Invariance of Connections and Measures in Loop Quantum Cosmology

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Symmetry Reduction on Classical Level

Ingredients: principal bundle P $+$ Symmetry on P

Config. Spaces	full theory	reduced theory	
Classical level	$\mathcal{A}$	$\mathcal{A}_{\mathrm{inv}}$	connections
Quantum level	$\overline{\mathcal{A}}$	$\overline{\mathcal{A}_{\mathrm{inv}}}$	compactification

- ▷ $\overline{\mathcal{A}} = \mathrm{Spec}(\mathfrak{C}_{\mathrm{full}})$ $\mathfrak{C}_{\mathrm{full}}$ – separating unital C^*-subalgebra of $B(\mathcal{A})$
- ▷ $\overline{\mathcal{A}_{\mathrm{inv}}} = \mathrm{Spec}(\mathfrak{C}_{\mathrm{inv}})$ $\mathfrak{C}_{\mathrm{inv}}$ – separating unital C^*-subalgebra of $B(\mathcal{A}_{\mathrm{inv}})$

Standard LQC: $\mathcal{A}_{\mathrm{inv}} \cong \mathbb{R}$ $\mathfrak{C}_{\mathrm{inv}} = C_{\mathrm{AP}}(\mathbb{R})$ $\overline{\mathcal{A}_{\mathrm{inv}}} = \mathbb{R}_{\mathrm{Bohr}}$

Fleischhack 2010:

$$\begin{tikzcd} \mathcal{A}_{\mathrm{inv}} \arrow[r, hook, "i"] \arrow[d] & \mathcal{A} \arrow[d] \\ \overline{\mathcal{A}_{\mathrm{inv}}} \arrow[r, hook, "\bar{i}"] & \overline{\mathcal{A}} \end{tikzcd}$$

- ▷ $\bar{i}$ exists and unique $\Longleftrightarrow$ $i^*(\mathfrak{C}_{\mathrm{full}}) \subseteq \mathfrak{C}_{\mathrm{inv}}$
- ▷ $\bar{i}$ embedding $\Longleftrightarrow$ $i^*(\mathfrak{C}_{\mathrm{full}}) \subseteq \mathfrak{C}_{\mathrm{inv}}$ dense

$\Longrightarrow$ can always choose $\mathfrak{C}_{\mathrm{inv}} := \overline{i^*(\mathfrak{C}_{\mathrm{full}})}$

$\Longrightarrow$ LQC: $\overline{\mathcal{A}_{\mathrm{inv}}} = \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$

Synopsis

Introduce notion of lifted action and apply it to LQG

- Quantum-reduction of LQG configuration spaces.
- Singling out measures by means of invariance properties.

Mathematical Setting

- X – set together with action $\theta\colon G \times X \to X$
  $\Longrightarrow \quad X_\theta := \{x \in X \,|\, \theta_g(x) = x \;\forall\, g \in G\}$ = reduced class. conf. space
- $\mathfrak{C}$ – C^*-subalgebra of $B(X)$ ($B(X)$ – bounded functions on X)
- $\overline{X}$ – $\mathrm{Spec}(\mathfrak{C})$ = quantum space $\iota\colon X \hookrightarrow \overline{X},\ \iota(x)\colon f \mapsto f(x)$

Crucial Observation (MH13)

If $\theta_g^*(\mathfrak{C}) \subseteq \mathfrak{C}$ for all $g \in G$, have unique lift $\Theta\colon G \times \overline{X} \to \overline{X}$ with

- Θ_g continuous $\forall\, g \in G \quad \Longrightarrow \quad \overline{X}_\Theta$ closed (compact if $\mathfrak{C}$ unital)
- $\Theta_g \circ \iota = \iota \circ \theta_g \quad \Longrightarrow \quad \overline{\iota(X_\theta)} \subseteq \overline{X}_\Theta$

LQG – Principal Fibre Bundle (P, π, M, S)

