The Asymptotics of 4d Spin Foam Models from SU(2) BF Asymptotics.

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The Papers

- “Asymptotic analysis of the EPRL four-simplex amplitude” (arXiv:0902.1170)
- “Lorentzian spin foam amplitudes: graphical calculus and asymptotics” (arXiv:0907.2440)

with John W. Barrett, Richard J. Dowdall, Winston J. Fairbairn, Henrique Gomes, Roberto Pereira

using:

Livine, Speziale: A new spinfoam vertex for quantum gravity [0705.0674]
Engle, Pereira, Rovelli: Flipped spinfoam vertex and loop gravity [0708.1236]
Freidel, Krasnov: A New Spin Foam Model for 4d Gravity [0708.1595]
EPR + Livine: LQG vertex with finite Immirzi parameter [0711.0146]
Pereira: Lorentzian LQG vertex amplitude [0710.5043]
SU(2) Statesum (1)

SU(2) BF theory Spin Foam is:

\[ Z_{SU(2)}(\mathcal{T}) = \sum_{j_t} d_{j_t} \int d n_{t\mathcal{T}} \prod_{\sigma} 15j(j_t, n_{t\mathcal{T}}) \]  

(1)

3d analogue
SU(2) Statesum (2)

SU(2) BF theory Spin Foam is:

$$Z_{SU(2)}(\mathcal{T}) = \sum_{j_t} d_{j_t} \int d n_{t\tau} \prod_{\sigma} 15j(j_t, n_{t\tau})$$

\hspace{1cm} (2)

$$j_t \in \text{Irrep}(SU(2))$$

\hspace{1cm} (3)

$$n_{t\tau} \in S^2$$
SU(2) Statesum (3)

The measure is \( \mathrm{d}j_t = ((-1)^{2j_t}(2j_t + 1))^{(\#_{\tau \in t} + 1)}. \)

\[
15j(j_t, n_{t\tau}) = \pm \int \prod_{\tau} \mathrm{d}X \prod_{t} \langle Jn_{t\tau} | X^\dagger_{\tau} X_{\tau'} | n_{t\tau'} \rangle^{2j_t}_{\tau, \tau' \in t} \tag{4}
\]

\( n_{t\tau} \cdot L | n_{t\tau} \rangle = \frac{1}{2} i | n_{t\tau} \rangle \) with the generators \( L^i \in \mathfrak{su}(2) \) (phase unspecified)

\( J : J^2 = (-1)^{2j} \) is the antilinear structure on SU(2)

\( \langle | \rangle_{1, 2 \in t_{12}} : \langle Jn_{12} | X^\dagger_1 X_2 | n_{21} \rangle^{2j_{12}} \)
SU(2) Asymptotics (1):

$15j(j_t, n_{t\tau})$ is an integral:

$$15j(j_t, n_{t\tau}) = \int \prod_{\tau} dX \exp(S^{BF})$$  \hspace{1cm} (5)

with action

$$S^{BF}(j_t, n_{t\tau}) = \sum_t 2j_t \left( \ln \langle J_{n_{t\tau}} | X_\tau^\dagger X_{\tau'} | n_{t\tau'} \rangle \right) .$$  \hspace{1cm} (6)

The reality and stationary point equations for this integral in terms of

$$b_{t\tau} := j_t X_t n_{t\tau}$$  \hspace{1cm} (7)

are:

$$\frac{\delta S}{\delta X_\tau} = 0 : \sum_{t \in \tau} b_{t\tau} = 0$$

$$\text{Re}(S) = 0 : b_{t\tau} = -b_{t\tau'}$$  \hspace{1cm} (8)
SU(2) Asymptotics (2):

$$\sum_{t \in \tau} b_{t \tau} = 0$$

$$b_{t \tau} = -b_{t \tau}'$$  \hspace{1cm} (9)

**Theorem:**

$b_{t \tau}$ are in one to one correspondence with constant, vector valued 2-forms $b$ on a 4-simplex.

**Proof:** $b_{t \tau} = \int_t b$ for the oriented triangles meeting at a vertex is a basis for the constant 2-forms on a 4-simplex, the other faces follow by closure/Stokes'.

**Note:** These are the BC dominant configurations.
SU(2) Asymptotics (3):

\[ \sum_{t \in \tau} b_{t\tau} = 0 \]

\[ b_{t\tau} = -b_{t\tau}' \quad (10) \]

If \( j_t, n_{t\tau} \) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

**Theorem:**

There are, up to symmetry, at most 2 solutions \( b_{t\tau}^\pm \) corresponding to the selfdual sectors of two parity related geometric 4-simplices \( \sigma, P\sigma \) with boundary geometry matching that defined by the \( j_t, n_{t\tau} \).

Convention: Call \( b^+ \) the solution for which boundary orientation and 4-simplex orientation agree.
Interlude: Bivector geometry

Up to inversion \( \text{inv} : V^I \rightarrow -V^I \) a non-degenerate 4-simplex \( \sigma \) in \( \mathbb{R}^4 \) with edge vectors \( V^I_e \) is characterized by the triangle bivectors constructed from these:

\[
(V_e \wedge V_{e'})^{IJ} = V_e^I V_{e'}^J = B_t^{IJ}.
\] (11)

These are proportional to the triangle areas.

