The Asymptotics of 4d Spin Foam Models from SU(2) BF Asymptotics.

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The Papers

- “Asymptotic analysis of the EPRL four-simplex amplitude” (arXiv:0902.1170)
- “Lorentzian spin foam amplitudes: graphical calculus and asymptotics” (arXiv:0907.2440)
- New Stuff: “SU(2) Paper” (arXiv:tba)

with John W. Barrett, Richard J. Dowdall, Winston J. Fairbairn, Henrique Gomes, Roberto Pereira

using:

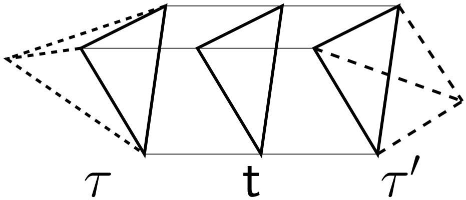
Livine, Speziale: A new spinfoam vertex for quantum gravity [0705.0674]
Engle, Pereira, Rovelli: Flipped spinfoam vertex and loop gravity [0708.1236]
Freidel, Krasnov: A New Spin Foam Model for 4d Gravity [0708.1595]
EPR + Livine: LQG vertex with finite Immirzi parameter [0711.0146]
Pereira: Lorentzian LQG vertex amplitude [0710.5043]

SU(2) Statesum (1)

SU(2) BF theory Spin Foam is:

$$Z_{\mathrm{SU}(2)}(\mathcal{T}) = \sum_{j_t} \mathrm{d}_{j_t} \int \mathrm{d}n_{t\tau} \prod_{\sigma} 15\mathrm{j}(j_t, n_{t\tau}) \tag{1}$$

3d analogue

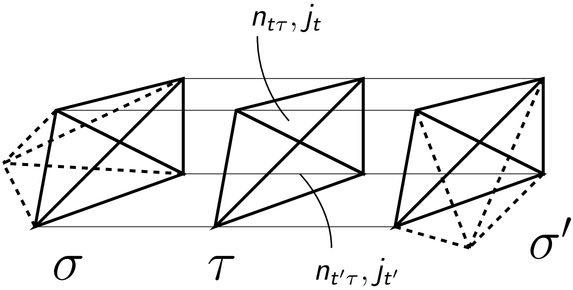


SU(2) Statesum (2)

SU(2) BF theory Spin Foam is:

$$Z_{\mathrm{SU}(2)}(\mathcal{T}) = \sum_{j_t} \mathrm{d}_{j_t} \int \mathrm{d}n_{t\tau} \prod_{\sigma} 15\mathrm{j}(j_t, n_{t\tau}) \tag{2}$$

$$\begin{aligned} j_t &\in Irrep(\mathrm{SU}(2)) \\ n_{t\tau} &\in S^2 \end{aligned} \tag{3}$$



SU(2) Statesum (3)

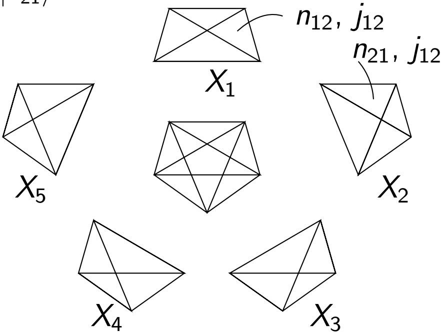
The measure is $\mathrm{d}_{j_t} = \left((-1)^{2j_t}(2j_t+1)\right)^{(\#_{\tau\in t}+1)}$.

$$15\mathrm{j}(j_t, n_{t\tau}) = \pm \int \prod_{\tau} \mathrm{d}X \prod_{t} \langle J n_{t\tau} | \, X^\dagger_\tau X_{\tau'} \, | n_{t\tau'} \rangle^{2j_t}_{\tau,\tau' \ni t} \qquad (4)$$

$n_{t\tau} \cdot L \, |n_{t\tau}\rangle = \frac{1}{2} i \, |n_{t\tau}\rangle$ with the generators $L^i \in \mathfrak{su}(2)$ (phase unspecified)

$J : J^2 = (-1)^{2j}$ is the antilinear structure on $\mathrm{SU}(2)$

$\langle|\rangle_{1,2\ni t_{12}} : \langle J n_{12} | \, X_1^\dagger X_2 \, | n_{21} \rangle^{2j_{12}}$



SU(2) Asymptotics (1):

