

The Asymptotics of 4d Spin Foam Models from $SU(2)$ BF Asymptotics.

Frank Hellmann

University of Nottingham

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The Papers

- “Asymptotic analysis of the EPRL four-simplex amplitude” (arXiv:0902.1170)
- “Lorentzian spin foam amplitudes: graphical calculus and asymptotics” (arXiv:0907.2440)
- New Stuff: “SU(2) Paper” (arXiv:tba)

with John W. Barrett, Richard J. Dowdall, Winston J. Fairbairn, Henrique Gomes, Roberto Pereira

using:

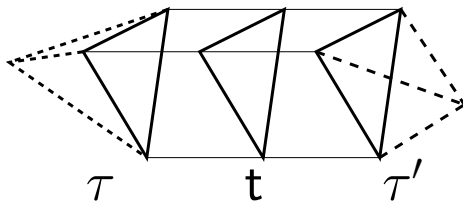
Livine, Speziale: A new spinfoam vertex for quantum gravity [0705.0674]
Engle, Pereira, Rovelli: Flipped spinfoam vertex and loop gravity [0708.1236]
Freidel, Krasnov: A New Spin Foam Model for 4d Gravity [0708.1595]
EPR + Livine: LQG vertex with finite Immirzi parameter [0711.0146]
Pereira: Lorentzian LQG vertex amplitude [0710.5043]

SU(2) Statesum (1)

SU(2) BF theory Spin Foam is:

$$Z_{\text{SU}(2)}(\mathcal{T}) = \sum_{j_t} d_{j_t} \int dn_{t\tau} \prod_{\sigma} 15j(j_t, n_{t\tau}) \quad (1)$$

3d analogue

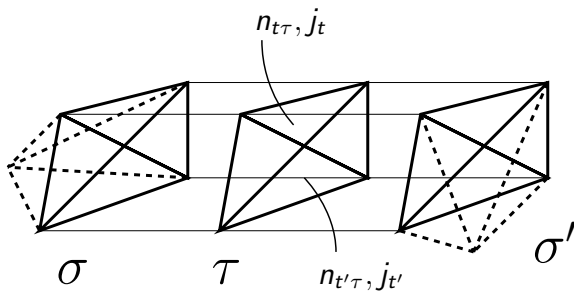


SU(2) Statesum (2)

SU(2) BF theory Spin Foam is:

$$Z_{\text{SU}(2)}(\mathcal{T}) = \sum_{j_t} d_{j_t} \int dn_{t\tau} \prod_{\sigma} 15j(j_t, n_{t\tau}) \quad (2)$$

$$\begin{aligned} j_t &\in \text{Irrep}(\text{SU}(2)) \\ n_{t\tau} &\in S^2 \end{aligned} \quad (3)$$



SU(2) Statesum (3)

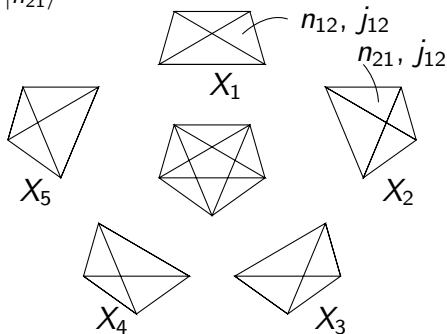
The measure is $d_{j_t} = ((-1)^{2j_t}(2j_t + 1))^{\#\tau \in t+1}$.

$$15j(j_t, n_{t\tau}) = \pm \int \prod_{\tau} dX \prod_t \langle J n_{t\tau} | X_{\tau}^{\dagger} X_{\tau'} | n_{t\tau'} \rangle_{\tau, \tau' \ni t}^{2j_t} \quad (4)$$

$n_{t\tau} \cdot L |n_{t\tau}\rangle = \frac{1}{2}i |n_{t\tau}\rangle$ with the generators $L^i \in \mathfrak{su}(2)$ (phase unspecified)

$J : J^2 = (-1)^{2j}$ is the antilinear structure on SU(2)

$\langle | \rangle_{1,2 \ni t_{12}} : \langle J n_{12} | X_1^{\dagger} X_2 | n_{21} \rangle^{2j_{12}}$



SU(2) Asymptotics (1):

