Path Integral Formulation of LQG, Semiclassical Limit, and Cosmological Perturbation Theory

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MH and Hongguang Liu, arXiv:1910.03763
MH and Hongguang Liu, arXiv:1912.08668
Outline

① Review / introduce the path integral formulation of reduced phase space LQG
② Semiclassical limit and equations of motion (EOMs)
③ Comparing with spin foam formulation
④ Cosmological perturbation theory from full LQG
⑤ Relation with Numerical Relativity
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Reduced phase space LQG

3 scenarios of deparametrized models: gravity coupled to

- Brown-Kuchar dust

\[
S_{B KD} \left[ \rho, g_{\mu\nu}, T, S^j, W_j \right] = -\frac{1}{2} \int d^4x \sqrt{|\text{det}(g)|} \rho \left[ g^{\mu\nu} U_\mu U_\nu + 1 \right], \quad U_\mu = -\partial_\mu T + W_j \partial_\mu S^j
\]

- Gaussian dust

\[
S_{GD} \left[ \rho, g_{\mu\nu}, T, S^j, W_j \right] = -\int d^4x \sqrt{|\text{det}(g)|} \left[ \frac{\rho}{2} \left( g^{\mu\nu} \partial_\mu T \partial_\nu T + 1 \right) + g^{\mu\nu} \partial_\mu T \left( W_j \partial_\nu S^j \right) \right]
\]

- Massless real scalar field

\[
S_\phi \left[ g_{\mu\nu}, \phi \right] = -\frac{1}{2} \int d^4x \sqrt{|\text{det}(g)|} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi
\]

Dirac observables = parametrizing gravity variables with values of dust fields

\[
T(x) \equiv \tau, \quad S^j(x) \equiv \sigma^j
\]

\(\tau\): physical time variable

\(\sigma\): physical space variable

Gravity Dirac observables

\[
A(\sigma, \tau) = A(x) \bigg|_{T(x) \equiv \tau, S^j(x) \equiv \sigma^j}, \quad E(\sigma, \tau) = E(x) \bigg|_{T(x) \equiv \tau, S^j(x) \equiv \sigma^j}
\]
Reduced phase space LQG

Canonical structure of Dirac observables:

\[ \left\{ E^i_a(\sigma, \tau), A^b_j(\sigma', \tau) \right\} = \frac{1}{2} \kappa \beta \delta^i_j \delta^b_a \delta^3(\sigma, \sigma') \]

Solving constraints (Abelianized constraints):

\[ C^{tot} = P + h(p, q, \partial_a T) \approx 0, \quad C^j_{tot} = P_j + S^a_j \left[ C_a(p, q) + P \partial_a T \right] \approx 0 \]

Physical Hamiltonian (generating \( \tau \) evolution):

\[ \frac{df}{d\tau} = \{ H_0, f \}, \quad H_0 = \int_{\mathcal{S}} d^3 \sigma \ h \]

- Brown-Kuchar dust  
  \[ H_0 = \int_{\mathcal{S}} d^3 \sigma \left( \mathcal{C}(\sigma, \tau)^2 - \frac{1}{4} \sum_{a=1}^{3} \mathcal{C}_a(\sigma, \tau)^2 \right) \]

- Gaussian dust  
  \[ H_0 = \int_{\mathcal{S}} d^3 \sigma \ \mathcal{C}(\sigma, \tau) \]

Physical requirements:  

\[ \mathcal{C} < 0, \quad \text{and} \quad \mathcal{C}^2 - \frac{1}{4} \sum_{a=1}^{3} \mathcal{C}_a^2 > 0 \quad (\text{for BK dust}) \]

\[ \mathcal{C} = \frac{1}{\kappa} \left[ F^a_{jk} - (\beta^2 + 1) \epsilon_{adk} K^d_j K^e_k \right] \epsilon_{abc} \frac{E^l_b E^k_c}{\sqrt{\det(q)}} + \frac{2 \Lambda}{\kappa} \sqrt{\det(q)} \]

\[ \mathcal{C}_a = \frac{4}{\kappa \beta} F^b_{jk} \frac{E^l_b E^k_c}{\sqrt{\det(q)}} \]

\[ Giesel \text{ and Thiemann 2007} \]

\[ Giesel \text{ and Thiemann 2015} \]
The quantization is on a fixed graph $\gamma$ on the space without boundary (e.g. $\gamma$ is a cubic lattice partitioning $3$-torus, $\mathcal{S} \simeq T^3$)

Holonomy and flux at every edge (Dirac observables)

$$h(e) := \mathcal{P} \exp \int_e A, \quad \text{and} \quad p^a(e) := -\frac{1}{2\beta a^2} \operatorname{tr} \left[ \tau^a \int_{\mathcal{S}_e} \epsilon_{ijk} d\sigma^i \wedge d\sigma^j h\left(\rho_e(\sigma)\right) E^k_b(\sigma) \tau^b h\left(\rho_e(\sigma)\right)^{-1} \right]$$

$$\tau^a = -i(\text{Pauli matrix})^a$$

$$\mathcal{H}^0_\gamma = \bigotimes_e L^2(\text{SU}(2)) \quad \longrightarrow \quad \mathcal{H}_\gamma$$

Gauss constraint is imposed quantum mechanically

$\mathcal{H}_\gamma$ is already physical Hilbert space because it is constructed with Dirac observables
Reduced phase space LQG

