

Invariance of Connections and Measures in Loop Quantum Cosmology

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soon to join at Florida Atlantik University

Symmetry Reduction on Classical Level

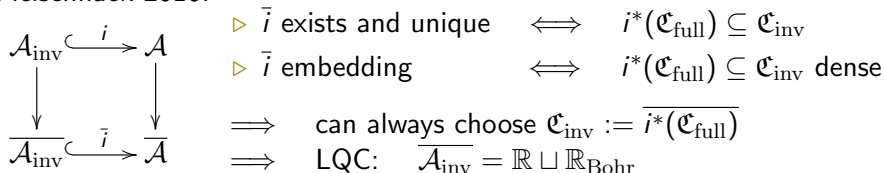
Ingredients: principal bundle P + Symmetry on P

Config. Spaces	full theory	reduced theory	
Classical level	\mathcal{A}	\mathcal{A}_{inv}	connections
Quantum level	$\overline{\mathcal{A}}$	$\overline{\mathcal{A}_{\text{inv}}}$	compactification

- ▷ $\overline{\mathcal{A}} = \text{Spec}(\mathfrak{C}_{\text{full}})$ $\mathfrak{C}_{\text{full}}$ – separating unital C^* -subalgebra of $B(\mathcal{A})$
- ▷ $\overline{\mathcal{A}_{\text{inv}}} = \text{Spec}(\mathfrak{C}_{\text{inv}})$ $\mathfrak{C}_{\text{inv}}$ – separating unital C^* -subalgebra of $B(\mathcal{A}_{\text{inv}})$

Standard LQC: $\mathcal{A}_{\text{inv}} \cong \mathbb{R}$ $\mathfrak{C}_{\text{inv}} = C_{\text{AP}}(\mathbb{R})$ $\overline{\mathcal{A}_{\text{inv}}} = \mathbb{R}_{\text{Bohr}}$

Fleischhack 2010:



Synopsis

Introduce notion of lifted action and apply it to LQG

- ▷ Quantum-reduction of LQG configuration spaces.
- ▷ Singling out measures by means of invariance properties.

Mathematical Setting

- ▷ X – set together with action $\theta: G \times X \rightarrow X$
 $\implies X_\theta := \{x \in X \mid \theta_g(x) = x \ \forall g \in G\} = \text{reduced class. conf. space}$
- ▷ \mathfrak{C} – C^* -subalgebra of $B(X)$ ($B(X)$ – bounded functions on X)
- ▷ \bar{X} – $\text{Spec}(\mathfrak{C}) = \text{quantum space}$ $\iota: X \hookrightarrow \bar{X}, \iota(x): f \mapsto f(x)$

Crucial Observation (MH13)

If $\theta_g^*(\mathfrak{C}) \subseteq \mathfrak{C}$ for all $g \in G$, have **unique** lift $\Theta: G \times \bar{X} \rightarrow \bar{X}$ **with**

- ▷ Θ_g continuous $\forall g \in G \implies \bar{X}_\Theta$ closed (compact if \mathfrak{C} unital)
- ▷ $\Theta_g \circ \iota = \iota \circ \theta_g \implies \overline{\iota(X_\theta)} \subseteq \bar{X}_\Theta$

LQG – Principal Fibre Bundle (P, π, M, S)

- ▷ $X = \mathcal{A}$: smooth connections on P
- ▷ \mathfrak{C} : cylindrical functions on \mathcal{A} generated by $f \circ h_\gamma$ for γ in fixed set \mathcal{P} .
- ▷ $\overline{X} = \overline{\mathcal{A}} = \text{Spec}(\mathfrak{C})$: generalized connections

Action θ comes from Lie group of automorphisms

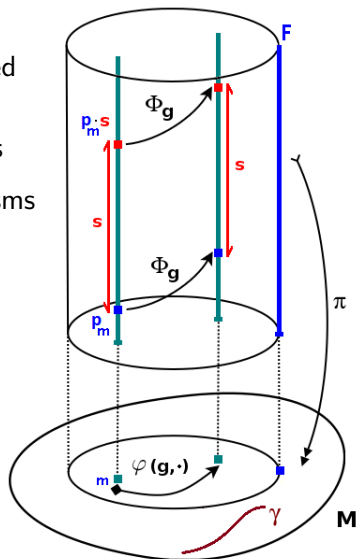
$\Phi: G \times P \rightarrow P =$ left action with

$$\Phi(g, p \cdot s) = \Phi(g, p) \cdot s$$

for all $g \in G, p \in P, s \in S$.