$\mp 15j(j_t, n_{t\tau})$ is an integral:

$$15j(j_t, n_{t\tau}) = \int \prod_{\tau} dX \exp(S^{BF}) \quad (5)$$

with action

$$S^{BF}(j_t, n_{t\tau}) = \sum_t 2j_t (\ln \langle J n_{t\tau} | X_{\tau}^{\dagger} X_{\tau'} | n_{t\tau'} \rangle). \quad (6)$$

The reality and stationary point equations for this integral in terms of

$$b_{t\tau} := j_t X_t n_{t\tau} \quad (7)$$

are:

$$\begin{aligned} \frac{\delta S}{\delta X_{\tau}} = 0 & : \sum_{t \in \tau} b_{t\tau} = 0 \\ \text{Re}(S) = 0 & : b_{t\tau} = -b_{t\tau'} \end{aligned} \quad (8)$$

SU(2) Asymptotics (2):

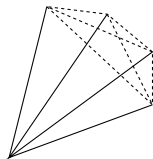
$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \quad (9)$$

Theorem:

$b_{t\tau}$ are in one to one correspondence with constant, vector valued 2-forms b on a 4-simplex.

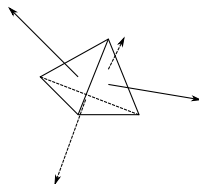
Proof: $b_{t\tau} = \int_t b$ for the oriented triangles meeting at a vertex is a basis for the constant 2-forms on a 4-simplex, the other faces follow by closure/Stokes'.

Note: These are the BC dominant configurations.



SU(2) Asymptotics (3):

$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \quad (10)$$



If j_t , $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

Theorem:

There are, up to symmetry, at most 2 solutions $b_{t\tau}^{\pm}$ corresponding to the selfdual sectors of two parity related geometric 4-simplices σ , $P\sigma$ with boundary geometry matching that defined by the j_t , $n_{t\tau}$.

Convention: Call b^+ the solution for which boundary orientation and 4-simplex orientation agree.

Interlude: Bivector geometry

Up to inversion ($inv : V^I \rightarrow -V^I$) a non-degenerate 4-simplex σ in \mathbb{R}^4 with edge vectors V_e^I is characterized by the triangle bivectors constructed from these:

$$(V_e \wedge V_{e'})^{IJ} = V_e^{[I} V_{e'}^{J]} = B_t^{IJ}. \quad (11)$$

These are proportional to the triangle areas.

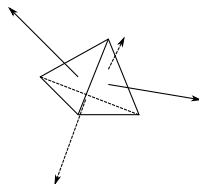
B^{IJ} are antisymmetric 4×4 matrices (6-dimensional space) and thus in $\mathfrak{so}(4)$: $\exp(B^{IJ}) \in SO(4)$. The double cover of $SO(4)$, $Spin(4)$, decomposes into $SU(2) \times SU(2)$.

We can decompose the 6-dimensional bivector space into two 3-dimensional spaces generating rotations purely in the left or the right factor. The left sector is called selfdual the right sector antiselfdual. We can then write

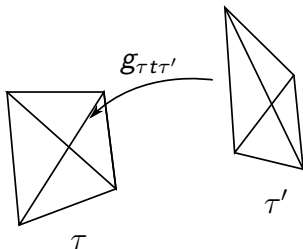
$$B_t = (b_t^s, b_t^a) \quad (12)$$

SU(2) Asymptotics (4):

$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \quad (13)$$

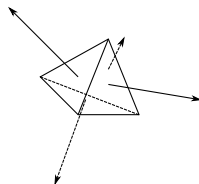


If j_{cd} , n_{cd} satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then define Regge gluing rotation $g_{\tau\tau'}$ by:



SU(2) Asymptotics (5):

$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \quad (14)$$



If j_{cd} , n_{cd} satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

Regge state

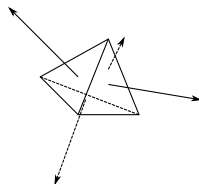
Using the geometric phase choice $|n_{t\tau}\rangle = g_{\tau t\tau'} |Jn_{t\tau'}\rangle$ on the boundary the action evaluates to

$$S^{BF}|_{b^\pm} = i \sum_t 2A_t (\pm\Theta_t) = \pm i 2S_{\text{Regge}}(\sigma), \quad (15)$$

with A_t the areas and Θ_t the dihedral angles of the geometric 4-simplex σ .