- $X = \mathcal{A}$: smooth connections on P
- $\mathfrak{C}$: cylindrical functions on $\mathcal{A}$ generated by $f \circ h_\gamma$ for γ in fixed set $\mathcal{P}$.
- $\overline{X} = \overline{\mathcal{A}} = \mathrm{Spec}(\mathfrak{C})$: generalized connections

Action θ comes from Lie group of automorphisms

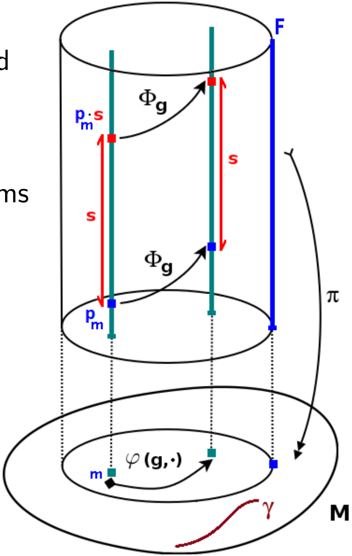
$\Phi\colon G \times P \to P$ = left action with

$$\Phi(g, p \cdot s) = \Phi(g, p) \cdot s$$

for all $g \in G$, $p \in P$, $s \in S$.

$\Longrightarrow$ Have two actions:

- $\theta\colon G \times \mathcal{A} \to \mathcal{A}$, $(g, \omega) \mapsto \Phi^*_{g^{-1}}\omega$
- $\varphi\colon G \times M \to M$, $(g, m) \mapsto (\pi \circ \Phi)(g, p)$ where $p \in F_m$



LQG – Invariance on Classical Level

Reduction on classical level:

- $\mathcal{A}_\theta = \{\omega \in \mathcal{A} \mid \theta(g,\omega) = \omega \; \forall\, g \in G\}$ = invariant connections on P
- Quantized reduced classical space $\overline{\mathcal{A}_\theta} := \mathrm{Spec}(\overline{\mathfrak{C}|_{\mathcal{A}_\theta}}) \cong \overline{\iota(\mathcal{A}_\theta)} \subseteq \overline{\mathcal{A}}$

Example: Homogeneous Isotropic LQC

$P = \mathbb{R}^3 \times \mathrm{SU}(2)$ Φ – euclidean group action $\mathcal{A}_\theta \cong \mathbb{R}$

	ABL [2003]	Fleischhack [2010]
$\mathcal{P}$	linear curves	embedded analytic curves
$\overline{\mathfrak{C}\|_{\mathcal{A}_\theta}}$	$C_{\mathrm{AP}}(\mathbb{R})$	$C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$
$\overline{\mathcal{A}_\theta}$	$\overline{\mathbb{R}}_{\mathrm{l}} = \mathbb{R}_{\mathrm{Bohr}}$	$\overline{\mathbb{R}}_\omega = \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$

Inclusion $i\colon \mathbb{R} \cong \mathcal{A}_\theta \hookrightarrow \mathcal{A}$:

- Extends to embedding $\bar{i}\colon \overline{\mathbb{R}}_\omega \to \overline{\mathcal{A}}$. (embedding of states)
- No such extension for $\overline{\mathbb{R}}_{\mathrm{l}}$. (Brunnemann, Fleischhack 2009)

Homogeneous Isotropic LQC – The Bohr Measures

Finite Radon measure μ on $\overline{\mathbb{R}}_\omega$: $\mathfrak{B}(\overline{\mathbb{R}}_\omega) = \mathfrak{B}(\mathbb{R}) \sqcup \mathfrak{B}(\mathbb{R}_{\mathrm{Bohr}})$

$\Longrightarrow \quad \mu(A) = \mu_1(A \cap \mathbb{R}) + \mu_2(A \cap \mathbb{R}_{\mathrm{Bohr}})$ $\qquad \mu_1$ on $\mathbb{R}$, μ_2 on $\mathbb{R}_{\mathrm{Bohr}}$

Invariance: $\mathcal{A}_\theta \cong \mathbb{R}$ $\qquad \overline{\mathcal{A}_\theta} \cong \overline{\mathbb{R}}_{\mathrm{l}}, \overline{\mathbb{R}}_\omega$ $\qquad$ (MH14)