\implies Have two actions:

- $\theta: G \times \mathcal{A} \rightarrow \mathcal{A}, (g, \omega) \mapsto \Phi_{g^{-1}}^* \omega$
- $\varphi: G \times M \rightarrow M, (g, m) \mapsto (\pi \circ \Phi)(g, p)$
where $p \in F_m$



LQG – Invariance on Classical Level

Reduction on classical level:

- ▷ $\mathcal{A}_\theta = \{\omega \in \mathcal{A} \mid \theta(g, \omega) = \omega \forall g \in G\}$ = invariant connections on P
- ▷ Quantized reduced classical space $\overline{\mathcal{A}}_\theta := \text{Spec}(\overline{\mathcal{C}}|_{\mathcal{A}_\theta}) \cong \overline{i(\mathcal{A}_\theta)} \subseteq \overline{\mathcal{A}}$

Example: Homogeneous Isotropic LQC

$$P = \mathbb{R}^3 \times \text{SU}(2) \quad \Phi - \text{euclidean group action} \quad \mathcal{A}_\theta \cong \mathbb{R}$$

	ABL [2003]	Fleischhack [2010]
\mathcal{P}	linear curves	embedded analytic curves
$\overline{\mathcal{C}} _{\mathcal{A}_\theta}$	$C_{\text{AP}}(\mathbb{R})$	$C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R})$
$\overline{\mathcal{A}}_\theta$	$\overline{\mathbb{R}}_1 = \mathbb{R}_{\text{Bohr}}$	$\overline{\mathbb{R}}_\omega = \mathbb{R} \sqcup \mathbb{R}_{\text{Bohr}}$

Inclusion $i: \mathbb{R} \cong \mathcal{A}_\theta \hookrightarrow \mathcal{A}$:

- ▷ Extends to embedding $\overline{i}: \overline{\mathbb{R}}_\omega \rightarrow \overline{\mathcal{A}}$. (embedding of states)
- ▷ No such extension for $\overline{\mathbb{R}}_1$. (Brunnemann, Fleischhack 2009)

Homogeneous Isotropic LQC – The Bohr Measures

Finite Radon measure μ on $\overline{\mathbb{R}}_\omega$: $\mathfrak{B}(\overline{\mathbb{R}}_\omega) = \mathfrak{B}(\mathbb{R}) \sqcup \mathfrak{B}(\mathbb{R}_{\text{Bohr}})$
 $\implies \mu(A) = \mu_1(A \cap \mathbb{R}) + \mu_2(A \cap \mathbb{R}_{\text{Bohr}})$ μ_1 on \mathbb{R} , μ_2 on \mathbb{R}_{Bohr}

Invariance: $\mathcal{A}_\theta \cong \mathbb{R}$ $\overline{\mathcal{A}}_\theta \cong \overline{\mathbb{R}}_1, \overline{\mathbb{R}}_\omega$ (MH14)

▷ Additive action $+$: $\mathbb{R} \times \mathcal{A}_\theta \rightarrow \mathcal{A}_\theta$ extends to $\overline{+}$: $\mathbb{R} \times \overline{\mathcal{A}}_\theta \rightarrow \overline{\mathcal{A}}_\theta$.

▷ μ_{Bohr} **unique** invariant normalized Radon measure. (exp. flux)

\implies In both cases single out $L^2(\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}})$ – standard **ABL** LQC

Embedding of distributional states: (Engle 2013)

$$C_{\text{AP}}(\mathbb{R})^* \hookrightarrow [C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R})]^* \xrightarrow{\chi \mapsto \chi \circ i^*} \mathfrak{C}_\omega^*$$

Assumpt. on $\langle \cdot, \cdot \rangle_\mu$: ▷ $\langle f_0, f_{\text{AP}} \rangle_\mu = 0 \implies \mu_1 = 0$

▷ $\langle f_{\text{AP}}, g_{\text{AP}} \rangle_\mu = \langle f_{\text{AP}}, g_{\text{AP}} \rangle_{\mu_{\text{Bohr}}} \implies \mu_2 = \mu_{\text{Bohr}}$