Non-graph-changing Hamiltonian: Positive and self-adjoint

Brown-Kuchar/Gaussian dust

\[ \hat{H} = \sum_{v \in V(\gamma)} \left[ \hat{M}^+ (v) \hat{M}^- (v) \right]^{1/4} = \hat{C}_v \hat{C}_v - \frac{\alpha}{4} \hat{C}_{a,v} \hat{C}_{a,v}, \]

\[ \hat{C}_{\mu,v} := \frac{2}{i \beta k \ell_P^2} \sum_{s_1 s_2 s_3 = \pm 1} s_1 s_2 s_3 e^{i I_s \ell_P^2} \text{Tr} \left( \tau_{\mu} \hat{h} \left( \alpha_v \left[ e_{v;I_1} \right] \right) \hat{h} \left( e_{v;I_2} \right) \hat{h} \left( e_{v;I_3} \right) \right), \quad \mu = 0,1,2,3 \]

Giesel-Thiemann’s Hamiltonian

\[ \hat{C}_v = \hat{C}_{0,v} + \frac{1 + \beta^2}{2} \hat{C}_{L,v} + \frac{2 \Lambda}{\kappa} V_v, \quad \hat{K} = \frac{i}{\hbar \beta^2} \left[ \sum_{v \in V(\gamma)} \hat{C}_{0,v}, \sum_{v \in V(\gamma)} V_v \right] \]

\[ \hat{C}_{L,v} = \frac{16}{3 \kappa \left( i \beta k \ell_P^2 / 2 \right)^3} \sum_{s_1 s_2 s_3 = \pm 1} s_1 s_2 s_3 e^{i I_s \ell_P^2} \text{Tr} \left( \hat{h} \left( e_{v;I_1} \right) \hat{h} \left( e_{v;I_2} \right) \hat{h} \left( e_{v;I_3} \right) \right) \]

Note: For Brown-Kuchar dust, \( \hat{H} \) is a quantization of

\[ \int_S d^3 \sigma \sqrt{C(\sigma, \tau)^2 - \frac{1}{4} \sum_{a=1}^3 C_a(\sigma, \tau)^2} \]

which extend

\[ H_0 = \int_S d^3 \sigma \sqrt{C(\sigma, \tau)^2 - \frac{1}{4} \sum_{a=1}^3 C_a(\sigma, \tau)^2} \] to the entire phase space beyond the physical requirements

\[ C < 0 \quad \text{and} \quad C^2 - \frac{1}{4} \sum_{a=1}^3 C_a^2 > 0 \]. Similar for Gaussian dust.
Path Integral Formulation

Given any non-graph-changing, positive, and self-adjoint physical Hamiltonian $\hat{H}$

Transition amplitude between 2 gauge invariant coherent states:

$$A_{[g],[g']} := \langle \Psi_{[g]}^T | U(T) | \Psi_{[g']}^T \rangle_{\mathcal{F}'} \quad U(\tau) := \exp \left( -\frac{i}{\hbar} T \hat{H} \right)$$

Gauge invariant coherent state: (labelled by gauge orbit $[g]$)

$$\Psi_{[g]}^T = \int dh \psi_{g}^T, \quad \text{where} \quad g^h = \left\{ h^{-1} g(e) h(e) \right\}_{e \in E(\gamma)} \quad dh = \prod_{v \in V(\gamma)} d\mu_H(h_v)$$

$$\psi_{g(e)}^T (h(e)) = \sum_{j_e} \left( 2j_e + 1 \right) e^{-i j_e (j_e + 1)/2} X_{j_e} (g(e) h(e)^{-1})$$

Holomorphic parametrization of LQG phase space

$$g(e) = e^{-ip^a(e) \tau^a/2} e^{\theta^a(e) \tau^a/2} \in \text{SL}(2,\mathbb{C}), \quad p^a(e), \theta^a(e) \in \mathbb{R}^3$$

Semiclassicality parameter (dimensionless):

$$t = \frac{\ell_P^2}{a^2}, \quad a \text{ is a length unit, e.g. 1cm, } t \to 0 \text{ as semiclassical limit}$$

$$\ell_P^2 = \hbar \kappa$$

Discretization and insert N+1 overcompleteness relations

$$\langle \psi_{g}^T | U(T) | \psi_{g}^T \rangle = \langle \psi_{g}^T | e^{-\frac{i}{\hbar} \Delta \hat{H}} \rangle \langle \psi_{g}^T \rangle, \quad \text{where} \quad \Delta \tau = \frac{T}{N} \quad \text{arbitrarily small.}$$

$$= \int d g_{N+1} \cdots d g_{1} \langle \psi_{g}^T | \tilde{\psi}_{g_{N+1}}^T \rangle \langle \tilde{\psi}_{g_{N+1}}^T | e^{-\frac{i}{\hbar} \Delta \hat{H}} \rangle \langle \tilde{\psi}_{g_{N}}^T | e^{-\frac{i}{\hbar} \Delta \hat{H}} \rangle \langle \tilde{\psi}_{g_{N}}^T | \cdots \langle \tilde{\psi}_{g_{2}}^T | e^{-\frac{i}{\hbar} \Delta \hat{H}} \rangle \langle \tilde{\psi}_{g_{1}}^T | \psi_{g}^T \rangle$$

Normalized coherent state

$$\tilde{\psi}_{g(e)}^T = \frac{\psi_{g(e)}^T}{\left\| \psi_{g(e)}^T \right\|}$$
Path Integral Formulation

MH and H. Liu, 2019

Discrete path integral on 4d hypercubic lattice $\gamma \times \text{(discrete time)}$

$$\frac{A_{[g], [g']}}{\| \Psi_{[g]} \| \| \Psi'_{[g']} \|} = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] e^{S[g,h]/t}$$

Semiclassicality parameter $t = \frac{\ell^2}{a^2} = \frac{\hbar \kappa}{a^2}$

$$\int dh : \text{integrating SU(2) gauge freedom}$$

$$g(e) = e^{-ip(e)e^{\tau/2}} e^{\theta(e)e^{\tau/2}} \in \text{SL}(2,\mathbb{C}), \quad p^\alpha(e), \theta^\alpha(e) \in \mathbb{R}^3$$