- Additive action $+ : \mathbb{R} \times \mathcal{A}_\theta \to \mathcal{A}_\theta$ extends to $\overline{+} : \mathbb{R} \times \overline{\mathcal{A}_\theta} \to \overline{\mathcal{A}_\theta}$.
- μ_{Bohr} **unique** invariant normalized Radon measure. (exp. flux)

$\Longrightarrow$ In both cases single out $L^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\mathrm{Bohr}})$ – standard ABL LQC

Embedding of distributional states: (Engle 2013)

$$C_{\mathrm{AP}}(\mathbb{R})^* \hookrightarrow [C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})]^* \xrightarrow{\chi \mapsto \chi \circ i^*} \mathfrak{C}_\omega^*$$

Assumpt. on $\langle \cdot, \cdot \rangle_\mu$:
- $\langle f_0, f_{\mathrm{AP}} \rangle_\mu = 0 \qquad \Longrightarrow \qquad \mu_1 = 0$
- $\langle f_{\mathrm{AP}}, g_{\mathrm{AP}} \rangle_\mu = \langle f_{\mathrm{AP}}, g_{\mathrm{AP}} \rangle_{\mu_{\mathrm{Bohr}}} \qquad \Longrightarrow \qquad \mu_2 = \mu_{\mathrm{Bohr}}$

LQG – Invariance on Quantum Level

Reduction on quantum level:

$$\theta_g^*(\mathfrak{C}) \subseteq \mathfrak{C}: \qquad \theta\colon G \times \mathcal{A} \to \mathcal{A} \quad \overset{\text{Lift}}{\Longrightarrow} \quad \Theta\colon G \times \overline{\mathcal{A}} \to \overline{\mathcal{A}}$$

$$\gamma \in \mathcal{P} \quad \Longrightarrow \quad \varphi_g \circ \gamma \in \mathcal{P} \qquad \forall\, g \in G,\ \forall\, \gamma \in \mathcal{P}.$$

- ▷ Invariant generalized connections

$$\overline{\mathcal{A}}_\Theta := \{\overline{\omega} \in \overline{\mathcal{A}} \,|\, \Theta(g, \overline{\omega}) = \overline{\omega} \quad \forall\, g \in G\}.$$

- ▷ Translations Θ_g continuous $\Longrightarrow$ $\overline{\mathcal{A}}_\Theta$ compact (S compact)
- ▷ Θ extends θ $\Longrightarrow$ $\overline{\mathcal{A}_\theta} \subseteq \overline{\mathcal{A}}_\Theta$

Quantization vs. Reduction: (MH13/14)

LQC	linear curves	embedded analytic curves
(Semi-)homogeneous	$\overline{\mathcal{A}_\theta} \subsetneq \overline{\mathcal{A}}_\Theta$	$\overline{\mathcal{A}_\theta} \subsetneq \overline{\mathcal{A}}_\Theta$
Homogeneous Isotropic	$\overline{\mathcal{A}_\theta} \cong \overline{\mathcal{A}}_\Theta$	$\overline{\mathcal{A}_\theta} \subsetneq \overline{\mathcal{A}}_\Theta$

Reduction – Summary (P, π, M, S) with S compact

$$
\begin{tikzcd}
& \boxed{\text{Symmetry } (G, \Phi) \text{ on } P} \arrow[dl, "\text{Choose } \mathcal{P}"'] \arrow[dr] & \\
\boxed{\overline{\mathcal{A}} = \mathrm{Spec}(\mathfrak{C})} \arrow[d, dash] & & \boxed{\theta\colon G \times \mathcal{A} \to \mathcal{A}} \arrow[d, dash] \arrow[dll, "\text{Lift}"'] \\
\boxed{\Theta\colon G \times \overline{\mathcal{A}} \to \overline{\mathcal{A}}} \arrow[d, dash] & \text{compact spaces} \arrow[dl, dash] \arrow[dr, dash] & \boxed{\mathcal{A}_\theta = \{\omega \in \mathcal{A} \,|\, \mathrm{Stab}_\theta(\omega) = G\}} \arrow[d, dash] \\
\boxed{\overline{\mathcal{A}}_\Theta := \{\overline{\omega} \in \overline{\mathcal{A}} \,|\, \mathrm{Stab}_\Theta(\overline{\omega}) = G\}} & \supseteq & \boxed{\overline{\mathcal{A}_\theta} = \overline{\iota(\mathcal{A}_\theta)} \subseteq \overline{\mathcal{A}}}
\end{tikzcd}
$$

