Path Integral Formulation of LQG, Semiclassical Limit, and Cosmological Perturbation Theory

Muxin Han

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MH and Hongguang Liu, arXiv:1910.03763

MH and Hongguang Liu, arXiv:1912.08668

MH and Hongguang Liu, arXiv:2004.xxxxx

MH, Haida Li and Hongguang Liu, arXiv:2004.xxxxx

Outline

① **Review / introduce the path integral formulation of reduced phase space LQG**

② **Semiclassical limit and equations of motion (EOMs)**

③ **Comparing with spin foam formulation**

④ **Cosmological perturbation theory from full LQG**

⑤ **Relation with Numerical Relativity**

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① Review / introduce the path integral formulation of reduced phase space LQG

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Reduced phase space LQG

3 scenarios of deparametrized models: gravity coupled to

- **Brown-Kuchar dust**

Brown and Kuchar 1994
Giesel and Thiemann 2007

$$S_{BKD}\left[\rho, g_{\mu\nu}, T, S^j, W_j\right] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{|\det(g)|}\,\rho\left[g^{\mu\nu}U_\mu U_\nu + 1\right], \quad U_\mu = -\partial_\mu T + W_j\partial_\mu S^j$$

- **Gaussian dust**

Kuchar and Torre 1990
Giesel and Thiemann 2015

$$S_{GD}\left[\rho, g_{\mu\nu}, T, S^j, W_j\right] = -\int \mathrm{d}^4x\sqrt{|\det(g)|}\left[\frac{\rho}{2}\left(g^{\mu\nu}\partial_\mu T\partial_\nu T + 1\right) + g^{\mu\nu}\partial_\mu T\left(W_j\partial_\nu S^j\right)\right]$$

- **Massless real scalar field**

Rovelli and Smolin 1993
Domagala, Giesel, Kaminski, and Lewandowski 2010

$$S_\phi\left[g_{\mu\nu}, \phi\right] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{|\det(g)|}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

Dirac observables = parametrizing gravity variables with values of dust fields

$$T(x) \equiv \tau, \quad S^j(x) \equiv \sigma^j$$

τ: physical time variable
σ: physical space variable

Rovelli 2001
Dittrich 2004
Thiemann 2004

Gravity Dirac observables

$$A(\sigma,\tau) = A(x)\Big|_{T(x)\equiv\tau, S^j(x)\equiv\sigma^j}, \quad E(\sigma,\tau) = E(x)\Big|_{T(x)\equiv\tau, S^j(x)\equiv\sigma^j}$$

Reduced phase space LQG

Canonical structure of Dirac observables:

$$\left\{E_a^i(\sigma,\tau), A_j^b(\sigma',\tau)\right\} = \frac{1}{2}\kappa\beta\delta_j^i\delta_a^b\delta^3(\sigma,\sigma')$$

***a,b,c…*: SU(2) indices**
***i,j,k,…*: spatial indices of the dust space $\mathcal{S}$**
(space of σ's, slice with constant τ)

$\kappa = 16\pi G$

Solving constraints (Abelianized constraints):

$$C^{tot} = P + h\left(p, q, \partial_\alpha T\right) \approx 0, \quad C_j^{tot} = P_j + S_j^\alpha\left[C_\alpha(p,q) + P\partial_\alpha T\right] \approx 0$$

P : **momentum of** T

P_j : **momentum of** S_j

Physical Hamiltonian (generating τ evolution):

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \{\mathbf{H}_0, f\}, \quad \mathbf{H}_0 = \int_{\mathcal{S}} d^3\sigma\, h$$

- **Brown-Kuchar dust** — Giesel and Thiemann 2007

$$\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma\sqrt{\mathscr{C}(\sigma,\tau)^2 - \frac{1}{4}\sum_{a=1}^{3}\mathscr{C}_a(\sigma,\tau)^2}$$

- **Gaussian dust** — Giesel and Thiemann 2015

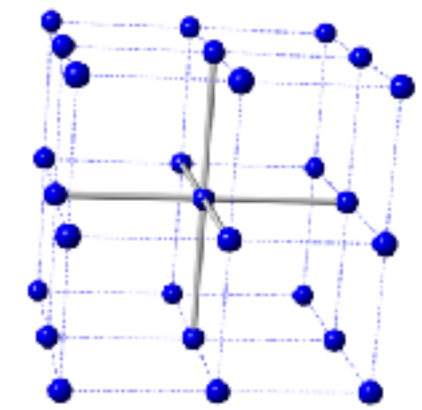
$$\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma\; \mathscr{C}(\sigma,\tau)$$

$$\mathscr{C} = \frac{1}{\kappa}\left[F_{jk}^a - \left(\beta^2+1\right)\varepsilon_{ade}K_j^dK_k^e\right]\varepsilon_{abc}\frac{E_b^jE_c^k}{\sqrt{\det(q)}} + \frac{2\Lambda}{\kappa}\sqrt{\det(q)}$$

$$\mathscr{C}_a = \frac{4}{\kappa\beta}F_{jk}^b\frac{E_a^jE_b^k}{\sqrt{\det(q)}}$$

Physical requirements: $\mathscr{C} < 0$, **and** $\mathscr{C}^2 - \frac{1}{4}\sum_{a=1}^{3}\mathscr{C}_a^2 > 0$ (**for BK dust**)

Reduced phase space LQG



The quantization is on a fixed graph γ on the space without boundary

(e.g. γ is a cubic lattice partitioning 3-torus, $\mathscr{S} \simeq T^3$)

Holonomy and flux at every edge (Dirac observables)

$$h(e) := \mathscr{P} \exp \int_e A, \quad \textbf{and} \quad p^a(e) := -\frac{1}{2\beta a^2} \operatorname{tr}\left[\tau^a \int_{S_e} \varepsilon_{ijk} \mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j h\left(\rho_e(\sigma)\right) E_b^k(\sigma) \tau^b h\left(\rho_e(\sigma)\right)^{-1}\right]$$

$\tau^a = -i(\textbf{Pauli matrix})^a$

$$\mathscr{H}_\gamma^0 = \otimes_e L^2(\mathrm{SU}(2)) \longrightarrow \mathscr{H}_\gamma$$

Gauss constraint is imposed quantum mechanically

$\mathscr{H}_\gamma$ **is already physical Hilbert space because it is constructed with Dirac observables**

Reduced phase space LQG

Non-graph-changing Hamiltonian: Positive and self-adjoint

Brown-Kuchar/Gaussian dust

$\alpha = 1$ **or** 0

$$\hat{\mathbf{H}} = \sum_{v\in V(\gamma)} \left[\hat{M}_-^\dagger(v)\hat{M}_-(v)\right]^{1/4} \qquad \hat{M}_-(v) = \hat{C}_v^\dagger \hat{C}_v - \frac{\alpha}{4}\hat{C}_{a,v}^\dagger \hat{C}_{a,v},$$

$$\hat{C}_{\mu,v} := \frac{2}{i\beta\kappa\ell_p^2} \sum_{s_1,s_2,s_3=\pm 1} s_1 s_2 s_3 \varepsilon^{I_1 I_2 I_3} \operatorname{Tr}\left(\tau_\mu \hat{h}\left(\alpha_{v;I_1 s_1, I_2 s_2}\right)\hat{h}\left(e_{v;I_3 s_3}\right)\left[\hat{h}\left(e_{v;I_3 s_3}\right)^{-1}, \hat{V}_v\right]\right), \quad \mu = 0,1,2,3$$