$\nu[g]$: measure factor (independent of $t$)

Classical action:

$$S[g,h] = \sum_{i=0}^{N+1} \sum_{e \in E(\gamma)} \left[ z_{i+1}(e)^2 - \frac{1}{2} p_{i+1}(e)^2 - \frac{1}{2} p_i(e)^2 \right] - \frac{i \kappa}{a^2} \sum_{i=1}^{N} \Delta \tau \left[ \frac{\langle \psi^i_{g_i+1} | \hat{H} | \psi^i_{g_i} \rangle}{\langle \psi^i_{g_i+1} | \psi^i_{g_i} \rangle} + i \epsilon_{i+1,i} \left( \frac{\Delta \tau}{\hbar} \right) \right]$$

$$z_{i+1}(e) = \arccosh \left( x_{i+1}(e) \right), \quad x_{i+1}(e) = \frac{1}{2} \text{Tr} \left[ g_{i+1}(e)^4 g_i(e) \right]$$

Remarks:

- The path integral formulation is rigorously derived from the canonical LQG (in reduced phase space)
- The path integral computes the LQG amplitude between boundary states
- The path integral is manifestly finite, and manifestly unitary
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Semiclassical Limit and EOMs

MH and H. Liu, 2019

Discrete path integral

\[
A_{[g], [g']} \left/ \|\Psi_{[g]}\| \|\Psi_{[g']}\| \right. = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] e^{S[g,h]/t}
\]

Semiclassical limit \( \hbar \to 0 \) or \( t \to 0 \) and stationary phase approximation

path integral is dominated by critical points satisfying semiclassical EOMs \( \delta S[g, h] = 0 \)

Discrete semiclassical EOMs:

- \( A_{[g], [g']} \left/ \|\Psi_{[g]}\| \|\Psi_{[g']}\| \right. = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] e^{S[g,h]/t} \)

Semiclassicality parameter \( t = \frac{\epsilon^2}{\alpha^2} = \frac{\hbar k}{\alpha^2} \)

Holomorphic deformation in \( SL(2, \mathbb{C}) \)

\( g_i(e) \mapsto g_i^\tau(e) = g_i(e)e^{e_i^\tau(e)\tau^a}, \quad e_i^\tau(e) \in \mathbb{C} \)

(A) \[
\frac{\delta S}{\delta g} = 0 : \quad \frac{D_1 \left( g_{i+1}(e), g_i(e) \right)}{\Delta \tau} = \frac{i \kappa}{a^2} \frac{\partial}{\partial e_i^\tau(e)} \left. \frac{\langle \psi_{g_i^{\tau}(e+\Delta \tau)} | \tilde{H} | \psi_{g_i^{\tau}(e)} \rangle}{\langle \psi_{g_i^{\tau}(e) \uparrow} | \psi_{g_i^{\tau}(e) \uparrow} \rangle} \right|_{\epsilon=0}
\]

(B) \[
\frac{\delta S}{\delta g^*} = 0 : \quad \frac{D_2 \left( g_i(e), g_{i-1}(e) \right)}{\Delta \tau} = - \frac{i \kappa}{a^2} \frac{\partial}{\partial e_i^\tau(e)^*} \left. \frac{\langle \psi_{g_i^{\tau}(e)} | \tilde{H} | \psi_{g_i^{\tau}(e-\Delta \tau)} \rangle}{\langle \psi_{g_i^{\tau}(e) \uparrow} | \psi_{g_i^{\tau}(e) \uparrow} \rangle} \right|_{\epsilon=0}.
\]

(C) \[
\frac{\delta S}{\delta \hbar} = 0 : \quad G^\alpha = \sum_{e_i^\tau(e) = \nu} \Lambda^\alpha_\nu(\theta) p_\nu^\alpha(e) - \sum_{e_i^\tau(e) = \nu} p_\nu^\alpha(e) = 0
\]

closure condition on initial data

\[ z_{i+1,i}(e) = \text{arccosh} \left( x_{i+1,i}(e) \right), \quad x_{i+1,i}(e) = \frac{1}{2} \text{Tr} \left[ g_{i+1}(e) g_i(e) \right], \quad e^{\theta^a \tau^a/2} e^{-\theta^a \tau^a/2} = \Lambda^\alpha_\nu(\theta) \tau^b \]
Semiclassical Limit and EOMs

\[
\frac{A_{[g],[g']}}{\|\Psi'_{[g]}\| \|\Psi'_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] \ e^{\mathcal{S}[g,h]/t}
\]

\[
\delta S = \frac{D_1 (g_{i+1}(e), g_i(e))}{\Delta \tau} = \frac{i \kappa}{a^2} \frac{\partial}{\partial e^a_i(e)} \frac{\langle \psi^t_{g_i+1} | \hat{H} | \psi^t_{g_i} \rangle}{\langle \psi^t_{g_i+1} | \psi^t_{g_i} \rangle} \bigg|_{\epsilon=0},
\]

\[
\delta S = \frac{D_2 (g_i(e), g_{i-1}(e))}{\Delta \tau} = \frac{i \kappa}{a^2} \frac{\partial}{\partial e^a_i(e)^*} \frac{\langle \psi^t_{g_i} | \hat{H} | \psi^t_{g_{i-1}} \rangle}{\langle \psi^t_{g_i} | \psi^t_{g_{i-1}} \rangle} \bigg|_{\epsilon=0},
\]

- Path integral is dominated by neighborhood of \( \| g_{i+1} - g_i \| \sim O(\sqrt{t}) \)
- \( g_{i+1} = g_i \) and \( g_i = g_{i-1} \) are isolated roots of \( D_1(g_{i+1}, g_i) = 0 \) and \( D_2(g_i, g_{i-1}) = 0 \) in the neighborhood

For any discrete solution \( \{g_i\}_{i=1}^{N+1} \) of EOMs, given finiteness of right-hand sides of (A) and (B),

\( \Delta \tau \to 0 \) forces \( D_1, D_2 \to 0 \), which forces \( g_{i+1} \to g_i \).