$\mp$15j($j_t, n_{t\tau}$) is an integral:

$$15\mathrm{j}(j_t, n_{t\tau}) = \int \prod_\tau \mathrm{d}X \, \exp(S^{BF}) \tag{5}$$

with action

$$S^{BF}(j_t, n_{t\tau}) = \sum_t 2j_t \left(ln \left\langle J n_{t\tau} \right| X_\tau^\dagger X_{\tau'} \left| n_{t\tau'} \right\rangle \right) . \tag{6}$$

The reality and stationary point equations for this integral in terms of

$$b_{t\tau} := j_t X_t n_{t\tau} \tag{7}$$

are:

$$\begin{aligned} \frac{\delta S}{\delta X_\tau} = 0 \quad &: \quad \sum_{t \in \tau} b_{t\tau} = 0 \\ \mathrm{Re}(S) = 0 \quad &: \quad b_{t\tau} = -b_{t\tau'} \end{aligned} \tag{8}$$

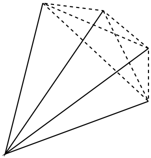
SU(2) Asymptotics (2):

$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \tag{9}$$

Theorem:

$b_{t\tau}$ are in one to one correspondence with constant, vector valued 2-forms b on a 4-simplex.

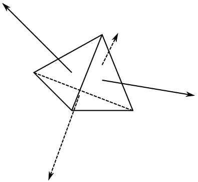
Proof: $b_{t\tau} = \int_t b$ for the oriented triangles meeting at a vertex is a basis for the constant 2-forms on a 4-simplex, the other faces follow by closure/Stokes'.



Note: These are the BC dominant configurations.

SU(2) Asymptotics (3):

$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad (10)$$



If j_t, $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

Theorem:

There are, up to symmetry, at most 2 solutions $b^{\pm}_{t\tau}$ corresponding to the selfdual sectors of two parity related geometric 4-simplices σ, $P\sigma$ with boundary geometry matching that defined by the j_t, $n_{t\tau}$.

Convention: Call b^+ the solution for which boundary orientation and 4-simplex orientation agree.

Interlude: Bivector geometry

Up to inversion ($inv : V^I \to -V^I$) a non-degenerate 4-simplex σ in $\mathbb{R}^4$ with edge vectors V_e^I is characterized by the triangle bivectors constructed from these:

$$(V_e \wedge V_{e'})^{IJ} = V_e^{[I} V_{e'}^{J]} = B_t^{IJ}. \tag{11}$$

These are proportional to the triangle areas.

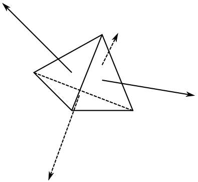
B^{IJ} are antisymmetric 4×4 matrices (6-dimensional space) and thus in $\mathfrak{so}(4)$: $\exp(B^{IJ}) \in \mathrm{SO}(4)$. The double cover of $\mathrm{SO}(4)$, $\mathrm{Spin}(4)$, decomposes into $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

We can decompose the 6-dimensional bivector space into two 3-dimensional spaces generating rotations purely in the left or the right factor. The left sector is called selfdual the right sector antiselfdual. We can then write

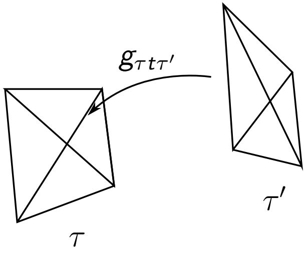
$$B_t = (b_t^s, b_t^a) \tag{12}$$

SU(2) Asymptotics (4):

$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad (13)$$

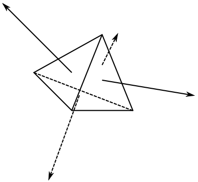


If j_{cd}, n_{cd} satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then define Regge gluing rotation $g_{\tau t \tau'}$ by:



SU(2) Asymptotics (5):

$$\sum_{t\in\tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad (14)$$



If j_{cd}, n_{cd} satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

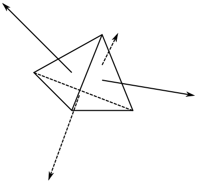
Regge state

Using the geometric phase choice $|n_{t\tau}\rangle = g_{\tau t\tau'}\,|Jn_{t\tau'}\rangle$ on the boundary the action evaluates to

$$S^{BF}|_{b^\pm} = i\sum_t 2A_t\,(\pm\Theta_t) = \pm i2S_{Regge}(\sigma), \qquad (15)$$

with A_t the areas and Θ_t the dihedral angles of the geometric 4-simplex σ.