Giesel-Thiemann's Hamiltonian

Thiemann 1996
Giesel and Thiemann 2006, 2007

$$\hat{C}_v = \hat{C}_{0,v} + \frac{1+\beta^2}{2}\hat{C}_{L,v} + \frac{2\Lambda}{\kappa}V_v, \quad \hat{K} = \frac{i}{\hbar\beta^2}\left[\sum_{v\in V(\gamma)} \hat{C}_{0,v}, \sum_{v\in V(\gamma)} V_v\right]$$

$$\hat{C}_{L,v} = \frac{16}{3\kappa\left(i\beta\ell_p^2/2\right)^3} \sum_{s_1,s_2,s_3=\pm 1} s_1 s_2 s_3 \varepsilon^{l_1 I_2 I_3} \operatorname{Tr}\left(\hat{h}\left(e_{v;I_1 s_1}\right)\left[\hat{h}\left(e_{v;h_1 s_1}\right)^{-1}, \hat{K}\right]\hat{h}\left(e_{v;I_2 s_2}\right)\left[\hat{h}\left(e_{v;i_2 s_2}\right)^{-1}, \hat{K}\right]\hat{h}\left(e_{v;I_3 s_3}\right)\left[\hat{h}\left(e_{v;I_3 s_3}\right)^{-1}, \hat{V}_v\right]\right)$$

Note: For Brown-Kuchar dust, $\hat{\mathbf{H}}$ is a quantization of $\int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{\left|\mathscr{C}(\sigma,\tau)^2 - \frac{1}{4}\sum_{a=1}^{3}\mathscr{C}_a(\sigma,\tau)^2\right|}$ **which extend**

$\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{\mathscr{C}(\sigma,\tau)^2 - \frac{1}{4}\sum_{a=1}^{3}\mathscr{C}_a(\sigma,\tau)^2}$ **to the entire phase space beyond the physical requirements**

$\mathscr{C} < 0$ **and** $\mathscr{C}^2 - \frac{1}{4}\sum_{a=1}^{3}\mathscr{C}_a^2 > 0$ **. Similar for Gaussian dust.**

Path Integral Formulation

Given any non-graph-changing, positive, and self-adjoint physical Hamiltonian $\hat{\mathbf{H}}$

Transition amplitude between 2 gauge invariant coherent states:

$$A_{[g],[g']} := \left\langle \Psi^t_{[g]} \middle| U(T) \middle| \Psi^t_{[g']} \right\rangle_{\mathscr{H}_\gamma}, \quad U(\tau) := \exp\left(-\frac{i}{\hbar} T \hat{\mathbf{H}}\right)$$

Gauge invariant coherent state: (labelled by gauge orbit $[g]$)

$$\Psi^t_{[g]} = \int \mathrm{d}h\, \psi^t_{g^h}, \quad \text{where} \quad g^h = \left\{ h^{-1}_{s(e)} g(e) h_{t(e)} \right\}_{e \in E(\gamma)}, \quad \mathrm{d}h = \prod_{v \in V(\gamma)} \mathrm{d}\mu_H(h_v)$$

$$\psi^t_{g(e)}\left(h(e)\right) = \sum_{j_e} \left(2j_e + 1\right) e^{-t j_e (j_e+1)/2} \chi_{j_e}\left(g(e) h(e)^{-1}\right)$$

Holomorphic parametrization of LQG phase space

$$g(e) = e^{-i p^a(e)\tau^a/2} e^{\theta^a(e)\tau^a/2} \in \mathrm{SL}(2,\mathbb{C}), \quad p^a(e), \theta^a(e) \in \mathbb{R}^3$$

$p^a(e)$: flux

$e^{\theta^a(e)\tau^a/2}$: holonomy

Semiclassicality parameter (dimensionless): $t = \frac{\ell_P^2}{a^2}$, a is a length unit, e.g. 1cm, $t \to 0$ as semiclassical limit

$(\ell_p^2 = \hbar\kappa)$

Discretization and insert N+1 overcompleteness relations

$$\left\langle \psi^t_g \middle| U(T) \middle| \psi^t_{g'^h} \right\rangle = \left\langle \psi^t_g \middle| \left[e^{-\frac{i}{\hbar}\Delta\tau \hat{\mathbf{H}}} \right]^N \middle| \psi^t_{g'^h} \right\rangle, \quad \text{where} \quad \Delta\tau = \frac{T}{N} \quad \text{arbitrarily small,}$$

$$= \int \mathrm{d}g_{N+1} \cdots \mathrm{d}g_1 \langle \psi^t_g | \tilde{\psi}^t_{g_{N+1}} \rangle \langle \tilde{\psi}^t_{g_{N+1}} | e^{-\frac{i}{\hbar}\Delta\tau \hat{\mathbf{H}}} | \tilde{\psi}^t_{g_N} \rangle \langle \tilde{\psi}^t_{g_N} | e^{-\frac{i}{\hbar}\Delta\tau \hat{\mathbf{H}}} | \tilde{\psi}^t_{g_{N-1}} \rangle \cdots \langle \tilde{\psi}^t_{g_2} | e^{-\frac{i}{\hbar}\Delta\tau \hat{\mathbf{H}}} | \tilde{\psi}^t_{g_1} \rangle \langle \tilde{\psi}^t_{g_1} | \psi^t_{g'^h} \rangle$$

Overcompleteness:

$$\int_{G^{\mathbb{C}}} \mathrm{d}g(e) \left| \tilde{\psi}^t_{g(e)} \right\rangle \left\langle \tilde{\psi}^t_{g(e)} \right| = 1_{\mathscr{H}_e}, \quad \mathrm{d}g(e) = \frac{c}{t^3} \mathrm{d}\mu_H(h(e)) \mathrm{d}^3 p(e), \quad c = \frac{2}{\pi} + o\left(t^\infty\right)$$

Normalized coherent state

$$\tilde{\psi}^t_{g(e)} = \frac{\psi^t_{g(e)}}{\left\| \psi^t_{g(e)} \right\|}$$

Path Integral Formulation

MH and H. Liu, 2019

Discrete path integral on 4d hypercubic lattice $\gamma \times$ (**disrete time**)

Semiclassicality parameter $t = \frac{\ell_P^2}{a^2} = \frac{\hbar\kappa}{a^2}$

$$\frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\| \; \|\Psi^t_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i \, \nu[g] \; e^{S[g,h]/t}$$

$\int dh$: **integrating SU(2) gauge freedom**

$g(e) = e^{-ip^a(e)\tau^a/2} e^{\theta^a(e)\tau^a/2} \in \mathrm{SL}(2,\mathbb{C}), \quad p^a(e), \theta^a(e) \in \mathbb{R}^3$

$\nu[g]$: **measure factor (independent of** t**)**

Classical action:

$$S[g,h] = \sum_{i=0}^{N+1} \sum_{e \in E(\gamma)} \left[z_{i+1,i}(e)^2 - \frac{1}{2} p_{i+1}(e)^2 - \frac{1}{2} p_i(e)^2 \right] - \frac{i\kappa}{a^2} \sum_{i=1}^{N} \Delta\tau \left[\frac{\langle \psi^t_{g_{i+1}} | \hat{\mathbf{H}} | \psi^t_{g_i} \rangle}{\langle \psi^t_{g_{i+1}} | \psi^t_{g_i} \rangle} + i\epsilon_{i+1,i} \left(\frac{\Delta\tau}{\hbar} \right) \right]$$

$$z_{i+1,i}(e) = \operatorname{arccosh}\left(x_{i+1,i}(e) \right), \quad x_{i+1,i}(e) = \frac{1}{2} \mathrm{Tr}\left[g_{i+1}(e)^\dagger g_i(e) \right]$$

negligible when $\Delta\tau \to 0$

Remarks:

- **The path integral formulation is rigorously derived from the canonical LQG (in reduced phase space)**
- **The path integral computes the LQG amplitude between boundary states**
- **The path integral is manifestly finite, and manifestly unitary**

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Semiclassical Limit and EOMs

MH and H. Liu, 2019

Discrete path integral

$$\frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\|\ \|\Psi^t_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i\, \nu[g]\ e^{S[g,h]/t}$$

Semiclassicality parameter $t = \frac{\ell_P^2}{a^2} = \frac{\hbar\kappa}{a^2}$

Semiclassical limit $\hbar \to 0$ or $t \to 0$ and stationary phase approximation

path integral is dominated by critical points satisfying semiclassical EOMs $\delta S[g,h] = 0$

Discrete semiclassical EOMs: **Holomorphic deformation in $\mathrm{SL}(2,\mathbb{C})$** $g_i(e) \mapsto g_i^\varepsilon(e) = g_i(e) e^{\varepsilon_i^a(e)\tau^a}, \quad \varepsilon_i^a(e) \in \mathbb{C}$

(A) $$\frac{\delta S}{\delta g} = 0 : \quad \frac{D_1\left(g_{i+1}(e), g_i(e)\right)}{\Delta\tau} = \frac{i\kappa}{a^2}\frac{\partial}{\partial \varepsilon_i^a(e)} \frac{\langle \psi^t_{g^\varepsilon_{i+1}} | \hat{\mathbf{H}} | \psi^t_{g_i^\varepsilon}\rangle}{\langle \psi^t_{g^\varepsilon_{i+1}} | \psi^t_{g_i^\varepsilon}\rangle}\Bigg|_{\varepsilon=0}$$

$$D_1(g_{i+1}, g_i) \equiv \frac{z_{i+1,i}\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger g_i\right]}{\sqrt{x_{i+1,i}-1}\sqrt{x_{i+1,i}+1}} - \frac{p_i \mathrm{Tr}\left[\tau^a g_i^\dagger g_i\right]}{\sinh\left(p_i\right)}$$

(B) $$\frac{\delta S}{\delta g^*} = 0 : \quad \frac{D_2\left(g_i(e), g_{i-1}(e)\right)}{\Delta\tau} = -\frac{i\kappa}{a^2}\frac{\partial}{\partial \varepsilon_i^a(e)^*} \frac{\langle \psi^t_{g_i^\varepsilon} | \hat{\mathbf{H}} | \psi^t_{g^\varepsilon_{i-1}}\rangle}{\langle \psi^t_{g_i^\varepsilon} | \psi^t_{g^\varepsilon_{i-1}}\rangle}\Bigg|_{\varepsilon=0},$$

$$D_2(g_i, g_{i-1}) \equiv \frac{z_{i,i-1}\mathrm{Tr}\left[\tau^a g_i^\dagger g_{i-1}\right]}{\sqrt{x_{i,i-1}-1}\sqrt{x_{i,i-1}+1}} - \frac{p_i \mathrm{Tr}\left[\tau^a g_i^\dagger g_i\right]}{\sinh\left(p_i\right)}$$

(C) $$\frac{\delta S}{\delta h} = 0 : \quad G_v^a \equiv \sum_{e,t(e)=v} \Lambda^a{}_c(\theta)\, p_0^c(e) - \sum_{e,s(e)=v} p_0^a(e) = 0$$

closure condition on initial data

$$z_{i+1,i}(e) = \mathrm{arccosh}\left(x_{i+1,i}(e)\right), \quad x_{i+1,i}(e) = \frac{1}{2}\mathrm{Tr}\left[g_{i+1}(e)^\dagger g_i(e)\right], \quad e^{\theta^a\tau^a/2}\tau^a e^{-\theta^a\tau^a/2} = \Lambda^a{}_b(\theta)\tau^b$$

Semiclassical Limit and EOMs

MH and H. Liu, 2019

$$\frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\|\ \|\Psi^t_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i\, \nu[g]\ e^{S[g,h]/t}$$

Semiclassicality parameter $t = \frac{\ell_P^2}{a^2} = \frac{\hbar\kappa}{a^2}$

(A) $\frac{\delta S}{\delta g} = 0$: $\frac{D_1\left(g_{i+1}(e), g_i(e)\right)}{\Delta\tau} = \frac{i\kappa}{a^2}\frac{\partial}{\partial\varepsilon_i^a(e)}\frac{\langle\psi^t_{g^\varepsilon_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g^\varepsilon_i}\rangle}{\langle\psi^t_{g^\varepsilon_{i+1}}|\psi^t_{g^\varepsilon_i}\rangle}\Bigg|_{\varepsilon=0}$

$$D_1(g_{i+1}, g_i) \equiv \frac{z_{i+1,i}\mathrm{Tr}\left[\tau^a g^\dagger_{i+1} g_i\right]}{\sqrt{x_{i+1,i}-1}\sqrt{x_{i+1,i}+1}} - \frac{p_i \mathrm{Tr}\left[\tau^a g_i^\dagger g_i\right]}{\sinh\left(p_i\right)}$$

(B) $\frac{\delta S}{\delta g^*} = 0$: $\frac{D_2\left(g_i(e), g_{i-1}(e)\right)}{\Delta\tau} = -\frac{i\kappa}{a^2}\frac{\partial}{\partial\varepsilon_i^a(e)^*}\frac{\langle\psi^t_{g^\varepsilon_i}|\hat{\mathbf{H}}|\psi^t_{g^\varepsilon_{i-1}}\rangle}{\langle\psi^t_{g^\varepsilon_i}|\psi^t_{g^\varepsilon_{i-1}}\rangle}\Bigg|_{\varepsilon=0}$,

$$D_2(g_i, g_{i-1}) \equiv \frac{z_{i,i-1}\mathrm{Tr}\left[\tau^a g^\dagger_i g_{i-1}\right]}{\sqrt{x_{i,i-1}-1}\sqrt{x_{i,i-1}+1}} - \frac{p_i \mathrm{Tr}\left[\tau^a g_i^\dagger g_i\right]}{\sinh\left(p_i\right)}$$

- **Path integral is dominated by neighborhood of** $\|g_{i+1} - g_i\| \sim O(\sqrt{t})$
- $g_{i+1} = g_i$ **and** $g_i = g_{i-1}$ **are isolated roots of** $D_1(g_{i+1}, g_i) = 0$ **and** $D_2(g_i, g_{i-1}) = 0$ **in the neighborhood**

For any discrete solution $\{g_i\}_{i=1}^{N+1}$ **of EOMs, given finiteness of right-hand sides of (A) and (B),**

$\Delta\tau \to 0$ **forces** $D_1, D_2 \to 0$**, which forces** $g_{i+1} \to g_i$**.**

Namely all discrete solution admits the approximation $g_i \simeq g(\tau)$ **as continuous function in** τ

$$\frac{D_1\left(g_{i+1}(e), g_i(e)\right)}{\Delta\tau},\quad \frac{D_2\left(g_i(e), g_{i-1}(e)\right)}{\Delta\tau} \quad \sim \quad \text{time derivatives}$$