Namely all discrete solution admits the approximation \( g_i \simeq g(\tau) \) as continuous function in \( \tau \)

\[
\frac{D_1 (g_{i+1}(e), g_i(e))}{\Delta \tau}, \quad \frac{D_2 (g_i(e), g_{i-1}(e))}{\Delta \tau} \sim \text{time derivatives}
\]

Semiclassicality parameter \( t = \frac{\ell^2}{a^2} = \frac{\hbar \kappa}{a^2} \)

\[
D_1(g_{i+1}, g_i) \equiv \frac{z_{i+1} \text{Tr} \left[ \tau^a g_{i+1}^\dagger g_i \right]}{\sqrt{x_{i+1} - 1} \sqrt{x_{i+1} + 1}} - \frac{p_i \text{Tr} \left[ \tau^a g_i g_i^\dagger \right]}{\sinh (p_i)}
\]

\[
D_2(g_i, g_{i-1}) \equiv \frac{z_{i-1} \text{Tr} \left[ \tau^a g_i^\dagger g_{i-1} \right]}{\sqrt{x_{i-1} - 1} \sqrt{x_{i-1} + 1}} - \frac{p_i \text{Tr} \left[ \tau^a g_i g_i^\dagger \right]}{\sinh (p_i)}
\]
Semiclassical Limit and EOMs

Time continuous limit $\Delta \tau \to 0$ of

(A) \[ \frac{\delta S}{\delta g_{i+1}(e)} = 0 : \quad D_1 \left( g_{i+1}(e), g_i(e) \right) = \frac{\delta}{\delta \epsilon_i(e)} \left. \langle \psi_{g_i}^{t_i} | \hat{\mathbf{H}} | \psi_{g_i}^{t_i} \rangle \right|_{\epsilon=0} \]

(B) \[ \frac{\delta S}{\delta g_i^*} = 0 : \quad D_2 \left( g_i(e), g_{i-1}(e) \right) = -\frac{\delta}{\delta \epsilon_i(e)^*} \left. \langle \psi_{g_i}^{t_i} | \hat{\mathbf{H}} | \psi_{g_{i-1}}^{t_{i-1}} \rangle \right|_{\epsilon=0} \]

- Right-hand sides: Matrix elements of $\hat{\mathbf{H}}$ (hard to compute) are reduced to expectation values of $\hat{\mathbf{H}}$ (easier to compute).

\[ \langle \psi_{g_i}^{t_i} | \hat{\mathbf{H}} | \psi_{g_{i-1}}^{t_{i-1}} \rangle = \mathbf{H} \left[ g_i^e \right] + O(\hbar) \]

Giesel and Thiemann 2006

Classical discrete Hamiltonian

- Left-hand sides: Reduce to derivatives of holonomies and fluxes.

\[ p^a(e), \theta^a(e) \in \mathbb{R}^3 \]

\[ g(e) = e^{-ip^a(e)\tau^a/2}e^{\theta^a(e)\tau^a/2} \in \text{SL}(2,\mathbb{C}) \]

\[ p^a(e) : \text{flux} \]

\[ e^{\theta^a(e)\tau^a/2} : \text{holonomy} \]

(A) and (B) reduces to

\[ \left( \frac{d\bar{p}(e)/d\tau}{d\bar{\theta}(e)/d\tau} \right) = \frac{i\kappa}{a^2} T^{-1}(p, \theta) \left( \frac{\partial \mathbf{H}}{\partial \bar{p}(e)} \left/ \frac{\partial \mathbf{H}}{\partial \bar{\theta}(e)} \right. \right) \]

All discrete solution of (A) and (B) admits the approximation $g_i \simeq g(\tau)$ as differentiable function

6 × 6 matrix $T(p, \theta)$ contains long formulae, see https://github.com/LQG-Florida-Atlantic-University/Classical-EOM
Relation with Hamilton’s Equations

Time continuous limit of semiclassical EOMs

\[
\begin{pmatrix}
\frac{d\vec{p}(e)}{d\tau} \\
\frac{d\vec{\theta}(e)}{d\tau}
\end{pmatrix}
= \frac{i\kappa}{a^2} T^{-1}(p, \theta)
\begin{pmatrix}
\partial H / \partial \vec{p}(e) \\
\partial H / \partial \vec{\theta}(e)
\end{pmatrix}
\]

\[p^a(e): \text{flux}\]

\[e^{\theta^a(e)\tau^a/2}: \text{holonomy}\]

Relate to Poisson brackets

\[
\{p^a(e), H\} = \{p^a(e), p^b(e)\} \frac{\partial H}{\partial p^b(e)} + \{p^a(e), \theta^b(e)\} \frac{\partial H}{\partial \theta^b(e)}
\]

\[
\{\theta^a(e), H\} = \{\theta^a(e), p^b(e)\} \frac{\partial H}{\partial p^b(e)} + \{\theta^a(e), \theta^b(e)\} \frac{\partial H}{\partial \theta^b(e)}
\]

\[P(p, \theta) = \begin{pmatrix}
\{p^a(e), p^b(e)\} \\
\{\theta^a(e), p^b(e)\} \\
\{p^a(e), \theta^b(e)\}
\end{pmatrix}
\]