SU(2) Asymptotics (5):

$$\sum_{t \in \tau} b_{t\tau} = 0$$
$$b_{t\tau} = -b_{t\tau'} \quad (16)$$



If j_{cd} , n_{cd} satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then with the geometric phase choice for $|n_{ab}\rangle$:

$$\mp 15j(\lambda j_t, n_{t\tau})$$
$$= \lambda^{-6} (N_+ \exp(S^{BF}|_{b^+}) + N_- \exp(S^{BF}|_{b^-})) + O(\lambda^{-7})$$
$$= \lambda^{-6} (N_+ \exp(i2S_{Regge}(\sigma)) + N_- \exp(-i2S_{Regge}(\sigma))) + O(\lambda^{-7}) \quad (17)$$

New models as $SU(2)^2$ constrained (1):

$$P(b^s, b^a) = (b^a, b^s)$$

- ⇒ Selfdual part of 4-simplex = antiselfdual part of parity related 4-simplex.
- ⇒ $Z_{SU(2)} \times Z'_{SU(2)}$ has a sector where both the selfdual and antiselfdual part of the geometry of (usually different) 4-simplices are present.
If they are of the same 4-simplex then we have $n_{t\tau} = n'_{t\tau}$ and $j_t = j'_t$.
- ⇒ Impose these as constraints on the statesum (simplicity constraints).

New models as $SU(2)^2$ constrained (2):

Write:

$$Z_{SU(2)} \times Z'_{SU(2)}(O) = \sum_{j_t, j'_t} d_{j_t} d_{j'_t} \int dn_{t\tau} dn'_{t\tau} \prod_v 15j(j_t, n_{t\tau}) 15j(j'_t, n'_{t\tau}) O(n_{t\tau}, n'_{t\tau}, j_t, j'_t)$$

Then for certain choices of face amplitudes and $c_\gamma = \frac{1+\gamma}{|1-\gamma|}$:

$$\begin{aligned} Z_{EPR} &= Z_{SU(2)} \times Z'_{SU(2)} \left(\prod \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, j'_t) \right) \\ Z_{FK} &= Z_{SU(2)} \times \overline{Z'_{SU(2)}} \left(\prod \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, j'_t) \right) \\ Z_{EPRL_{\gamma < 1}} = Z_{FK_{\gamma < 1}} &= Z_{SU(2)} \times Z'_{SU(2)} \left(\prod \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, c_\gamma j'_t) \right) \\ Z_{FK_{\gamma > 1}} &= Z_{SU(2)} \times \overline{Z'_{SU(2)}} \left(\prod \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, c_\gamma j'_t) \right) \quad (18) \end{aligned}$$

New models asymptotics (1):

$$\begin{aligned}\sum_{t \in \tau} b_{t\tau} &= 0 & \sum_{t \in \tau} b'_{t\tau} &= 0 \\ b_{t\tau} &= -b_{t\tau'} & b'_{t\tau} &= -b'_{t\tau'}\end{aligned}\tag{19}$$

B -field sector: The solutions correspond to two constant vector valued 2-forms b, b' related by

$$\int_t b = c_\gamma g_t \int_t b'.\tag{20}$$

Any $SU(2)$ BF solution is a solution of the constrained theory with $b = c_\gamma b'$.

New models asymptotics (2):

$$\begin{aligned} \sum_{t \in \tau} b_{t\tau} &= 0 & \sum_{t \in \tau} b'_{t\tau} &= 0 \\ b_{t\tau} &= -b_{t\tau'} & b'_{t\tau} &= -b'_{t\tau'} \end{aligned} \tag{21}$$

If $j_t, n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

Theorem:

There are, up to symmetry, at most 2 solutions $b_{t\tau}^{\pm}$ corresponding to the selfdual and anti-selfdual sectors of a 4-simplex σ with boundary geometry matching that defined by the $j_t, n_{t\tau}$.