- Lift Θ exists if $\mathcal{P}$ invariant, i.e., $\varphi_g \circ \gamma \in \mathcal{P}$ for all $\gamma \in \mathcal{P}$, $g \in G$.
- Usually fulfilled because φ and $\mathcal{P}$ smooth or analytic.

Quantum Reduction – A Measure on $\mathrm{Hom}_\Theta(\mathcal{P}_\omega, \mathrm{Iso}_F)$

Setting:

- S compact and connected
- $\mathcal{P} = \mathcal{P}_\omega$ – embedded analytic curves in M

$\Longrightarrow$ $\overline{\mathcal{A}}_\Theta \cong \mathrm{Hom}_\Theta(\mathcal{P}_\omega, \mathrm{Iso}_F)$ = space of invariant homomorphisms:

- $\mathrm{A}(\gamma)\colon F_{\gamma(0)} \to F_{\gamma(1)}$ (generalized parallel transport)
- $\mathrm{A}(\varphi_g \circ \gamma)(\Phi_g(p)) = (\Phi_g \circ \mathrm{A})(\gamma)(p)$ (invariance)

Strategy:

- Split up $\mathcal{P}_\omega = \bigsqcup_{\alpha \in I} \mathcal{P}_\alpha$ into natural subsets $\mathcal{P}_\alpha$ closed under inversion and decomposition, and obtain
- $\mathrm{Hom}_\Theta(\mathcal{P}_\omega, \mathrm{Iso}_F) \cong \prod_{\alpha \in I} \mathrm{Hom}_\Theta(\mathcal{P}_\alpha, \mathrm{Iso}_F)$.
- Define a normalized Radon measure μ_α on each $\mathrm{Hom}_\Theta(\mathcal{P}_\alpha, \mathrm{Iso}_F)$ and take the Radon product measure on $\mathrm{Hom}_\Theta(\mathcal{P}_\omega, \mathrm{Iso}_F)$.

Analytic and Pointwise Proper Case (MH14)

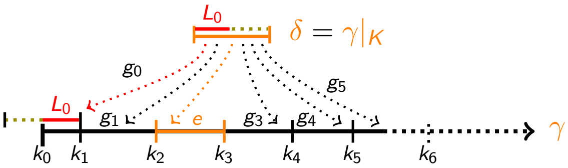
If φ analytic and φ_x proper for each $x \in M$:

- Decomposition $\mathcal{P}_\omega = \mathcal{P}_{\mathrm{cont}} \sqcup \mathcal{P}_{\mathrm{disc}}$ into continuously and discretely generated curves.

Here, $\gamma \in \mathcal{P}_{\mathrm{disc}} \iff \exists\, K \subseteq \mathrm{dom}[\gamma]$ such that for $\delta := \gamma|_K$:

$$\mathrm{im}[\delta] \cap \mathrm{im}[\varphi_g \circ \delta] \text{ infinite for } g \in G \implies \varphi_g \circ \delta = \delta.$$

$\implies$ Can find K such that γ can be decomposed into translates of δ:



- Guarantees that $\mathcal{P}_{\mathrm{cont}}, \mathcal{P}_{\mathrm{disc}}$ closed under decompositions and inversions.

Now, measures on $\mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{cont}}, \mathrm{Iso}_F)$ and $\mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc}}, \mathrm{Iso}_F)$.