Semiclassical Limit and EOMs

Time continuous limit $\Delta\tau \to 0$ of

(A) $$\frac{\delta S}{\delta g} = 0: \quad \frac{D_1\left(g_{i+1}(e), g_i(e)\right)}{\Delta\tau} = \frac{i\kappa}{a^2}\frac{\partial}{\partial\varepsilon_i^a(e)}\frac{\langle\psi^t_{g^\varepsilon_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g^\varepsilon_i}\rangle}{\langle\psi^t_{g^\varepsilon_{i+1}}|\psi^t_{g^\varepsilon_i}\rangle}\Bigg|_{\varepsilon=0}$$

$$D_1(g_{i+1}, g_i) \equiv \frac{z_{i+1,i}\mathrm{Tr}\left[\tau^a g^\dagger_{i+1}g_i\right]}{\sqrt{x_{i+1,i}-1}\sqrt{x_{i+1,i}+1}} - \frac{p_i \mathrm{Tr}\left[\tau^a g_i^\dagger g_i\right]}{\sinh\left(p_i\right)}$$

(B) $$\frac{\delta S}{\delta g^*} = 0: \quad \frac{D_2\left(g_i(e), g_{i-1}(e)\right)}{\Delta\tau} = -\frac{i\kappa}{a^2}\frac{\partial}{\partial\varepsilon_i^a(e)^*}\frac{\langle\psi^t_{g^\varepsilon_i}|\hat{\mathbf{H}}|\psi^t_{g^\varepsilon_{i-1}}\rangle}{\langle\psi^t_{g^\varepsilon_i}|\psi^t_{g^\varepsilon_{i-1}}\rangle}\Bigg|_{\varepsilon=0},$$

$$D_2(g_i, g_{i-1}) \equiv \frac{z_{i,i-1}\mathrm{Tr}\left[\tau^a g^\dagger_i g_{i-1}\right]}{\sqrt{x_{i,i-1}-1}\sqrt{x_{i,i-1}+1}} - \frac{p_i \mathrm{Tr}\left[\tau^a g_i^\dagger g_i\right]}{\sinh\left(p_i\right)}$$

- **Right-hand sides: Matrix elements of $\hat{\mathbf{H}}$ (hard to compute) are reduced to expectation values of $\hat{\mathbf{H}}$ (easier to compute).**

$$\langle\tilde{\psi}^t_{g^\varepsilon}|\hat{\mathbf{H}}|\tilde{\psi}^t_{g^\varepsilon}\rangle = \mathbf{H}\left[g^\varepsilon\right] + O(\hbar)$$

Giesel and Thiemann 2006

Classical discrete Hamiltonian

- **Left-hand sides: Reduce to derivatives of holonomies and fluxes.**

$g(e) = e^{-ip^a(e)\tau^a/2}e^{\theta^a(e)\tau^a/2} \in \mathrm{SL}(2,\mathbb{C}), \quad p^a(e), \theta^a(e) \in \mathbb{R}^3$

$p^a(e)$**: flux**

$e^{\theta^a(e)\tau^a/2}$**: holonomy**

(A) and (B) reduces to $$\begin{pmatrix} \mathrm{d}\vec{p}(e)/\mathrm{d}\tau \\ \mathrm{d}\vec{\theta}(e)/\mathrm{d}\tau \end{pmatrix} = \frac{i\kappa}{a^2}T^{-1}(p,\theta)\begin{pmatrix} \partial\mathbf{H}/\partial\vec{p}(e) \\ \partial\mathbf{H}/\partial\vec{\theta}(e) \end{pmatrix}$$

All discrete solution of (A) and (B) admits the approximation $g_i \simeq g(\tau)$ as differentiable function

6×6 **matrix $T(p,\theta)$ contains long formulae, see https://github.com/LQG-Florida-Atlantic-University/Classical-EOM**

Relation with Hamilton's Equations

Time continuous limit of semiclassical EOMs

$$\begin{pmatrix} \mathrm{d}\vec{p}(e)/\mathrm{d}\tau \\ \mathrm{d}\vec{\theta}(e)/\mathrm{d}\tau \end{pmatrix} = \frac{i\kappa}{a^2} T^{-1}(p,\theta) \begin{pmatrix} \partial \mathbf{H}/\partial \vec{p}(e) \\ \partial \mathbf{H}/\partial \vec{\theta}(e) \end{pmatrix}$$

$p^a(e)$: **flux**

$e^{\theta^a(e)\tau^a/2}$: **holonomy**

Relate to Poisson brackets

$$\left\{p^a(e), \mathbf{H}\right\} = \left\{p^a(e), p^b(e)\right\} \frac{\partial \mathbf{H}}{\partial p^b(e)} + \left\{p^a(e), \theta^b(e)\right\} \frac{\partial \mathbf{H}}{\partial \theta^b(e)}$$

$$\left\{\theta^a(e), \mathbf{H}\right\} = \left\{\theta^a(e), p^b(e)\right\} \frac{\partial \mathbf{H}}{\partial p^b(e)} + \left\{\theta^a(e), \theta^b(e)\right\} \frac{\partial \mathbf{H}}{\partial \theta^b(e)}$$

$$P(p,\theta) = \begin{pmatrix} \left\{p^a(e), p^b(e)\right\} & \left\{p^a(e), \theta^b(e)\right\} \\ \left\{\theta^a(e), p^b(e)\right\} & 0 \end{pmatrix}$$

We check that

$$-\frac{ia^2}{\kappa} P(p,\theta) T(p,\theta) = 1_{6\times 6}$$

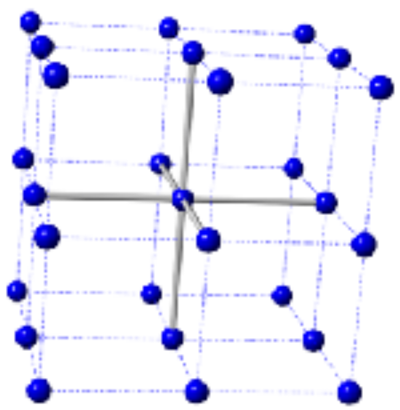
Time continuous limit of semiclassical EOMs is equivalent to Hamilton's equations of holonomies and fluxes

$$\frac{\mathrm{d}p^a(e)}{\mathrm{d}\tau} = \left\{p^a(e), \mathbf{H}\right\}, \quad \frac{\mathrm{d}\theta^a(e)}{\mathrm{d}\tau} = \left\{\theta^a(e), \mathbf{H}\right\}$$

Gauge invariance $\quad \frac{\mathrm{d}G_v^a}{\mathrm{d}\tau} = \left\{G_v^a, \mathbf{H}\right\} = 0.$

- **Semiclassical dynamics is given by Hamiltonian flow generated by physical discrete Hamiltonian**
- **Semiclassical dynamics becomes an initial value problem in the phase space**
- **Initial conditions of p^a, θ^a uniquely determines a solution within the region where $\mathbf{H}$ is regular**

Lattice Continuum Limit of Semiclassical EOMs



μ

μ : coordinate length of lattice edge

Lattice continuum limit: $\mu \to 0, \quad |V(\gamma)| \to \infty,$ **while** $\mu^3 |V(\gamma)|$ **is fixed**

Lattice continuum limit is after the semiclassical limit $t = \dfrac{\ell_P^2}{a^2} \to 0$

We are in this regime: $\ell_P \ll \mu \ll a$

Holonomy and flux: $\theta^a\left(e_I(v)\right) = \mu\left[A_I^a(v) + O(\mu)\right], \quad p^a\left(e_I(v)\right) = \dfrac{2\mu^2}{\beta a^2}\left[E_a^I(v) + O(\mu)\right]$