SU(2) Asymptotics (5):



$$\sum_{t\in\tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad (16)$$

If j_{cd}, n_{cd} satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then with the geometric phase choice for $|n_{ab}\rangle$:

$$\begin{aligned} & \mp 15\mathrm{j}(\lambda j_t, n_{t\tau}) \\ = \quad & \lambda^{-6}\left(N_+ \exp(S^{BF}|_{b^+}) + N_- \exp(S^{BF}|_{b^-})\right) + O(\lambda^{-7}) \\ = \quad & \lambda^{-6}\left(N_+ \exp(i2S_{Regge}(\sigma)) + N_- \exp(-i2S_{Regge}(\sigma))\right) + O(\lambda^{-7}) \qquad (17) \end{aligned}$$

New models as $\mathrm{SU}(2)^2$ constrained (1):

$P(b^s, b^a) = (b^a, b^s)$

$\Rightarrow$ Selfdual part of 4-simplex = antiselfdual part of parity related 4-simplex.

$\Rightarrow$ $Z_{\mathrm{SU}(2)} \times Z'_{\mathrm{SU}(2)}$ has a sector where both the selfdual and antiselfdual part of the geometry of (usually different) 4-simplices are present.

If they are of the same 4-simplex then we have $n_{t\tau} = n'_{t\tau}$ and $j_t = j'_t$.

$\Rightarrow$ Impose these as constraints on the statesum (simplicity constraints).

New models as $\mathrm{SU}(2)^2$ constrained (2):

Write:

$$Z_{\mathrm{SU}(2)} \times Z'_{\mathrm{SU}(2)}(O) =$$
$$\sum_{j_t, j'_t} \mathrm{d}_{j_t} \mathrm{d}_{j'_t} \int \mathrm{d}n_{t\tau} \mathrm{d}n'_{t\tau} \prod_v 15\mathrm{j}(j_t, n_{t\tau})\, 15\mathrm{j}(j'_t, n'_{t\tau})\, O\left(n_{t\tau}, n'_{t\tau}, j_t, j'_t\right)$$

Then for certain choices of face amplitudes and $c_\gamma = \frac{1+\gamma}{|1-\gamma|}$:

$$\begin{aligned}
Z_{EPR} &= Z_{\mathrm{SU}(2)} \times Z'_{\mathrm{SU}(2)} \left(\prod \delta(n_{t\tau} - n'_{t\tau})\delta(j_t, j'_t)\right) \\
Z_{FK} &= Z_{\mathrm{SU}(2)} \times \overline{Z'_{\mathrm{SU}(2)}} \left(\prod \delta(n_{t\tau} - n'_{t\tau})\delta(j_t, j'_t)\right) \\
Z_{EPRL_{\gamma<1}} = Z_{FK_{\gamma<1}} &= Z_{\mathrm{SU}(2)} \times Z'_{\mathrm{SU}(2)} \left(\prod \delta(n_{t\tau} - n'_{t\tau})\delta(j_t, c_\gamma j'_t)\right) \\
Z_{FK_{\gamma>1}} &= Z_{\mathrm{SU}(2)} \times \overline{Z'_{\mathrm{SU}(2)}} \left(\prod \delta(n_{t\tau} - n'_{t\tau})\delta(j_t, c_\gamma j'_t)\right) \qquad (18)
\end{aligned}$$

New models asymptotics (1):

$$\sum_{t\in\tau} b_{t\tau} = 0 \qquad \sum_{t\in\tau} b'_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad b'_{t\tau} = -b'_{t\tau'} \tag{19}$$

B-field sector: The solutions correspond to two constant vector valued 2-forms b, b' related by

$$\int_t b = c_\gamma g_t \int_t b'. \tag{20}$$

Any $\mathrm{SU}(2)$ BF solution is a solution of the constrained theory with $b = c_\gamma b'$.

New models asymptotics (2):

$$\sum_{t\in\tau} b_{t\tau} = 0 \qquad \sum_{t\in\tau} b'_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad b'_{t\tau} = -b'_{t\tau'} \tag{21}$$

If j_t, $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

Theorem:

There are, up to symmetry, at most 2 solutions $b^{\pm}_{t\tau}$ corresponding to the selfdual and anti-selfdual sectors of a 4-simplex σ with boundary geometry matching that defined by the j_t, $n_{t\tau}$.