The Space $\mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc}}, \mathrm{Iso}_F)$ (MH14)

Measures by means of projective limit constructions:

- Surjectivity of $\mathrm{pr}_{\gamma_1,\dots,\gamma_k} : \mathrm{A} \mapsto (\mathrm{A}(\gamma_1), \dots, \mathrm{A}(\gamma_k)) \in S^k$ (up to $\cong$)
- Modification on γ_i by G_{γ_i}-invariant maps $\Psi \colon \mathbb{R} \to S$ (G_{γ_i} = stabilizer)

φ **not free:** Split $\mathcal{P}_{\mathrm{disc}} = \mathcal{P}_{\mathrm{disc,free}} \sqcup \mathcal{P}_{\mathrm{disc,stab}}$ ($G_\gamma = \{e\}$, $G_\gamma \neq \{e\}$)

$$\mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc}}, \mathrm{Iso}_F) \cong \mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc,free}}, \mathrm{Iso}_F) \times \mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc,stab}}, \mathrm{Iso}_F)$$

- Measure on $\mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc,free}}, \mathrm{Iso}_F)$ specializing to A-L one if $G = \{e\}$.
- Measure on $\mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc,stab}}, \mathrm{Iso}_F)$ from case to case.

LQC	$\mathcal{P}_{\mathrm{disc,free}}$	$\mathcal{P}_{\mathrm{disc,stab}}$	Measure on
(Semi-)homomogeneous	many	$\emptyset$	■ = ■
Spherically Symmetric	many	lines through 0	■ ≅ ■ × ■
Homogeneous Isotropic	many	$\emptyset$	■ = ■

The Space $\mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{cont}}, \mathrm{Iso}_F)$ (MH14)

Decompose: $\mathcal{P}_{\mathbf{cont}} = \mathcal{P}_{\mathrm{CNL}} \sqcup \mathcal{P}_{\mathfrak{g}}$: **(non-Lie $\sqcup$ Lie)**

- $\mathcal{P}_{\mathfrak{g}}$ – Lie algebra generated curves $\gamma^x_{\vec{g}}: t \mapsto \varphi(\exp(t \cdot \vec{g}), x)$
- $\mathcal{P}_{\mathrm{CNL}} = \mathcal{P}_{\mathbf{cont}} \backslash \mathcal{P}_{\mathfrak{g}}$

$\Longrightarrow$ $\mathbf{Hom}_\Theta(\mathcal{P}_{\mathbf{cont}}, \mathbf{Iso}_F) \cong \mathbf{Hom}_\Theta(\mathcal{P}_{\mathrm{CNL}}, \mathbf{Iso}_F) \times \mathbf{Hom}_\Theta(\mathcal{P}_{\mathfrak{g}}, \mathbf{Iso}_F)$

- Usually, not clear how $\mathcal{P}_{\mathrm{CNL}}$ looks like.
- φ transitive or proper + each G_x normal subgroup $\Longrightarrow \mathcal{P}_{\mathrm{CNL}} = \emptyset$.

LQC	$\mathcal{P}_{\mathrm{CNL}}$	$\mathcal{P}_{\mathfrak{g}}$
(Semi-)homogeneous	$\emptyset$	linear curves
Spherically Symmetric	?	circles
Homogeneous Isotropic	?	lines, circles, spirals

For now, ignore $\mathcal{P}_{\mathrm{CNL}}$ and concentrate on $\mathbf{Hom}_\Theta(\mathcal{P}_{\mathfrak{g}}, \mathbf{Iso}_F)$.

The Space $\mathrm{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \mathrm{Iso}_F)$ (MH14)

Idea: Calculate $\mathbf{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \mathbf{Iso}_F)$ explicitly.

- Need certain families of **stable** Lie algebra elements. (directions in $\mathfrak{g}$)
- Have such families in all 3 LQC cases discussed so far.
- $\vec{g} \in \mathfrak{g}\backslash\mathfrak{g}_x$ **stable** at x iff

$$\gamma^x_{\vec{g}}|_{[0,l]} \sim_\mathcal{A} \gamma^x_{\pm\mathrm{Ad}_h(\vec{g})}|_{[0,l']} \text{ for } h \in G_x \qquad \Longrightarrow \qquad \mathrm{Ad}_h(\vec{g}) = \pm\vec{g}$$

$\mathbf{SU(2)}$: $\mathrm{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \mathrm{Iso}_F)$ product of copies of

$$\{0_{\mathrm{Bohr}}\} \qquad \mathbb{R}_{\mathrm{Bohr}} \qquad \mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^1 \qquad \mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^2$$

with $(\psi, v) \sim (\psi', v')$ iff $\psi, \psi' = 0_{\mathrm{Bohr}}$ or $(\psi', v') = (-\psi, -v)$.