Lattice continuum limit of $\dfrac{\mathrm{d}p^a(e)}{\mathrm{d}\tau} = \left\{p^a(e), \mathbf{H}\right\}, \quad \dfrac{\mathrm{d}\theta^a(e)}{\mathrm{d}\tau} = \left\{\theta^a(e), \mathbf{H}\right\}$ **:**

$$-\frac{\mathrm{d}A_I^a(v)}{\mathrm{d}\tau} = \frac{\kappa\beta}{2}\frac{\delta\widetilde{\mathbf{H}}_0}{\delta E_a^I(v)} + O(\mu), \qquad \frac{\mathrm{d}E_a^I(v)}{\mathrm{d}\tau} = \frac{\kappa\beta}{2}\frac{\delta\widetilde{\mathbf{H}}_0}{\delta A_I^a(v)} + O(\mu)$$

$$\widetilde{\mathbf{H}}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{\left|\mathscr{C}(\sigma,\tau)^2 - \frac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a(\sigma,\tau)^2\right|}$$

V. S.

$$\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{\mathscr{C}(\sigma,\tau)^2 - \frac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a(\sigma,\tau)^2}$$

classical theory in the continuum

Brown-Kuchar/Gaussian dust

$\alpha = 1$ **or** 0

- **Semiclassical EOMs coincides with classical theory of gravity-dust when** $\mathscr{C}^2 - \dfrac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a^2 > 0$ **(physical requirement of gravity-dust)**
- **Irregularity of the semiclassical dynamics:** $\mathscr{C}^2 - \dfrac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a^2 = 0$

Lattice Continuum Limit of Semiclassical EOMs

$$-\frac{\mathrm{d}A_I^a(v)}{\mathrm{d}\tau} = \frac{\kappa\beta}{2}\frac{\delta\widetilde{\mathbf{H}}_0}{\delta E_a^I(v)} + O(\mu), \qquad \frac{\mathrm{d}E_a^I(v)}{\mathrm{d}\tau} = \frac{\kappa\beta}{2}\frac{\delta\widetilde{\mathbf{H}}_0}{\delta A_I^a(v)} + O(\mu)$$

$$\widetilde{\mathbf{H}}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{\left|\mathscr{C}(\sigma,\tau)^2 - \frac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a(\sigma,\tau)^2\right|}$$

V. S.

$$\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{\mathscr{C}(\sigma,\tau)^2 - \frac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a(\sigma,\tau)^2}$$

classical theory in the continuum

Brown-Kuchar/Gaussian dust

$\alpha = 1$ or 0

Semiclassical consistency and regularity are determined by the initial condition (initial state of path integral) $g(e) = e^{-ip^a(e)\tau^a/2}e^{\theta^a(e)\tau^a/2}$

If initial condition in the phase space are within the classically allowed regime, namely satisfy $\mathscr{C} < 0$ and $\mathscr{C}^2 - \frac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a^2 > 0$

Both $\mathscr{C} < 0$ and $\mathscr{C}^2 - \frac{\alpha}{4}\sum_{a=1}^{3}\mathscr{C}_a^2 > 0$ are preserved by time evolution when total time T is finite

The reason is that e.g. for Brown-Kuchar dust $\mathscr{C}^2 - \frac{1}{4}\sum_{a=1}^{3}\mathscr{C}_a^2$ and $\mathscr{C}_j$ are conserved in the continuum classical theory, and is approximately conserved up to $O(\mu)$ in the semiclassical theory

Consequently,

- Semiclassical EOMs is regular.
- The solution is uniquely determined by the initial condition.

If initial coherent states $\Psi^t_{[g]}$ relates to the classically forbidden regime of the phase space, $\Psi^t_{[g]}$ is not semiclassical

Solutions may exist, but not relate to classical theory (analog of negative energy states in QFT, anti-spacetimes)

But here they are controlled by initial states of the path integral.

Giesel and Thiemann 2007
Christodoulou, Riello, Rovelli 2012

Asymptotics of Transition Amplitude

Stationary phase approximation of the path integral as $t \to 0$:

If semiclassical boundary state labels $[g], [g']$ are connected by a trajectory $g(\tau)$ (in phase space) satisfying EOMs

$$\frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\| \, \|\Psi^t_{[g']}\|} = \int dh \int \prod_{i=1}^{N+1} dg_i \, \nu[g] \; e^{S[g,h]/t}$$

$$= \int \mathrm{d}h \, \frac{(2\pi t)^{\mathcal{N}/2}}{\sqrt{\det(-H)}} \nu[g(\tau), h] \; e^{S[g(\tau),h]/t} \; [1 + O(t)]$$

- **Solution of EOM is unique for semiclassical boundary states.**
- g **- integral behaves asymptotically as a single oscillatory exponential.**
- h **- integral is over all SU(2) gauge transformations of the initial data.**

$$\int dh = \prod_{v \in V(\gamma)} \int_{\mathrm{SU}(2)} d\mu_H(h_v)$$

Amplitude is suppressed if $[g], [g']$ cannot be connected by any trajectory.

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⑤ Relation with Numerical Relativity

Comparing with Spin Foam Formulation

$$\sum_{j_f, i_e} \prod_f A_f\left(j_f\right) \prod_e A_e\left(j_f, i_e\right) \prod_v A_v\left(j_f, i_e\right) \quad \text{V. S.} \quad \frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\| \; \|\Psi^t_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i \, \nu[g] \; e^{S[g,h]/t}$$

Similarities:

- **Both describe transition amplitudes between LQG spin-network states**
- **The path integral can be reformulated as hypercubic spin foams** — Kisielowski and Lewandowski 2018
- **Both relates to classical gravity (Regge calculus and gravity-dust theory respectively)**

Open issues of spin foam formulation:

Cosine problem	non-unique solution even with the phase space initial condition, different orientations.	
Flatness problem	amplitude is dominant by flat spacetime? or large-j limit is not the right limit? not clear.	
Relation with canonical LQG	not clear.	
Divergence	divergent if no Λ .	
Unitarity	not clear.	

Comparing with Spin Foam Formulation

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Similarities:

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Open issues of spin foam formulation:

Cosine problem	**non-unique solution even with the phase space initial condition, different orientations.**	**Solution is unique determined by a classically allowed initial condition. No cosine (thanks to the time continuous limit).**
Flatness problem	**amplitude is dominant by flat spacetime? or large-j limit is not the right limit? not clear.**	**All physically interesting curved spacetimes can be recovered (see below, cosmology and perturbations).**
Relation with canonical LQG	**not clear.**	**derived from canonical LQG**
Divergence	**divergent if no Λ .**	**equals to unitary transition amplitude, is finite irrelevant to Λ .**
Unitarity	**not clear.**	**equals to unitary transition amplitude**

Comparing with Spin Foam Formulation

$$\sum_{j_f,i_e}\prod_f A_f\left(j_f\right)\prod_e A_e\left(j_f,i_e\right)\prod_v A_v\left(j_f,i_e\right)$$

v. s.

$$\frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\|\ \|\Psi^t_{[g']}\|} = \int dh\prod_{i=1}^{N+1} dg_i\,\nu[g]\ e^{S[g,h]/t}$$

$$S[g,h] = \sum_{i=0}^{N+1}\sum_{e\in E(\gamma)}\left[z_{i+1,i}(e)^2 - \frac{1}{2}p_{i+1}(e)^2 - \frac{1}{2}p_i(e)^2\right] - \frac{i\kappa}{a^2}\sum_{i=1}^{N}\Delta\tau\left[\frac{\langle\psi^t_{g_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g_i}\rangle}{\langle\psi^t_{g_{i+1}}|\psi^t_{g_i}\rangle} + i\epsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right)\right]$$

Open issues of both formulation:

Computational complexity	The complexity grows fast for large j, and when number of vertices grows	Non-polynomial Hamiltonian operator, but perturbative computations are allowed
Lattice dependence	The definition depends on the triangulation	The definition depends on the cubic lattice γ

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Cosmological Perturbation Theory from Full LQG