If there are two solutions then:

$$(b_{t\tau}, c_\gamma b'_{t\tau}) \in \{(b^+_{t\tau}, b^+_{t\tau}), (b^+_{t\tau}, b^-_{t\tau}), (b^-_{t\tau}, b^+_{t\tau}), (b^-_{t\tau}, b^-_{t\tau})\} \tag{22}$$

New models asymptotics (3):

$$\sum_{t\in\tau} b_{t\tau} = 0 \qquad \sum_{t\in\tau} b'_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad b'_{t\tau} = -b'_{t\tau'} \tag{23}$$

If j_t, $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions $b^\pm$ corresponding to selfdual and antiselfdual parts of a 4-simplex σ then with the geometric phase choice we have:

$$S^{EPR,\gamma<1}|_{\pm,\pm} = S^{BF}|_{b^\pm} + c_\gamma S^{BF}|_{b^\pm} = i2\left(\pm S_{Regge} + c_\gamma(\pm S_{Regge})\right) \tag{24}$$

$$S^{FK,\gamma>1}|_{\pm,\pm} = S^{BF}|_{b^\pm} + c_\gamma \overline{S^{BF}|_{b^\pm}} = i2\left(\pm S_{Regge}(\sigma) - c_\gamma(\pm S_{Regge}(\sigma))\right) \tag{25}$$

New models asymptotics (4):

$$\sum_{t\in\tau} b_{t\tau} = 0 \qquad \sum_{t\in\tau} b'_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \qquad b'_{t\tau} = -b'_{t\tau'} \tag{26}$$

If j_t, $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions $b^{\pm}$ corresponding to selfdual and antiselfdual parts of a 4-simplex σ then with the geometric phase choice we have:

$$S^{EPR,\gamma<1}|_{\pm,\pm} = \frac{i2}{1-\gamma}\left(((\pm + \pm)S_{Regge}(\sigma) - \gamma(\pm - \pm)S_{Regge}(\sigma))\right) \tag{27}$$

$$S^{FK_{\gamma>1}}|_{\pm,\pm} = \frac{i2}{1-\gamma}\left((\pm + \pm)S_{Regge}(\sigma) - \gamma(\pm - \pm)S_{Regge}(\sigma)\right) \tag{28}$$

$$S^{FK}|_{\pm,\pm} = i2\left((\pm - \pm)S_{Regge}(\sigma)\right) \tag{29}$$

New models asymptotics (5):

$$
\begin{aligned}
&\sum_{t\in\tau} b_{t\tau} = 0 \qquad && \sum_{t\in\tau} b'_{t\tau} = 0 \\
&b_{t\tau} = -b_{t\tau'} && b'_{t\tau} = -b'_{t\tau'}
\end{aligned}
\tag{30}
$$

If j_t, $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions $b^{\pm}$ corresponding to selfdual and antiselfdual parts of a 4-simplex σ then with the geometric phase choice we have:

$$
\begin{aligned}
Z^{EPR,\gamma<1}(\lambda\sigma) &= \lambda^{-12}\sum_{\pm\pm} N_{\pm}N_{\pm}\exp(S^{EPR,\gamma<1}|_{\pm}) + O(\lambda^{-13}) \\
Z^{FK,\gamma>1}(\lambda\sigma) &= \lambda^{-12}\sum_{\pm\pm} N_{\pm}\overline{N_{\pm}}\exp(S^{FK,\gamma>1}|_{\pm}) + O(\lambda^{-13})
\end{aligned}
\tag{31}
$$

Conclusions

- Diagonal $\mathrm{SU}(2)$ BF solutions are general solutions of the simplicity constraints.
- Heuristically also for the continuum action: $S = bF^+ + c_\gamma bF^-$ if $F^+ = F^-$ that's just BF theory.
- "Degenerate" sector in BC model finally beginning to be understood.
- Go back and reinterpret old results in this light: There is a $\mathrm{SU}(2)$ BF theory floating around!

- To Do: Classify $\mathrm{SU}(2)$ BF solutions in the non geometric sector.
- To Do: Study whole statesums. k, n variation. Does geometricity propagate?

  Pathintegral reformulation ala Conrady and Freidel seems to indicate promise as well as problems (Bonzom).

  Clarify with complementary analysis of classical discrete EoMs. (currently ongoing)