If there are two solutions then:

$$(b_{t\tau}, c_{\gamma} b'_{t\tau}) \in \{(b_{t\tau}^+, b_{t\tau}^+), (b_{t\tau}^+, b_{t\tau}^-), (b_{t\tau}^-, b_{t\tau}^+), (b_{t\tau}^-, b_{t\tau}^-)\} \tag{22}$$

New models asymptotics (3):

$$\begin{aligned} \sum_{t \in \tau} b_{t\tau} &= 0 & \sum_{t \in \tau} b'_{t\tau} &= 0 \\ b_{t\tau} &= -b_{t\tau'} & b'_{t\tau} &= -b'_{t\tau'} \end{aligned} \quad (23)$$

If j_t , $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions b^\pm corresponding to selfdual and antiselfdual parts of a 4-simplex σ then with the geometric phase choice we have:

$$S^{EPR, \gamma < 1} |_{\pm, \pm} = S^{BF} |_{b^\pm} + c_\gamma S^{BF} |_{b^\pm} = i2 (\pm S_{Regge} + c_\gamma (\pm S_{Regge})) \quad (24)$$

$$S^{FK, \gamma > 1} |_{\pm, \pm} = S^{BF} |_{b^\pm} + c_\gamma \overline{S^{BF} |_{b^\pm}} = i2 (\pm S_{Regge}(\sigma) - c_\gamma (\pm S_{Regge}(\sigma))) \quad (25)$$

New models asymptotics (4):

$$\begin{aligned}\sum_{t \in \tau} b_{t\tau} &= 0 & \sum_{t \in \tau} b'_{t\tau} &= 0 \\ b_{t\tau} &= -b_{t\tau'} & b'_{t\tau} &= -b'_{t\tau'}\end{aligned}\tag{26}$$

If j_t , $n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions b^\pm corresponding to selfdual and antiselfdual parts of a 4-simplex σ then with the geometric phase choice we have:

$$S^{EPR, \gamma < 1} |_{\pm, \pm} = \frac{i2}{1 - \gamma} (((\pm + \pm) S_{Regge}(\sigma) - \gamma(\pm - \pm) S_{Regge}(\sigma)))\tag{27}$$

$$S^{FK, \gamma > 1} |_{\pm, \pm} = \frac{i2}{1 - \gamma} (((\pm + \pm) S_{Regge}(\sigma) - \gamma(\pm - \pm) S_{Regge}(\sigma))\tag{28}$$

$$S^{FK} |_{\pm, \pm} = i2 ((\pm - \pm) S_{Regge}(\sigma))\tag{29}$$

New models asymptotics (5):

$$\begin{aligned}\sum_{t \in \tau} b_{t\tau} &= 0 & \sum_{t \in \tau} b'_{t\tau} &= 0 \\ b_{t\tau} &= -b_{t\tau'} & b'_{t\tau} &= -b'_{t\tau'}\end{aligned}\tag{30}$$

If $j_t, n_{t\tau}$ satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions b^\pm corresponding to selfdual and antiselfdual parts of a 4-simplex σ then with the geometric phase choice we have:

$$\begin{aligned}Z^{EPR, \gamma < 1}(\lambda\sigma) &= \lambda^{-12} \sum_{\pm\pm} N_\pm N_\pm \exp(S^{EPR, \gamma < 1}|_\pm) + O(\lambda^{-13}) \\ Z^{FK, \gamma > 1}(\lambda\sigma) &= \lambda^{-12} \sum_{\pm\pm} N_\pm \overline{N}_\pm \exp(S^{FK, \gamma > 1}|_\pm) + O(\lambda^{-13})\end{aligned}\tag{31}$$

Conclusions

- Diagonal $SU(2)$ BF solutions are general solutions of the simplicity constraints.
- Heuristically also for the continuum action: $S = bF^+ + c_\gamma bF^-$ if $F^+ = F^-$ that's just BF theory.
- “Degenerate” sector in BC model finally beginning to be understood.
- Go back and reinterpret old results in this light: There is a $SU(2)$ BF theory floating around!

- To Do: Classify $SU(2)$ BF solutions in the non geometric sector.
- To Do: Study whole statesums. k, n variation. Does geometricity propagate? Pathintegral reformulation ala Conrady and Freidel seems to indicate promise as well as problems (Bonzom).
Clarify with complementary analysis of classical discrete EoMs. (currently ongoing)