$\Longrightarrow$ Radon product measure on $\mathrm{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \mathrm{Iso}_F)$:

(Semi-)homogeneous	$\left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^2\right]^{\lvert \dim[G] \times M/G \rvert}$
Spherically symmetric	$\left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^1\right]^{\lvert \mathbb{R}_{>0} \rvert} \times \left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^2\right]^{\lvert (0,\pi/2) \times \mathbb{R}_{>0} \rvert}$
Homogeneous Isotropic	$\mathbb{R}_{\mathrm{Bohr}} \times \left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^1\right]^{\lvert \mathbb{R} \times \mathbb{R}_{>0} \rvert}$

Measures – Summary (S compact and connected)

Constructions for $\mathcal{P}_\omega$ + φ analytic and pointwise proper.

$$
\begin{tikzcd}
\boxed{\mathcal{P}_{\mathrm{cont}}} \arrow[d, leftrightarrow, "\varphi \text{ trans. or proper}"', "\text{normal stabilizers}"] & & \boxed{\mathcal{P}_\omega} \arrow[ll, "\text{split}"'] \arrow[rr, "\text{split}"] & & \boxed{\mathcal{P}_{\mathrm{disc}}} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{split}"] & \\
\boxed{\mathcal{P}_{\mathfrak{g}}} & & & \boxed{\mathcal{P}_{\mathrm{disc,free}}} & & \boxed{\mathcal{P}_{\mathrm{disc,stab}}}
\end{tikzcd}
$$

Measure for $\mathrm{SU}(2)$, S^1
stable elements in $\mathfrak{g}$ needed.

Generalized
A-L measure.

Measure in several
situations.

(Semi-)homogeneous LQC: (φ proper and free)

- ▷ $\overline{\mathcal{A}}_\Theta \cong \left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^2\right]^{|\dim[G] \times M/G|} \times \mathrm{Hom}_\Theta(\mathcal{P}_{\mathrm{disc,free}}, \mathrm{Iso}_F)$
- ▷ Radon product measure on full space $\overline{\mathcal{A}}_\Theta$.

Spherically symmetric + homogeneous isotropic:

- ▷ If $\mathcal{P}_{\mathrm{cont}} = \mathcal{P}_{\mathfrak{g}}$, have measure on $\overline{\mathcal{A}}_\Theta$ as well.

$\Longrightarrow$ Show $\mathcal{P}_{\mathrm{cont}} = \mathcal{P}_{\mathfrak{g}}$ by hand or find further general conditions implying this.

Alternative: Invariance Modulo Gauge (Ongoing Work)

Instead of invariant homomorphisms $\Theta_g(\mathrm{A}) = \mathrm{A}$ consider homomorphisms invariant up to gauge:

- $\mathcal{G} := \{\sigma\colon P \to P \mid \pi \circ \sigma = \pi,\ \sigma(p \cdot s) = \sigma(p) \cdot s\}$
- $\mathrm{Hom}_{\Theta,\mathcal{G}}(\mathcal{P}_\omega, \mathrm{Iso}_F) := \left\{\mathrm{A} \,\middle|\, \forall\, g \in G : \exists\, \sigma_g \in \mathcal{G} : \Theta_g(\mathrm{A}) = \sigma_g(\mathrm{A})\right\}$

$$\text{Invariance :} \quad (\Phi_{g^{-1}} \circ \mathrm{A})(\varphi_g \circ \gamma)(\Phi_g(p)) = \mathrm{A}(\gamma)(p)$$

$$\text{Up to gauge :} \quad (\Phi_{g^{-1}} \circ \mathrm{A})(\varphi_g \circ \gamma)(\Phi_g(p)) = (\sigma_g \circ \mathrm{A})(\gamma)(\sigma_g^{-1}(p)).$$