LQG – Invariance on Quantum Level

Reduction on quantum level:

$$\theta_g^*(\mathcal{C}) \subseteq \mathcal{C}: \quad \theta: G \times \mathcal{A} \rightarrow \mathcal{A} \quad \xrightarrow{\text{Lift}} \quad \Theta: G \times \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}}$$

$$\gamma \in \mathcal{P} \quad \implies \quad \varphi_g \circ \gamma \in \mathcal{P} \quad \forall g \in G, \forall \gamma \in \mathcal{P}.$$

▷ Invariant generalized connections

$$\overline{\mathcal{A}}_\Theta := \{\overline{w} \in \overline{\mathcal{A}} \mid \Theta(g, \overline{w}) = \overline{w} \quad \forall g \in G\}.$$

▷ Translations Θ_g continuous $\implies \overline{\mathcal{A}}_\Theta$ compact (S compact)

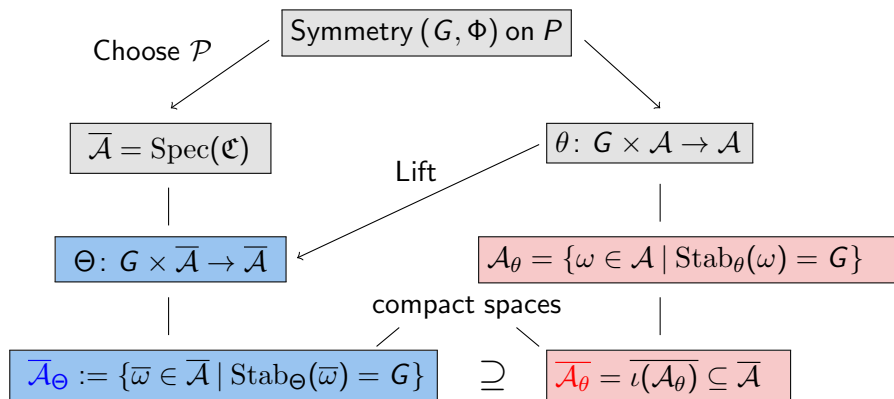
▷ Θ extends $\theta \implies \overline{\mathcal{A}}_\theta \subseteq \overline{\mathcal{A}}_\Theta$

Quantization vs. Reduction:

(MH13/14)

LQC	linear curves	embedded analytic curves
(Semi-)homogeneous	$\overline{\mathcal{A}}_\theta \subsetneq \overline{\mathcal{A}}_\Theta$	$\overline{\mathcal{A}}_\theta \subsetneq \overline{\mathcal{A}}_\Theta$
Homogeneous Isotropic	$\overline{\mathcal{A}}_\theta \cong \overline{\mathcal{A}}_\Theta$	$\overline{\mathcal{A}}_\theta \subsetneq \overline{\mathcal{A}}_\Theta$

Reduction – Summary

 (P, π, M, S) with S compact

- ▷ Lift Θ exists if \mathcal{P} invariant, i.e., $\varphi_g \circ \gamma \in \mathcal{P}$ for all $\gamma \in \mathcal{P}$, $g \in G$.
- ▷ Usually fulfilled because φ and \mathcal{P} smooth or analytic.

Quantum Reduction – A Measure on $\text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F)$

Setting:

- ▷ S compact and connected
 - ▷ $\mathcal{P} = \mathcal{P}_\omega$ – embedded analytic curves in M
- $\implies \bar{\mathcal{A}}_\Theta \cong \text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F)$ = space of invariant homomorphisms:
- ▷ $A(\gamma): F_{\gamma(0)} \rightarrow F_{\gamma(1)}$ (generalized parallel transport)
 - ▷ $A(\varphi_g \circ \gamma)(\Phi_g(p)) = (\Phi_g \circ A)(\gamma)(p)$ (invariance)

Strategy:

- ▷ Split up $\mathcal{P}_\omega = \bigsqcup_{\alpha \in I} \mathcal{P}_\alpha$ into natural subsets \mathcal{P}_α closed under inversion and decomposition, and obtain
- ▷ $\text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F) \cong \prod_{\alpha \in I} \text{Hom}_\Theta(\mathcal{P}_\alpha, \text{Iso}_F)$.
- ▷ Define a normalized Radon measure μ_α on each $\text{Hom}_\Theta(\mathcal{P}_\alpha, \text{Iso}_F)$ and take the Radon product measure on $\text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F)$.