Recall semiclassical EOMs (from the path integral):

$$\begin{pmatrix} \mathrm{d}\vec{p}(e)/\mathrm{d}\tau \\ \mathrm{d}\vec{\theta}(e)/\mathrm{d}\tau \end{pmatrix} = \frac{i\kappa}{a^2} T^{-1}(p,\theta) \begin{pmatrix} \partial \mathbf{H}/\partial \vec{p}(e) \\ \partial \mathbf{H}/\partial \vec{\theta}(e) \end{pmatrix}$$

OR

$$\frac{\mathrm{d}p^a(e)}{\mathrm{d}\tau} = \{p^a(e), \mathbf{H}\}, \quad \frac{\mathrm{d}\theta^a(e)}{\mathrm{d}\tau} = \{\theta^a(e), \mathbf{H}\}$$

$p^a(e)$: **flux**

$e^{\theta^a(e)\tau^a/2}$: **holonomy**

Insert the ansatz of cosmological perturbation theory

Homogenous and isotropic cosmology

$$\theta^a\left(e_I(v)\right) = \mu\left[\beta K_0 \delta_I^a + \mathscr{X}^a\left(e_I(v)\right)\right], \quad p^a\left(e_I(v)\right) = \frac{2\mu^2}{\beta a^2}\left[P_0\delta_I^a + \mathscr{Y}^a\left(e_I(v)\right)\right]$$

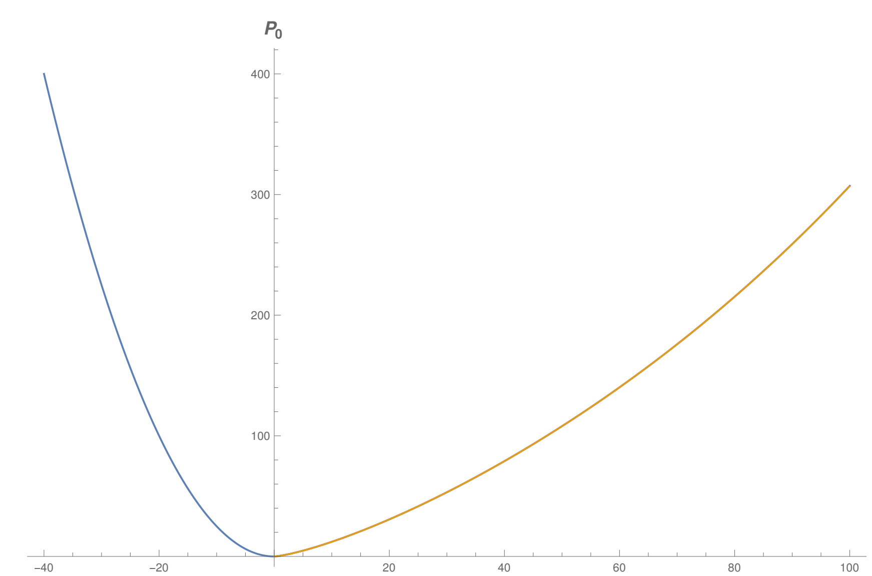
Perturbations of holonomy and fluxes

$$\begin{pmatrix} \mathrm{d}\vec{p}(e)/\mathrm{d}\tau \\ \mathrm{d}\vec{\theta}(e)/\mathrm{d}\tau \end{pmatrix} = \frac{i\kappa}{a^2} T^{-1}(p,\theta) \begin{pmatrix} \partial \mathbf{H}/\partial \vec{p}(e) \\ \partial \mathbf{H}/\partial \vec{\theta}(e) \end{pmatrix}$$

0-th order EOMs:

μ_0 **- scheme effective cosmology**

Unsymmetrical bounce



1-st order EOMs:

Linear EOMs of perturbations on μ_0 **- scheme effective cosmological background**

Linear EOMs of Cosmological Perturbations

Perturbations of fluxes and holonomies

$$V^\rho(v) = \left(\mathscr{Y}^a\left(e_I(v)\right), \mathscr{X}^a\left(e_I(v)\right)\right)^T, \quad \rho = 1,\cdots,18$$

Lattice Fourier transformation

$$V^\rho(\tau, \vec{\sigma}) = \int_{-\pi/\mu}^{\pi/\mu} \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{\sigma}} V^\rho(\tau, \vec{k}), \quad \vec{\sigma} \in (\mu\mathbb{Z})^3$$

Integral over 1st Brillouin zone

$|k^i| \leq \frac{\pi}{\mu}$ **UV cut-off**

18 linear EOMs of cosmological perturbations:

$$\frac{\mathrm{d}V^\rho(\tau,k)}{\mathrm{d}\tau} = \mathbf{U}^\rho_{\ \nu}(\mu,\tau,k)\ V^\nu(\tau,k) \qquad \vec{k} = (k,\ 0,\ 0)$$

18×18 **matrix** $\mathbf{U}(\mu,\tau,k)$ **contains long formulae, will appear at https://github.com/LQG-Florida-Atlantic-University/cos_pert**

3 linearized closure conditions (for initial conditions):

$$0 = P_0\left[\left(V^{15} - V^{18}\right)\sin\left(\beta\mu K_0\right) - \left(V^{16} + V^{17}\right)\left(\cos\left(\beta\mu K_0\right) - 1\right)\right] + \beta K_0\left[-iV^1\sin(k\mu) + V^1\cos(k\mu) - V^6\sin\left(\beta\mu K_0\right) + V^9\sin\left(\beta\mu K_0\right)\right.$$
$$\left. + V^7\cos\left(\beta\mu K_0\right) + V^8\cos\left(\beta\mu K_0\right) - V^1 - V^7 - V^8\right]$$

$$0 = P_0\left[\cos(k\mu)\left(V^{14}\sin\left(\beta\mu K_0\right) + V^{13}\cos\left(\beta\mu K_0\right) - V^{13}\right) - i\sin(k\mu)\left(V^{14}\sin\left(\beta\mu K_0\right) + V^{13}\cos\left(\beta\mu K_0\right) - V^{13}\right) - V^{17}\sin\left(\beta\mu K_0\right) + V^{18}\cos\left(\beta\mu K_0\right) - V^{18}\right]$$
$$+\beta K_0\left[iV^5\sin(k\mu)\sin\left(\beta\mu K_0\right) - \cos(k\mu)\left(V^5\sin\left(\beta\mu K_0\right) + V^4\cos\left(\beta\mu K_0\right)\right) + \left(-V^9 + iV^4\sin(k\mu)\right)\cos\left(\beta\mu K_0\right) + V^8\sin\left(\beta\mu K_0\right) + V^4 + V^9\right]$$

$$0 = P_0\left[-\cos(k\mu)\left(V^{13}\sin\left(\beta\mu K_0\right) - V^{14}\left(\cos\left(\beta\mu K_0\right) - 1\right)\right) + i\sin(k\mu)\left(V^{13}\sin\left(\beta\mu K_0\right) - V^{14}\cos\left(\beta\mu K_0\right) + V^{14}\right) + V^{16}\sin\left(\beta\mu K_0\right) + V^{15}\cos\left(\beta\mu K_0\right) - V^{15}\right]$$
$$+\beta K_0\left[\cos(k\mu)\left(V^4\sin\left(\beta\mu K_0\right) - V^5\cos\left(\beta\mu K_0\right)\right) - i\sin(k\mu)\left(V^4\sin\left(\beta\mu K_0\right) - V^5\cos\left(\beta\mu K_0\right)\right) - V^7\sin\left(\beta\mu K_0\right) - V^6\cos\left(\beta\mu K_0\right) + V^5 + V^6\right]$$

EOMs and closure conditions are solved numerically.