We check that

\[-i\frac{a^2}{\kappa} P(p, \theta) T(p, \theta) = 1_{6\times6}\]

Time continuous limit of semiclassical EOMs is equivalent to Hamilton’s equations of holonomies and fluxes

\[
\frac{dp^a(e)}{d\tau} = \{p^a(e), H\}, \quad \frac{d\theta^a(e)}{d\tau} = \{\theta^a(e), H\}
\]

Gauge invariance \(\frac{dG^a_v}{d\tau} = \{G^a_v, H\} = 0\)

- Semiclassical dynamics is given by Hamiltonian flow generated by physical discrete Hamiltonian
- Semiclassical dynamics becomes an initial value problem in the phase space
- Initial conditions of \(p^a, \theta^a\) uniquely determines a solution within the region where \(H\) is regular
Lattice Continuum Limit of Semiclassical EOMs

\( \mu : \) coordinate length of lattice edge

Lattice continuum limit: \( \mu \to 0, \quad |V(\gamma)| \to \infty, \) while \( \mu^3 |V(\gamma)| \) is fixed

Lattice continuum limit is after the semiclassical limit \( t = \frac{\ell_p^2}{a^2} \to 0 \)

We are in this regime: \( \ell_p \ll \mu \ll a \)

Holonomy and flux:

\[
\theta^a (e_I(\nu)) = \mu [A_I^a(\nu) + O(\mu)], \quad p^a (e_I(\nu)) = \frac{2\mu^2}{\beta a^2} [E_I^a(\nu) + O(\mu)]
\]

Lattice continuum limit of

\[
\frac{dp^a(e)}{d\tau} = \{p^a(e), H\}, \quad \frac{d\theta^a(e)}{d\tau} = \{\theta^a(e), H\}
\]

\[
-\frac{dA_I^a(\nu)}{d\tau} = \frac{\kappa \beta}{2} \frac{\delta \tilde{H}_0}{\delta E_I^a(\nu)} + O(\mu), \quad \frac{dE_I^a(\nu)}{d\tau} = \frac{\kappa \beta}{2} \frac{\delta \tilde{H}_0}{\delta A_I^a(\nu)} + O(\mu)
\]

\[
\tilde{H}_0 = \int_S d^3 \sigma \left[ \sqrt{C(\sigma, \tau)^2 - \frac{\alpha}{4} \sum_{a=1}^3 C_a(\sigma, \tau)^2} \right]
\]

V. S. \( \tilde{H}_0 = \int_S d^3 \sigma \sqrt{C(\sigma, \tau)^2 - \frac{\alpha}{4} \sum_{a=1}^3 C_a(\sigma, \tau)^2} \)

Brown-Kuchar/Gaussian dust \( \alpha = 1 \) or \( 0 \)

classical theory in the continuum

• Semiclassical EOMs coincides with classical theory of gravity-dust when \( C^2 - \frac{\alpha}{4} \sum_{a=1}^3 C_a^2 > 0 \) (physical requirement of gravity-dust)

• Irregularity of the semiclassical dynamics: \( C^2 - \frac{\alpha}{4} \sum_{a=1}^3 C_a^2 = 0 \)
Lattice Continuum Limit of Semiclassical EOMs

\[
\frac{\mathrm{d}A^a_v(v)}{\mathrm{d}\tau} = \frac{\kappa \beta}{2} \frac{\delta \overline{H}_0}{\delta E^I_a(v)} + O(\mu), \quad \frac{\mathrm{d}E^I_a(v)}{\mathrm{d}\tau} = \frac{\kappa \beta}{2} \frac{\delta \overline{H}_0}{\delta A^a_v(v)} + O(\mu)
\]

\[
\overline{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \left[ \mathcal{C}(\sigma, \tau)^2 - \frac{\alpha}{4} \sum_{a=1}^{3} \mathcal{C}_a(\sigma, \tau)^2 \right]
\]

V. S. \quad H_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \left[ \mathcal{C}(\sigma, \tau)^2 - \frac{\alpha}{4} \sum_{a=1}^{3} \mathcal{C}_a(\sigma, \tau)^2 \right]

Brown-Kuchar/Gaussian dust

\[\alpha = 1 \text{ or } 0\]

classical theory in the continuum

Semiclassical consistency and regularity are determined by the initial condition (initial state of path integral) \[g(e) = e^{-i p^a(e) \tau^a/2} e^{\theta^a(e) \tau^a/2}\]

If initial condition in the phase space are within the classically allowed regime, namely satisfy \[\mathcal{C} < 0\] and \[\mathcal{C}^2 - \frac{\alpha}{4} \sum_{a=1}^{3} \mathcal{C}_a^2 > 0\]

Both \[\mathcal{C} < 0\] and \[\mathcal{C}^2 - \frac{\alpha}{4} \sum_{a=1}^{3} \mathcal{C}_a^2 > 0\] are preserved by time evolution when total time \(T\) is finite

The reason is that e.g. for Brown-Kuchar dust \[\mathcal{C}^2 - \frac{1}{4} \sum_{a=1}^{3} \mathcal{C}_a^2\] and \[\mathcal{C}_j\] are conserved in the continuum classical theory, and is approximately conserved up to \(O(\mu)\) in the semiclassical theory

Consequently,

- Semiclassical EOMs is regular.
- The solution is uniquely determined by the initial condition.

If initial coherent states \(\Psi^{I}_{[g]}\) relates to the classically forbidden regime of the phase space, \(\Psi^{I}_{[g]}\) is not semiclassical

Solutions may exist, but not relate to classical theory (analog of negative energy states in QFT, anti-spacetimes)

But here they are controlled by initial states of the path integral.