Analytic and Pointwise Proper Case

(MH14)

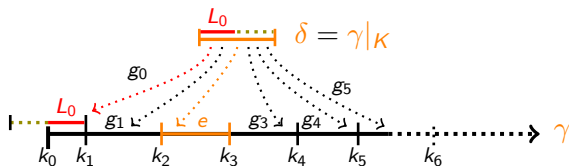
If φ analytic and φ_x proper for each $x \in M$:

- ▷ Decomposition $\mathcal{P}_\omega = \mathcal{P}_{\text{cont}} \sqcup \mathcal{P}_{\text{disc}}$ into **continuously** and **discretely** generated curves.

Here, $\gamma \in \mathcal{P}_{\text{disc}} \iff \exists K \subseteq \text{dom}[\gamma]$ such that for $\delta := \gamma|_K$:

$\text{im}[\delta] \cap \text{im}[\varphi_g \circ \delta]$ infinite for $g \in G \implies \varphi_g \circ \delta = \delta$.

\implies Can find K such that γ can be decomposed into translates of δ :



- ▷ Guarantees that $\mathcal{P}_{\text{cont}}, \mathcal{P}_{\text{disc}}$ closed under decompositions and inversions.

Now, measures on $\text{Hom}_\Theta(\mathcal{P}_{\text{cont}}, \text{ISO}_F)$ and $\text{Hom}_\Theta(\mathcal{P}_{\text{disc}}, \text{ISO}_F)$.

The Space $\text{Hom}_\Theta(\mathcal{P}_{\text{disc}}, \text{ISO}_F)$

(MH14)

Measures by means of projective limit constructions:

▷ Surjectivity of $\text{pr}_{\gamma_1, \dots, \gamma_k} : A \mapsto (A(\gamma_1), \dots, A(\gamma_k)) \in S^k$ (up to \cong)








▷ Modification on γ_i by G_{γ_i} -invariant maps $\Psi : \mathbb{R} \rightarrow S$ ($G_{\gamma_i} = \text{stabilizer}$)

φ not free: Split $\mathcal{P}_{\text{disc}} = \mathcal{P}_{\text{disc, free}} \sqcup \mathcal{P}_{\text{disc, stab}}$ ($G_\gamma = \{e\}$, $G_\gamma \neq \{e\}$)

$$\text{Hom}_\Theta(\mathcal{P}_{\text{disc}}, \text{ISO}_F) \cong \text{Hom}_\Theta(\mathcal{P}_{\text{disc, free}}, \text{ISO}_F) \times \text{Hom}_\Theta(\mathcal{P}_{\text{disc, stab}}, \text{ISO}_F)$$

▷ Measure on $\text{Hom}_\Theta(\mathcal{P}_{\text{disc, free}}, \text{ISO}_F)$ specializing to A-L one if $G = \{e\}$.

▷ Measure on $\text{Hom}_\Theta(\mathcal{P}_{\text{disc, stab}}, \text{ISO}_F)$ from case to case.

LQC	$\mathcal{P}_{\text{disc, free}}$	$\mathcal{P}_{\text{disc, stab}}$	Measure on
(Semi-)homogeneous	many	\emptyset	 = 
Spherically Symmetric	many	lines through 0	 \cong  \times 
Homogeneous Isotropic	many	\emptyset	 = 

The Space $\text{Hom}_\Theta(\mathcal{P}_{\text{cont}}, \text{ISO}_F)$

(MH14)

Decompose: $\mathcal{P}_{\text{cont}} = \mathcal{P}_{\text{CNL}} \sqcup \mathcal{P}_{\mathfrak{g}}$: (non-Lie \sqcup Lie)

▷ $\mathcal{P}_{\mathfrak{g}}$ – Lie algebra generated curves $\gamma_{\vec{g}}^x: t \mapsto \varphi(\exp(t \cdot \vec{g}), x)$

▷ $\mathcal{P}_{\text{CNL}} = \mathcal{P}_{\text{cont}} \setminus \mathcal{P}_{\mathfrak{g}}$

$\implies \text{Hom}_\Theta(\mathcal{P}_{\text{cont}}, \text{ISO}_F) \cong \text{Hom}_\Theta(\mathcal{P}_{\text{CNL}}, \text{ISO}_F) \times \text{Hom}_\Theta(\mathcal{P}_{\mathfrak{g}}, \text{ISO}_F)$

▷ Usually, not clear how \mathcal{P}_{CNL} looks like.