Continuum Limit of Cosmological Perturbations

18 linear EOMs of cosmological perturbations: $$\frac{\mathrm{d}V^\rho(\tau,k)}{\mathrm{d}\tau} = \mathbf{U}^\rho{}_\nu(\mu,\tau,k)\ V^\nu(\tau,k)\ , \qquad \mu\to 0$$

3 linearized closure conditions $\mu\to 0$ reduces to linearized Gauss constraint

3-metric perturbations: $$q_{IJ}(\tau,k) = P_0(\tau)\delta_{IJ} + \delta h_{IJ}(\tau,k) \qquad \delta h_{IJ} = \begin{pmatrix} -V^1+V^2+V^3 & -V^4-V^7 & -V^5-V^8 \\ -V^4-V^7 & V^1-V^2+V^3 & -V^6-V^9 \\ -V^5-V^8 & -V^6-V^9 & V^1+V^2-V^3 \end{pmatrix}.$$

SVT decomposition: $$\delta h_{IJ} = P_0\left(2\psi\delta_{IJ} + 2\partial_I\partial_J E + 2\partial_{(I}\mathscr{F}_{J)} + h^T_{IJ}\right) \qquad \delta N_I = \sqrt{P_0}(\partial_I B + S_I)$$

Scalar modes:
(conformal time η)

$$2\mathscr{H}(\eta)\frac{\mathrm{d}\psi(\eta,k)}{\mathrm{d}\eta} + \frac{\mathrm{d}^2\psi(\eta,k)}{\mathrm{d}\eta^2} = 0$$

$$\frac{\mathrm{d}^2E(\eta,k)}{\mathrm{d}\eta^2} + 2\mathscr{H}(\eta)\frac{\mathrm{d}E(\eta,k)}{\mathrm{d}\eta} - \alpha\mathscr{H}(\eta)B(\eta,k) - \psi(\eta,k) = 0$$

Vector modes: $$2\mathscr{H}(\eta)\frac{\mathrm{d}\partial_{(I}\mathscr{F}_{J)}(\eta,k)}{\mathrm{d}\eta} + \frac{\mathrm{d}\partial_{(I}\mathscr{F}_{J)}(\eta,k)}{\mathrm{d}\eta^2} - \alpha\mathscr{H}(\eta)\partial_{(I}S_{J)}(\eta,k) = 0$$

Tensor modes: $$k^2h^T_{IJ}(\eta,k) + 2\mathscr{H}\frac{\mathrm{d}h^T_{IJ}(\eta,k)}{\mathrm{d}\eta} + \frac{\mathrm{d}^2h^T_{IJ}(\eta,k)}{\mathrm{d}\eta^2} = 0$$

Reproduce the classical gauge invariant cosmological perturbation theory in [Giesel, Hofmann, Thiemann, and Winkler 2007], when $\alpha = 1$ (Brown-Kuchar dust)

We obtain gravitons as spin-2 particles emergent from full LQG

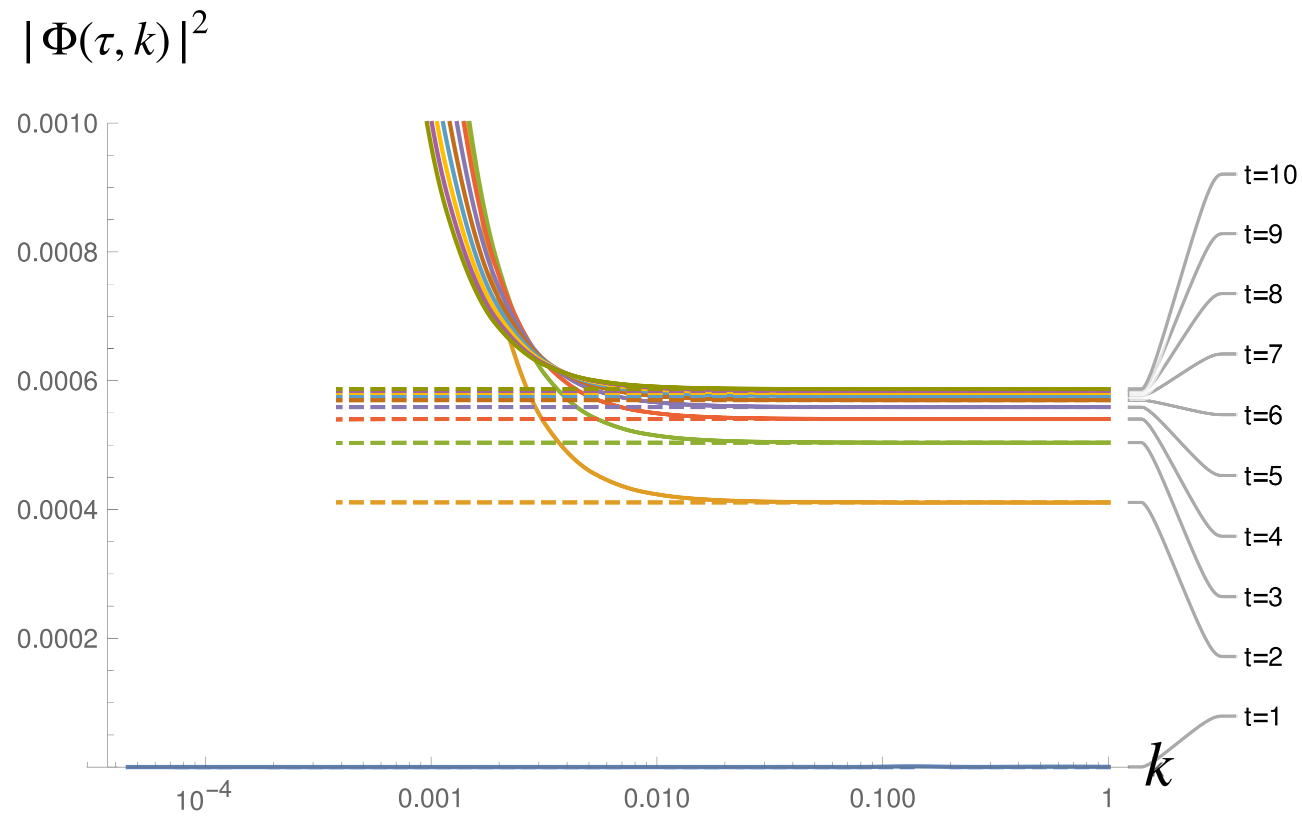
Note: Warsaw's Hamiltonian by Alesci, Assanioussi, Lewandowski, and Makinen seems not good in the continuum limit

Linear EOMs of Cosmological Perturbations

Scalar mode power spectrum:

Matter contribution: only dust, no radiation

We didn't take into account the inflation



$\Lambda = 10^{-5}, \quad \mu = 10^{-2}, \quad \alpha = 1$

Bardeen potential

$$\Phi = \psi + \mathscr{H}(B - E')$$

Initial condition

- $\Phi(1,k) = 0$,
- **No energy perturbation**
- $\delta N_I(1,k) = 0$
- $\psi(1,k) = 0.001$

Tensor modes: Late time $K_0(\tau) \to 0$ dispersion relation ($\Lambda \to 0$)

$$\omega^2 = \frac{\sin^2(k\mu)\left(\left(\beta^2+1\right)\cos(k\mu) - \beta^2\right)}{\mu^2} = k^2\left[1 - \frac{1}{6}\mu^2 k^2\left(3\beta^2+5\right) + O\left(\mu^3\right)\right]$$