Properties of $\mathrm{Hom}_{\Theta,\mathcal{G}}(\mathcal{P}_\omega, \mathrm{Iso}_F)$:

- Compact if structure group compact.
- More physical than $\mathrm{Hom}_{\Theta}(\mathcal{P}_\omega, \mathrm{Iso}_F)$.
- Contains $\mathrm{Hom}_{\Theta}(\mathcal{P}_\omega, \mathrm{Iso}_F)$ and usually much larger:

  $\sigma \in \mathcal{G}$ and $\mathrm{A} \in \mathrm{Hom}_{\Theta}(\mathcal{P}_\omega, \mathrm{Iso}_F) \implies \sigma^*\mathrm{A} \in \mathrm{Hom}_{\Theta,\mathcal{G}}(\mathcal{P}_\omega, \mathrm{Iso}_F)$ for

  $$(\sigma^*\mathrm{A})(\gamma)(p) := \sigma \circ \mathrm{A}(\sigma^{-1}(p))$$
- Seems easier to construct measures on this space.

Ongoing and Future Work

Quantization vs. Reduction:

- As for (Semi-)homogeneous LQC, use modification results to find conditions for non-commutativity of quantization and reduction.

Measures:

- Find further conditions showing that $\mathcal{P}_{\mathrm{cont}} = \mathcal{P}_{\mathfrak{g}}$ (or determine $\mathcal{P}_{\mathrm{CNL}}$) in order to get measures on $\overline{\mathcal{A}}_{\Theta} = \mathrm{Hom}_{\Theta}(\mathcal{P}_{\omega}, \mathrm{Iso}_{F})$.
- Construct measure on $\mathrm{Hom}_{\Theta,\mathcal{G}}(\mathcal{P}_{\omega}, \mathrm{Iso}_{F})$.
- Investigate invariances of measures to get uniqueness results.

Dynamics:

- Reduction of holonomy-flux algebra and representation on the respective reduced Hilbert spaces.
- Embedding of states into respective symmetric sectors of LQG.

Summary

Homogeneous Isotropic LQC:

- Haar measure on $\mathbb{R}_{\mathrm{Bohr}}$ = unique normalized Radon measure on $\overline{\mathcal{A}_\theta} = \overline{\mathbb{R}}_{\mathrm{l}}, \overline{\mathbb{R}}_{\omega}$ invariant under lift of additive action $+\colon \mathbb{R} \times \mathcal{A}_\theta \to \mathcal{A}_\theta$.
  $\Longrightarrow$ $L^2(\mathbb{R}_{\mathrm{Bohr}}, \mu_{\mathrm{Bohr}})$ = ABL standard
- Lie algebra part of quantum-reduced $\mathrm{Hom}_\Theta(\mathcal{P}_\omega, \mathrm{Iso}_F)$:

$$\mathrm{Hom}_\Theta(\mathcal{P}_{\mathfrak{g}}, \mathrm{Iso}_F) \cong \mathbb{R}_{\mathrm{Bohr}} \times \left[\mathbb{R}_{\mathrm{Bohr}} \widetilde{\times} \mathrm{S}^1\right]^{|\mathbb{R}\times\mathbb{R}_{>0}|}$$

Reduction on Quantum Level:

- Usually gives more than quantization of reduced classical space.
  $\longrightarrow$ (Semi-)homogeneous + homogeneous isotropic LQC
- Splitting up $\mathcal{P}_\omega$ allows to factorize $\mathrm{Hom}_\Theta(\mathcal{P}_\omega, \mathrm{Iso}_F)$
  $\longrightarrow$ define measure on each factor separately.
  Here, non-trivial conditions on symmetry and structure group.
- Factor $\mathrm{Hom}_\Theta(\mathcal{P}_{\mathfrak{g}}, \mathrm{Iso}_F)$ (Lie algebra generated curves) homeomorphic to Tychonoff product of spaces involving $\mathbb{R}_{\mathrm{Bohr}}$.

Thank you for your attention !