Giesel and Thiemann 2007
Christodoulou, Riello, Rovelli 2012
Asymptotics of Transition Amplitude

Stationary phase approximation of the path integral as \( t \to 0 \):

If semiclassical boundary state labels \([g], [g']\) are connected by a trajectory \( g(\tau) \) (in phase space) satisfying EOMs

\[
\frac{A_{[g],[g']}}{\|\Psi_{[g']}\| \|\Psi_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] e^{S_{[g,h]} / t} \\
= \int dh \frac{(2\pi t)^{N/2}}{\sqrt{\det(-H)}} \nu[g(\tau), h] e^{S_{[g(\tau),h]} / t} [1 + O(t)]
\]

- Solution of EOM is unique for semiclassical boundary states.
- \( g \) - integral behaves asymptotically as a single oscillatory exponential.
- \( h \) - integral is over all SU(2) gauge transformations of the initial data.

\[
\int dh = \prod_{\nu \in V(\gamma)} \int_{\text{SU}(2)} d\mu_H(h_{\nu})
\]

Amplitude is suppressed if \([g], [g']\) cannot be connected by any trajectory.
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⑤ Relation with Numerical Relativity
Comparing with Spin Foam Formulation

\[
\sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_f, i_e) \prod_v A_v(j_f, i_e) \quad \text{v. s.} \quad \frac{A_{[g],[g']}}{||\Psi_{[g]}|| \cdot ||\Psi_{[g']}||} = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] \cdot e^{S_{[g,h]} / \hbar}
\]

Similarities:
- Both describe transition amplitudes between LQG spin-network states
- The path integral can be reformulated as hypercubic spin foams
- Both relates to classical gravity (Regge calculus and gravity-dust theory respectively)

Open issues of spin foam formulation:

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<tr>
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Comparing with Spin Foam Formulation

\[ \sum_{j_f, i_e} \prod_{f} A_f(j_f) \prod_{e} A_e(j_f, i_e) \prod_{v} A_v(j_f, i_e) \quad \text{v. s.} \quad \frac{A_{[g],[g']}}{||\Psi_{[g]}||^2 ||\Psi_{[g']}||^2} = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] e^{S[g,h]/\hbar} \]

Similarities:

- Both describe transition amplitudes between LQG spin-network states
- The path integral can be reformulated as hypercubic spin foams (Kisielowski and Lewandowski 2012)
- Both relates to classical gravity (Regge calculus and gravity-dust theory respectively)

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<td>Solution is unique determined by a classically allowed initial condition. No cosine (thanks to the time continuous limit).</td>
</tr>
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<td>All physically interesting curved spacetimes can be recovered (see below, cosmology and perturbations).</td>
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Comparing with Spin Foam Formulation

$$\sum \prod_{f} A_{f}(j_f) \prod_{e} A_{e}(j_f, i_e) \prod_{v} A_{v}(j_f, i_e)$$

\[ \text{v. s.} \quad \frac{A_{[g],[g']}}{||\Psi_{[g]}|| \ ||\Psi'_{[g']}||} = \int dh \prod_{i=1}^{N+1} d g_{i} \nu[g] \ e^{S[g,h]/l} \]

$$S[g,h] = \sum_{i=0}^{N+1} \sum_{e \in \mathcal{E}(\gamma)} \left[ z_{i+1,i}(e)^{2} - \frac{1}{2} p_{i+1}(e)^{2} - \frac{1}{2} p_{i}(e)^{2} \right] - \frac{i \kappa}{a^{2}} \sum_{i=1}^{N} \Delta \tau \left[ \frac{\langle \psi_{g_{i+1}}^{t} | \hat{H} | \psi_{g_{i}}^{t} \rangle}{\langle \psi_{g_{i+1}}^{t} | \psi_{g_{i}}^{t} \rangle} + i \epsilon_{i+1,i} \left( \frac{\Delta \tau}{\hbar} \right) \right]$$

Open issues of both formulation:

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<tr>
<th>Computational complexity</th>
<th>The complexity grows fast for large $j$, and when number of vertices grows</th>
<th>Non-polynomial Hamiltonian operator, but perturbative computations are allowed</th>
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Cosmological Perturbation Theory from Full LQG

Recall semiclassical EOMs (from the path integral):

$$
\left( \frac{d \vec{p}(e) / d \tau}{d \vec{\theta}(e) / d \tau} \right) = \frac{i \kappa}{a^2} T^{-1}(p, \theta) \left( \frac{\partial H / \partial \vec{p}(e)}{\partial H / \partial \vec{\theta}(e)} \right)
$$

OR

$$
\frac{d p^a(e)}{d \tau} = \{ p^a(e), H \}, \quad \frac{d \theta^a(e)}{d \tau} = \{ \theta^a(e), H \}
$$

$p^a(e)$: flux

$e^{\theta^a(e) e^{\theta^a(e)/2}}$: holonomy

Insert the ansatz of cosmological perturbation theory

Homogenous and isotropic cosmology

$$
\theta^a \left( e_I(v) \right) = \mu \left[ \beta K_0 \delta^a + \chi^a \left( e_I(v) \right) \right], \quad p^a \left( e_I(v) \right) = \frac{2 \mu^2}{\beta a^2} \left[ P_0 \delta^a + \gamma^a \left( e_I(v) \right) \right]
$$

Perturbations of holonomy and fluxes

0-th order EOMs:

$\mu_0$ - scheme effective cosmology

Unsymmetrical bounce

1-st order EOMs:

Linear EOMs of perturbations on $\mu_0$ - scheme
effective cosmological background
Linear EOMs of Cosmological Perturbations

Perturbations of fluxes and holonomies

\[ V^\rho(v) = \left( \mathcal{Y}^a \left( e_\rho(v) \right), \mathcal{X}^a \left( e_\rho(v) \right) \right)^T, \quad \rho = 1, \ldots, 18 \]