▷ φ transitive or proper + each G_x normal subgroup $\implies \mathcal{P}_{\text{CNL}} = \emptyset$.

LQC	\mathcal{P}_{CNL}	$\mathcal{P}_{\mathfrak{g}}$
(Semi-)homogeneous	\emptyset	linear curves
Spherically Symmetric	?	circles
Homogeneous Isotropic	?	lines, circles, spirals

For now, ignore \mathcal{P}_{CNL} and concentrate on $\text{Hom}_\Theta(\mathcal{P}_{\mathfrak{g}}, \text{ISO}_F)$.

The Space $\text{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \text{Iso}_F)$

(MH14)

Idea: Calculate $\text{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \text{Iso}_F)$ explicitly.

- ▷ Need certain families of **stable** Lie algebra elements. (directions in \mathfrak{g})
- ▷ Have such families in all 3 LQC cases discussed so far.
- ▷ $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$ **stable** at x iff

$$\gamma_{\vec{g}}^x|_{[0,1]} \sim_{\mathcal{A}} \gamma_{\pm \text{Ad}_h(\vec{g})}^x|_{[0,1]} \text{ for } h \in G_x \quad \implies \quad \text{Ad}_h(\vec{g}) = \pm \vec{g}$$

SU(2): $\text{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \text{Iso}_F)$ product of copies of

$$\{0_{\text{Bohr}}\} \quad \mathbb{R}_{\text{Bohr}} \quad \mathbb{R}_{\text{Bohr}} \tilde{\times} S^1 \quad \mathbb{R}_{\text{Bohr}} \tilde{\times} S^2$$

with $(\psi, \nu) \sim (\psi', \nu')$ iff $\psi, \psi' = 0_{\text{Bohr}}$ or $(\psi', \nu') = (-\psi, -\nu)$.

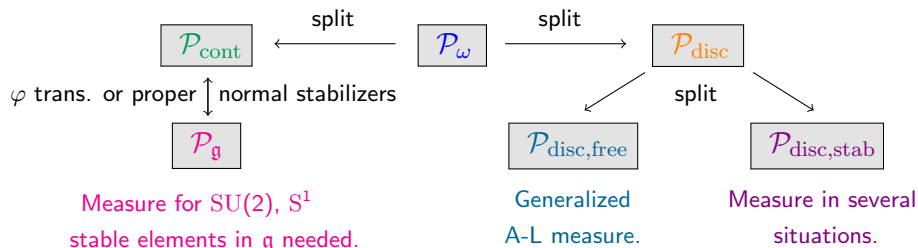
\implies Radon product measure on $\text{Hom}_\Theta(\mathcal{P}_\mathfrak{g}, \text{Iso}_F)$:

(Semi-)homogeneous	$[\mathbb{R}_{\text{Bohr}} \tilde{\times} S^2]^{ \dim[G] \times M/G }$
Spherically symmetric	$[\mathbb{R}_{\text{Bohr}} \tilde{\times} S^1]^{ \mathbb{R}_{>0} } \times [\mathbb{R}_{\text{Bohr}} \tilde{\times} S^2]^{ (0, \pi/2) \times \mathbb{R}_{>0} }$
Homogeneous Isotropic	$\mathbb{R}_{\text{Bohr}} \times [\mathbb{R}_{\text{Bohr}} \tilde{\times} S^1]^{ \mathbb{R} \times \mathbb{R}_{>0} }$

Measures – Summary

(S compact and connected)

Constructions for $\mathcal{P}_\omega + \varphi$ analytic and pointwise proper.



(Semi-)homogeneous LQC: (φ proper and free)

- ▷ $\overline{\mathcal{A}}_\Theta \cong [\mathbb{R}_{\text{Bohr}} \widetilde{\times} S^2]^{|\dim[G] \times M/G|} \times \text{Hom}_\Theta(\mathcal{P}_{\text{disc,free}}, \text{ISO}_F)$
- ▷ Radon product measure on full space $\overline{\mathcal{A}}_\Theta$.

Spherically symmetric + homogeneous isotropic:

- ▷ If $\mathcal{P}_{\text{cont}} = \mathcal{P}_g$, have measure on $\overline{\mathcal{A}}_\Theta$ as well.
 \implies Show $\mathcal{P}_{\text{cont}} = \mathcal{P}_g$ by hand or find further general conditions implying this.