Same as the one obtained by Dapor and Liegener 2020

Vector modes are interfered by tensor modes at the discrete level
Interference disappears at the Lattice continuum limit $\mu \to 0$

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Relation to Numerical Relativity

$$\begin{pmatrix} \mathrm{d}\vec{p}(e)/\mathrm{d}\tau \\ \mathrm{d}\vec{\theta}(e)/\mathrm{d}\tau \end{pmatrix} = \frac{i\kappa}{a^2} T^{-1}(p,\theta) \begin{pmatrix} \partial \mathbf{H}/\partial \vec{p}(e) \\ \partial \mathbf{H}/\partial \vec{\theta}(e) \end{pmatrix}$$

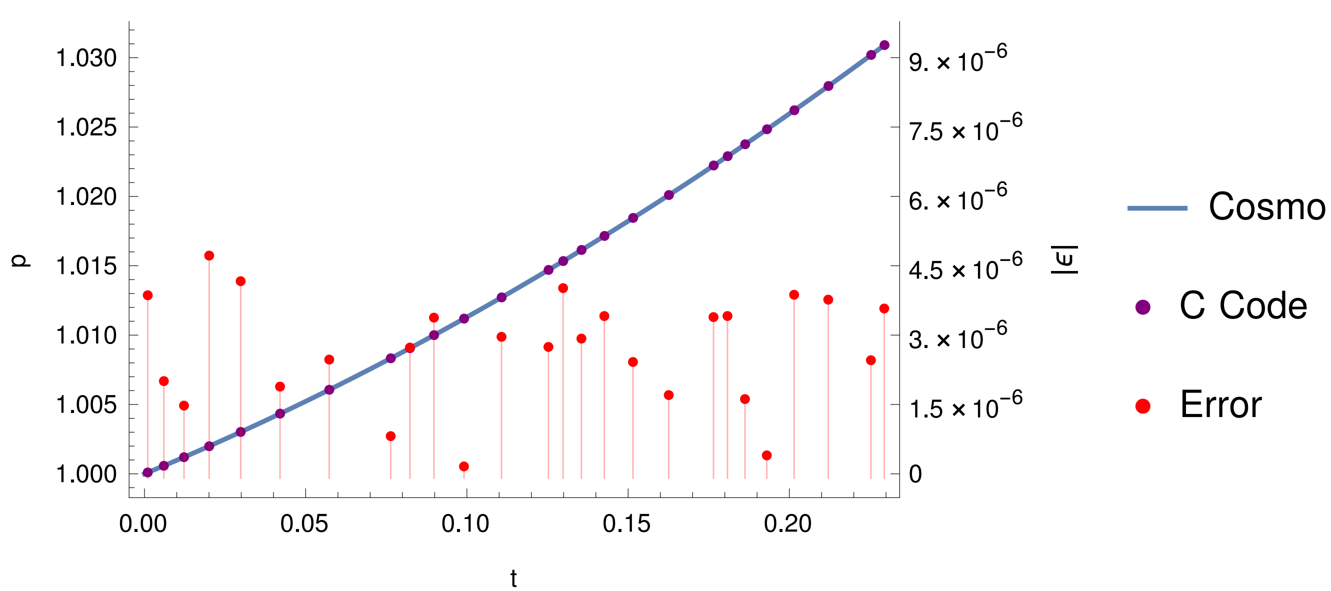
→ **Symmetry reduction, linearization, etc to simply equations** → **Solve simplified equations**

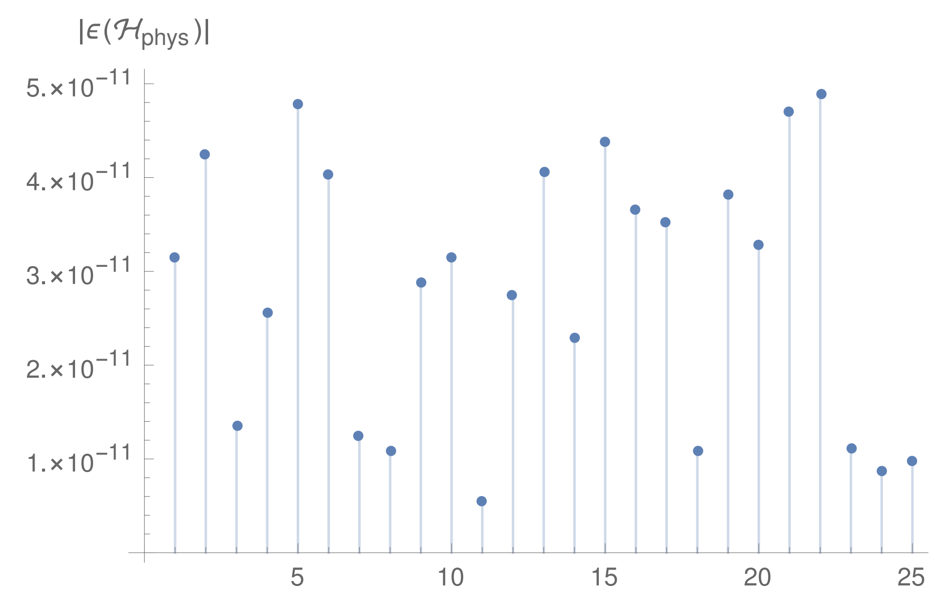
→ **Numerically solve the full EOMs, symmetry of solution is imposed by initial conditions**

- **Semiclassical EOMs of the full LQG is of the same type as it has been studied in Numerical Relativity.**
- **Given various initial data, numerical methods can create spacetimes from semiclassical LQG.**

We have made a C++ package for numerically evolving the full EOMs, and run various tests for codes.

Revisit homogeneous isotropic cosmology





Conclusion

- We have present a new path integral formulation of LQG transition amplitude

$$\frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\|\ \|\Psi^t_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i\, \nu[g]\ e^{S[g,h]/t}$$

- Semiclassical limit reproduces the classical gravity-dust theory on the continuum

- Comparing with spin foam formulation, the new formulation has advantages including finiteness, unitarity, relation with canonical LQG, and absence of cosine and flatness problems

- Derive cosmological perturbation theory from the full LQG theory: scalar, vector, tensor modes

  Scalar modes: power spectrum

  Tensor modes: graviton as spin-2 excitations from LQG

- Semiclassical dynamics of full LQG relates to Numerical Relativity

Outlook

- **Computation of LQG transition amplitude: matrix elements / expectation values of Hamiltonian.**

$$\frac{A_{[g],[g']}}{\|\Psi^t_{[g]}\| \; \|\Psi^t_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i \, \nu[g] \; e^{S[g,h]/t}$$

$$S[g,h] = \sum_{i=0}^{N+1} \sum_{e \in E(\gamma)} \left[z_{i+1,i}(e)^2 - \frac{1}{2} p_{i+1}(e)^2 - \frac{1}{2} p_i(e)^2 \right] - \frac{i\kappa}{a^2} \sum_{i=1}^{N} \Delta\tau \left[\frac{\langle \psi^t_{g_{i+1}} | \hat{\mathbf{H}} | \psi^t_{g_i} \rangle}{\langle \psi^t_{g_{i+1}} | \psi^t_{g_i} \rangle} + i\epsilon_{i+1,i} \left(\frac{\Delta\tau}{\hbar} \right) \right]$$

- **Behavior of lattice refinement at quantum level.**

- **Cosmological perturbation theory with inflaton, and phenomenology.**

- **Other semiclassical spacetimes using numerical solutions: Black holes and other generic spacetimes**

- **Cosmological perturbation theory from $\bar{\mu}$ - scheme LQG effective dynamics** **MH and H. Liu 2019**

Outlook

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Thanks for your attention !