Lattice Fourier transformation

\[ V^\rho(\tau, \overline{\sigma}) = \int_{-\pi/\mu}^{\pi/\mu} \frac{d^3k}{(2\pi)^3} e^{i \vec{k} \cdot \overline{\sigma}} V^\rho(\tau, -\vec{k}), \quad \overline{\sigma} \in (\mu \mathbb{Z})^3 \]

Integral over 1st Brillouin zone

\[ |k^i| \leq \frac{\pi}{\mu} \quad \text{UV cut-off} \]

18 linear EOMs of cosmological perturbations:

\[
\frac{dV^\rho(\tau, k)}{d\tau} = U^\rho_\nu(\mu, \tau, k) V^\nu(\tau, k) \quad \overline{k} = (k, 0, 0)
\]

18 × 18 matrix \( U(\mu, \tau, k) \) contains long formulae, will appear at https://github.com/LQG-Florida-Atlantic-University/cos_pert

3 linearized closure conditions (for initial conditions):

\[
0 = P_0 \left[ (V^{15} - V^{18}) \sin(\beta \mu K_0) - (V^{16} + V^{17}) \left( \cos(\beta \mu K_0) - 1 \right) \right] + \beta K_0 \left[ -i V^4 \sin(k \mu) + V^1 \cos(k \mu) - V^6 \sin(\beta \mu K_0) + V^9 \sin(\beta \mu K_0) 
+ V^7 \cos(\beta \mu K_0) + V^8 \cos(\beta \mu K_0) - V^1 - V^3 - V^7 - V^8 \right] \\
0 = P_0 \left[ \cos(k \mu) \left( V^{14} \sin(\beta \mu K_0) + V^{13} \cos(\beta \mu K_0) - V^{12} \right) - i \sin(k \mu) \left( V^{14} \sin(\beta \mu K_0) + V^{13} \cos(\beta \mu K_0) - V^{13} \right) - V^4 \sin(\beta \mu K_0) - V^5 \cos(\beta \mu K_0) - V^4 + V^9 \right] + \beta K_0 \left[ i V^5 \sin(k \mu) \sin(\beta \mu K_0) - \cos(k \mu) \left( V^5 \sin(\beta \mu K_0) + V^4 \cos(\beta \mu K_0) \right) + \left( -V^9 + i V^4 \sin(k \mu) \right) \cos(\beta \mu K_0) + V^8 \sin(\beta \mu K_0) + V^4 + V^9 \right] \\
0 = P_0 \left[ -\cos(k \mu) \left( V^{13} \sin(\beta \mu K_0) - V^{14} \cos(\beta \mu K_0) - 1 \right) + i \sin(k \mu) \left( V^{13} \sin(\beta \mu K_0) - V^{14} \cos(\beta \mu K_0) + V^{14} \right) + V^{16} \sin(\beta \mu K_0) + V^{15} \cos(\beta \mu K_0) - V^{15} \right] + \beta K_0 \left[ \cos(k \mu) \left( V^4 \sin(\beta \mu K_0) - V^5 \cos(\beta \mu K_0) \right) - i \sin(k \mu) \left( V^4 \sin(\beta \mu K_0) - V^5 \cos(\beta \mu K_0) \right) - V^3 \sin(\beta \mu K_0) - V^5 \cos(\beta \mu K_0) + V^5 + V^6 \right]
\]

EOMs and closure conditions are solved numerically.
Continuum Limit of Cosmological Perturbations

18 linear EOMs of cosmological perturbations:

\[
\frac{dV^\mu(\tau, k)}{d\tau} = U^\mu_\nu(\mu, \tau, k) V^\nu(\tau, k), \quad \mu \to 0
\]

3 linearized closure conditions \( \mu \to 0 \) reduces to linearized Gauss constraint

3-metric perturbations:

\[ q_{IJ}(\tau, k) = P_0(\tau) \delta_{IJ} + \delta h_{IJ}(\tau, k) \]

\[ \delta h_{IJ} = \begin{pmatrix}
-V^1 + V^2 + V^3 & -V^4 - V^7 & -V^5 - V^8 \\
-V^4 - V^7 & V^1 - V^2 + V^3 & -V^6 - V^9 \\
-V^5 - V^8 & -V^6 - V^9 & V^1 + V^2 - V^3
\end{pmatrix}.
\]

SVT decomposition:

\[ \delta h_{IJ} = P_0 \left( 2 \psi \delta_{IJ} + 2 \partial_I \partial_J \mathcal{E} + 2 \partial_I \mathcal{F}_J + h^T_{IJ} \right) \]

\[ \delta N_I = \sqrt{P_0} (\partial_I B + S_I) \]

Scalar modes:
(angular momentum \( \mathcal{I} \))

\[ 2 \mathcal{H}(\eta) \frac{d\psi(\eta, k)}{d\eta} + \frac{d^2 \psi(\eta, k)}{d\eta^2} = 0 \]

\[ \frac{d^2 E(\eta, k)}{d\eta^2} + 2 \mathcal{H}(\eta) \frac{dE(\eta, k)}{d\eta} - \alpha \mathcal{H}(\eta) B(\eta, k) - \psi(\eta, k) = 0 \]

Vector modes:

\[ 2 \mathcal{H}(\eta) \frac{d\partial_I \mathcal{F}_J(\eta, k)}{d\eta} + \frac{d\partial_I \mathcal{F}_J(\eta, k)}{d\eta^2} - \alpha \mathcal{H}(\eta) \partial_I S_J(\eta, k) = 0 \]