Alternative: Invariance Modulo Gauge (Ongoing Work)

Instead of invariant homomorphisms $\Theta_g(A) = A$ consider homomorphisms invariant **up to gauge**:

$$\triangleright \mathcal{G} := \{\sigma : P \rightarrow P \mid \pi \circ \sigma = \pi, \sigma(p \cdot s) = \sigma(p) \cdot s\}$$

$$\triangleright \text{Hom}_{\Theta, \mathcal{G}}(\mathcal{P}_\omega, \text{Iso}_F) := \{A \mid \forall g \in G : \exists \sigma_g \in \mathcal{G} : \Theta_g(A) = \sigma_g(A)\}$$

$$\text{Invariance : } (\Phi_{g^{-1}} \circ A)(\varphi_g \circ \gamma)(\Phi_g(p)) = A(\gamma)(p)$$

$$\text{Up to gauge : } (\Phi_{g^{-1}} \circ A)(\varphi_g \circ \gamma)(\Phi_g(p)) = (\sigma_g \circ A)(\gamma)(\sigma_g^{-1}(p)).$$

Properties of $\text{Hom}_{\Theta, \mathcal{G}}(\mathcal{P}_\omega, \text{Iso}_F)$:

\triangleright Compact if structure group compact.

\triangleright More physical than $\text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F)$.

\triangleright Contains $\text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F)$ and usually much larger:

$$\sigma \in \mathcal{G} \text{ and } A \in \text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F) \implies \sigma^* A \in \text{Hom}_{\Theta, \mathcal{G}}(\mathcal{P}_\omega, \text{Iso}_F) \text{ for}$$

$$(\sigma^* A)(\gamma)(p) := \sigma \circ A(\sigma^{-1}(p))$$

\triangleright Seems easier to construct measures on this space.

Ongoing and Future Work

Quantization vs. Reduction:

- ▷ As for (Semi-)homogeneous LQC, use modification results to find conditions for non-commutativity of quantization and reduction.

Measures:

- ▷ Find further conditions showing that $\mathcal{P}_{\text{cont}} = \mathcal{P}_{\mathfrak{g}}$ (or determine \mathcal{P}_{CNL}) in order to get measures on $\overline{\mathcal{A}}_{\Theta} = \text{Hom}_{\Theta}(\mathcal{P}_{\omega}, \text{Iso}_{\mathcal{F}})$.
- ▷ Construct measure on $\text{Hom}_{\Theta, \mathcal{G}}(\mathcal{P}_{\omega}, \text{Iso}_{\mathcal{F}})$.
- ▷ Investigate invariances of measures to get uniqueness results.

Dynamics:

- ▷ Reduction of holonomy-flux algebra and representation on the respective reduced Hilbert spaces.
- ▷ Embedding of states into respective symmetric sectors of LQG.

Summary

Homogeneous Isotropic LQC:

- ▷ Haar measure on \mathbb{R}_{Bohr} = unique normalized Radon measure on $\overline{\mathcal{A}_\theta} = \overline{\mathbb{R}}_1, \overline{\mathbb{R}}_\omega$ invariant under lift of additive action $+: \mathbb{R} \times \mathcal{A}_\theta \rightarrow \mathcal{A}_\theta$.
 $\implies L^2(\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}) = \text{ABL standard}$
- ▷ Lie algebra part of quantum-reduced $\text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F)$:

$$\text{Hom}_\Theta(\mathcal{P}_g, \text{Iso}_F) \cong \mathbb{R}_{\text{Bohr}} \times [\mathbb{R}_{\text{Bohr}} \tilde{\times} S^1]^{|\mathbb{R} \times \mathbb{R}_{>0}|}$$

Reduction on Quantum Level:

- ▷ Usually gives more than quantization of reduced classical space.
 \longrightarrow (Semi-)homogeneous + homogeneous isotropic LQC
- ▷ Splitting up \mathcal{P}_ω allows to factorize $\text{Hom}_\Theta(\mathcal{P}_\omega, \text{Iso}_F)$
 \longrightarrow define measure on each factor separately.
 Here, non-trivial conditions on symmetry and structure group.
- ▷ Factor $\text{Hom}_\Theta(\mathcal{P}_g, \text{Iso}_F)$ (Lie algebra generated curves) homeomorphic to Tychonoff product of spaces involving \mathbb{R}_{Bohr} .

Thank you for your attention !