t}{6} \right) \sum_{a<b} a_{ab} \in 2\pi \hbar \mathbb{Z} \) fulfilled by LQG \( a \sim j \)

**Interesting:** \( t \in i\mathbb{R} \) no quantization condition needed

In LQG, corresponding to the limit: Barbero-Immirzi parameter \( \rightarrow \) infinity
Relation with Loop Quantum Gravity

SL(2,C) CS theory on $S^3$ with certain Wilson graph operator

$$Z_{\Gamma_5} = \int [DAD\bar{A}] e^{iCS[S^3|A,\bar{A}]_{\Gamma_5}[A, \bar{A}]}$$

Wilson graph operator imposes the right boundary condition on $\partial(S^3 \setminus \Gamma_5) = \Sigma_{g=6}$

$$x_{ab} = \exp \left[ \frac{2\pi i\hbar}{t} (1 + i\gamma) j_{ab} \right], \quad \gamma = \frac{\text{Im}(t)}{\text{Re}(t)}, \quad j_{ab} \in \mathbb{Z}/2$$

Barbero-Immirzi parameter

- **semiclassical limit = double-scaling limit** $j \to \infty, \hbar \to 0, \ j\hbar$ fixed

- $Z_{\Gamma_5}$ has the same semiclassical limit as the 3d block

$$Z_{\Gamma_5} \sim Z^{(\alpha)} \left( S^3 \setminus \Gamma_5 \left| u \right. \right) Z^{(\bar{\alpha})} \left( S^3 \setminus \Gamma_5 \left| \bar{u} \right. \right)$$

gives classical Einstein-Regge action as the leading order.
Deformation of EPRL Spinfoam Amplitude

\[ Z_{\Gamma_5} \xrightarrow{\hbar \to 0, \ j \to \infty, \ j\hbar \ \text{fixed}} \; e^{\frac{i}{\ell_P} S_{\text{Regge}}} + e^{-\frac{i}{\ell_P} S_{\text{Regge}}} \]

\[ Z_{\text{EPRL}} \xrightarrow{j \to \infty} \; e^{\frac{i}{\ell_P} S_{\text{Regge}}} + e^{-\frac{i}{\ell_P} S_{\text{Regge}}} \]

Promote CS 3d block to be a wave-function/spinfoam-amplitude of 4d LQG

\[ Z^{(\alpha)} \left( S^3 \setminus \Gamma_5 \big| u \right) Z^{(\alpha)} \left( S^3 \setminus \Gamma_5 \big| \bar{u} \right) \]

Finiteness

Identify/generalize spin-network data to flat connection data on closed 2-surface + the quantization condition

relate to Rovelli, Vidotto 2015
Generalize to 4d Simplicial Complex

3-manifold obtained from gluing graph complements through 4-holed sphere

Flat connections on 3-manifold = Simplicial geometry on 4-manifold

\[ Z^{(\alpha)}(M_3\mid u) \sim \exp \left[ \frac{i}{\hbar} S_{\text{Regge}}^\wedge + o(\log \hbar) \right] \]

Einstein-Regge action on the entire simplicial complex
Quantum Curves

\[ Z^{(\alpha)}(M_3|u) \]

- Loop Quantum Gravity
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3d-3d correspondence

M-theory in 11d:

M5-brane \(\rightarrow\) IR dynamics: 6d SCFT with gauge group \(G\)

(6-dim) \[16\] supercharges (maximal SUSY)

Compactify M5 on \(M_3 \times S^3_b\)

3d ellipsoid

(or \(S^2 \times_q S^1\) or \(\mathbb{R}^2 \times_q S^1\))

\(G_C\) CS on \(M_3\)

\(\Leftrightarrow\) 3d \(\mathcal{N} = 2\) SUSY gauge theory \(T_{M_3}\)

(SCFT with 4 Q’s)

- \(Z_{CS}(M_3) = Z_{T_{M_3}}^{N=2}(S^3_b)\)

- \(\mathcal{M}_{flat}(M_3, G_C) \simeq \mathcal{M}_{SUSY}(T_{M_3})\)

- \(Z^{(\alpha)}(M_3) = Z_{T_{M_3}}^{N=2}(\mathbb{R}^2 \times_q S^1)\) with boundary SUSY ground state \(\alpha\)

Dimofte, Gaiotto, Gukov 2011
C. Beem, T. Dimofte, S. Pasquetti 2012
Cordova, Jafferis 2013
Lee, Yamazaki 2013
Chung, Dimofte, Gukov, Sułkowski 2014
Dimofte-Gaiotto-Gukov (DGG) Construction

\[ T_{DGG,M_3} \quad 3d \mathcal{N} = 2 \text{ SCFT with Abelian gauge group } U(1)^n \]

\[ (\text{Gauge theories labelled by 3-manifolds}) \]

\[ M_3 \quad \text{Ideal triangulation} \rightarrow \{ \} \]

\[ T_\Delta = 3d \mathcal{N} = 2 \text{ chiral multiplet} ; \quad \text{gluing} \rightarrow \text{gauging} + \text{superpotential} \]

\[ \rightarrow \quad \text{Pachner move} = 3d \text{ mirror symmetry} \]

\[ \mathcal{M}_{flat}(M_3, SL(2, \mathbb{C})) \leftrightarrow \mathcal{M}_{SUSY}(T_{DGG,M_3}) \]

\[ Z'_{CS}(M_3) = Z_{DGG,M_3}(S^3_b) \]

\[ Z^{(\alpha)}(M_3) = Z_{DGG,M_3}(\mathbb{R}^2 \times_q S^1) \text{ with boundary SUSY ground state } \alpha \]
4d LQG and 3d SCFT

LQG vacua = Simplicial geometries = Flat conn on $M_3$ = SUSY vacua in $T_{M_3}$

LQG partition function = CS partition function of $M_3$ = SUSY partition function of $T_{M_3}$
(Spinfoam Amplitude)

\[ \sim \exp \left[ i S_{\text{Regge}}^{\Lambda} + \cdots \right] \]
4d LQG and 3d SCFT

LQG vacua = Simplicial geometries = Flat conn on $M_3 = \text{SUSY vacua in } T_{M_3}$

LQG partition function = CS partition function of $M_3 = \text{SUSY partition function of } T_{M_3}$
(Scinfoam Amplitude)

$$\sim \exp \left[ i S^\Lambda_{\text{Regge}} + \cdots \right]$$

The end

Thanks for your attention!