Tensor modes:

\[ k^2 h^T_{IJ}(\eta, k) + 2 \mathcal{H} \frac{dh^T_{IJ}(\eta, k)}{d\eta} + \frac{d^2 h^T_{IJ}(\eta, k)}{d\eta^2} = 0 \]

We obtain gravitons as spin-2 particles emergent from full LQG

Reproduce the classical gauge invariant cosmological perturbation theory in [Giesel, Hofmann, Thiemann, and Winkler 2007], when \( \alpha = 1 \) (Brown-Kuchar dust)

Note: Warsaw’s Hamiltonian by Alesci, Assanioussi, Lewandowski, and Makinen seems not good in the continuum limit
Scalar mode
power spectrum:

Matter contribution:
only dust, no radiation

We didn’t take into
account the inflation

Tensor modes: Late time \( K_0(\tau) \rightarrow 0 \) dispersion relation \((\Lambda \rightarrow 0)\)

\[
\omega^2 = \frac{\sin^2(k\mu)\left( (\beta^2 + 1) \cos(k\mu) - \beta^2 \right)}{\mu^2} = k^2 \left[ 1 - \frac{1}{6} \mu^2 k^2 (3\beta^2 + 5) + O(\mu^3) \right]
\]

Vector modes are interfered by tensor modes at the discrete level
Interference disappears at the Lattice continuum limit \( \mu \rightarrow 0 \)

Bardeen potential
\( \Phi = \psi + \mathcal{H}(B - E') \)

Initial condition
- \( \Phi(1, k) = 0 \),
- No energy perturbation
- \( \delta N_I(1, k) = 0 \)
- \( \psi(1, k) = 0.001 \)

\( \Lambda = 10^{-5}, \mu = 10^{-2}, \alpha = 1 \)
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Relation to Numerical Relativity

Symmetry reduction, linearization, etc to simply equations

\[
\left( \frac{d\bar{\rho}(e)}{d\tau} , \frac{d\bar{\theta}(e)}{d\tau} \right) = \frac{i\kappa}{\alpha^2} T^{-1}(p, \theta) \left( \frac{\partial H}{\partial \bar{\rho}(e)} , \frac{\partial H}{\partial \bar{\theta}(e)} \right)
\]

Solve simplified equations

Numerically solve the full EOMs, symmetry of solution is imposed by initial conditions

- Semiclassical EOMs of the full LQG is of the same type as it has been studied in Numerical Relativity.
- Given various initial data, numerical methods can create spacetimes from semiclassical LQG.

We have made a C++ package for numerically evolving the full EOMs, and run various tests for codes.

Revisit homogeneous isotropic cosmology
Conclusion

- We have presented a new path integral formulation of LQG transition amplitude

\[
\frac{A_{[g],[g']}}{\|\Psi_{[g]}\| \|\Psi_{[g']}\|} = \int \frac{dh}{N+1} \prod_{i=1}^{N+1} dg_i \nu[g] e^{S[g,h]/t}
\]

- Semiclassical limit reproduces the classical gravity-dust theory on the continuum

- Comparing with spin foam formulation, the new formulation has advantages including finiteness, unitarity, relation with canonical LQG, and absence of cosine and flatness problems

- Derive cosmological perturbation theory from the full LQG theory: scalar, vector, tensor modes

  Scalar modes: power spectrum
  Tensor modes: graviton as spin-2 excitations from LQG

- Semiclassical dynamics of full LQG relates to Numerical Relativity
• Computation of LQG transition amplitude: matrix elements / expectation values of Hamiltonian.

\[
\frac{A_{[g],[g']}}{\|\Psi_{[g]}\| \|\Psi_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] \ e^{S_{[g,h]}/\hbar}
\]

\[
S_{[g,h]} = \sum_{i=0}^{N+1} \sum_{e \in E(\gamma)} \left[ z_{i+1,e}(e)^2 - \frac{1}{2} p_{i+1}(e)^2 - \frac{1}{2} p_i(e)^2 \right] - \frac{i\kappa}{a^2} \sum_{i=1}^{N} \Delta \tau \left[ \frac{\langle \psi_{\tilde{g}_{i+1}}^i | \hat{H} | \psi_{\tilde{g}_{i+1}}^i \rangle}{\langle \psi_{\tilde{g}_{i+1}}^i | \psi_{\tilde{g}_{i+1}}^i \rangle} + i\epsilon_{i+1,i} \left( \frac{\Delta \tau}{\hbar} \right) \right]
\]

• Behavior of lattice refinement at quantum level.

• Cosmological perturbation theory with inflaton, and phenomenology.

• Other semiclassical spacetimes using numerical solutions: Black holes and other generic spacetimes

• Cosmological perturbation theory from $\tilde{\mu}$ - scheme LQG effective dynamics  
  
  MH and H. Liu 2019
Outlook

- Computation of LQG transition amplitude: matrix elements / expectation values of Hamiltonian.

\[
\frac{A_{[g],[g']}}{\|\Psi_{[g]}\| \|\Psi_{[g']}\|} = \int dh \prod_{i=1}^{N+1} dg_i \nu[g] e^{S[g,h]t}
\]

\[
S[g,h] = \sum_{i=0}^{N+1} \sum_{e \in E_i} \left[ z_{i+1,e} (e^2 - \frac{1}{2} p_{i+1} (e)^2 - \frac{1}{2} p_i (e)^2) \right] - \frac{i \kappa}{a^2} \sum_{i=1}^{N} \Delta \tau \left[ \frac{\langle \psi_{g_i+1} \hat{H} \psi_{g_i} \rangle}{\langle \psi_{g_i+1} \psi_{g_i} \rangle} + i \epsilon_{i+1,i} \left( \frac{\Delta \tau}{\hbar} \right) \right]
\]

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Thanks for your attention!