\( B^{IJ} \) are antisymmetric \( 4 \times 4 \) matrices (6-dimensional space) and thus in \( \mathfrak{so}(4) \): \( \exp(B^{IJ}) \in \text{SO}(4) \). The double cover of \( \text{SO}(4) \), \( \text{Spin}(4) \), decomposes into \( \text{SU}(2) \times \text{SU}(2) \).

We can decompose the 6-dimensional bivector space into two 3-dimensional spaces generating rotations purely in the left or the right factor. The left sector is called selfdual the right sector antiselfdual. We can then write

\[
B_t = (b_t^s, b_t^a)
\] (12)
SU(2) Asymptotics (4):

\[ \sum_{t \in \tau} b_{t\tau} = 0 \]

\[ b_{t\tau} = -b_{t\tau'} \]  \hspace{1cm} (13)

If \( j_{cd}, n_{cd} \) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then define Regge gluing rotation \( g_{\tau t\tau'} \) by:
SU(2) Asymptotics (5):

\[
\sum_{t \in \tau} b_{t\tau} = 0
\]

\[
b_{t\tau} = -b_{t\tau}' \quad (14)
\]

If \(j_{cd}, \ n_{cd}\) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

### Regge state

Using the geometric phase choice \(|n_{t\tau}\rangle = g_{\tau t\tau'} |Jn_{t\tau'}\rangle\) on the boundary the action evaluates to

\[
S^{BF}|_{b^\pm} = i \sum_{t} 2A_t \left( \pm \Theta_t \right) = \pm i 2S_{\text{Regge}}(\sigma),
\]

with \(A_t\) the areas and \(\Theta_t\) the dihedral angles of the geometric 4-simplex \(\sigma\).
SU(2) Asymptotics (5):

\[ \sum_{t \in \tau} b_{t\tau} = 0 \]

\[ b_{t\tau} = -b_{t\tau}' \]  \hspace{1cm} (16)

If \( j_{cd} \), \( n_{cd} \) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then with the geometric phase choice for \( |n_{ab}\rangle \):

\[ \mp 15j(\lambda j_t, n_{t\tau}) \]

\[ = \lambda^{-6} \left( N_+ \exp(S^{BF}|_{b^+}) + N_- \exp(S^{BF}|_{b^-}) \right) + O(\lambda^{-7}) \]

\[ = \lambda^{-6} \left( N_+ \exp(i2S_{\text{Regge}}(\sigma)) + N_- \exp(-i2S_{\text{Regge}}(\sigma)) \right) + O(\lambda^{-7}) \]  \hspace{1cm} (17)
New models as $SU(2)^2$ constrained (1):

\[ P(b^s, b^a) = (b^a, b^s) \]

⇒ Selfdual part of 4-simplex = antiselfdual part of parity related 4-simplex.

⇒ $Z_{SU(2)} \times Z'_{SU(2)}$ has a sector where both the selfdual and antiselfdual part of the geometry of (usually different) 4-simplices are present.

If they are of the same 4-simplex then we have $n_{t\tau} = n'_{t\tau}$ and $j_t = j'_t$.

⇒ Impose these as constraints on the statesum (simplicity constraints).
New models as $SU(2)^2$ constrained (2):

Write:

$$Z_{SU(2)} \times Z'_{SU(2)}(O) =$$

$$\sum_{j_t, j'_t} d_{j_t} d_{j'_t} \int d n_{t\tau} d n'_{t\tau} \prod_v 15j(j_t, n_{t\tau}) 15j(j'_t, n'_{t\tau}) O (n_{t\tau}, n'_{t\tau}, j_t, j'_t)$$

Then for certain choices of face amplitudes and $c_\gamma = \frac{1+\gamma}{|1-\gamma|}$:

$$Z_{EPR} = Z_{SU(2)} \times Z'_{SU(2)} \left( \prod_v \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, j'_t) \right)$$

$$Z_{FK} = Z_{SU(2)} \times \overline{Z'}_{SU(2)} \left( \prod_v \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, j'_t) \right)$$

$$Z_{EPRL, \gamma<1} = Z_{FK, \gamma<1} = Z_{SU(2)} \times Z'_{SU(2)} \left( \prod_v \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, c_\gamma j'_t) \right)$$

$$Z_{FK, \gamma>1} = Z_{SU(2)} \times \overline{Z'}_{SU(2)} \left( \prod_v \delta(n_{t\tau} - n'_{t\tau}) \delta(j_t, c_\gamma j'_t) \right)$$

(18)
New models asymptotics (1):

\[ \sum_{t \in \tau} b_{t\tau} = 0 \quad \sum_{t \in \tau} b'_{t\tau} = 0 \]

\[ b_{t\tau} = -b_{t\tau}' \quad b'_{t\tau} = -b'_{t\tau}' \]  \hspace{1cm} (19)

**B-field sector:** The solutions correspond to two constant vector valued 2-forms \( b, b' \) related by

\[ \int_t b = c_\gamma g_t \int_t b'. \]  \hspace{1cm} (20)

Any SU(2) BF solution is a solution of the constrained theory with \( b = c_\gamma b' \).
New models asymptotics (2):

\[
\sum_{t \in \tau} b_{t\tau} = 0 \qquad \sum_{t \in \tau} b'_{t\tau} = 0 \\
b_{t\tau} = -b'_{t\tau} \qquad b'_{t\tau} = -b'_{t\tau}
\] (21)

If \( j_t, n_{t\tau} \) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree then:

**Theorem:**

There are, up to symmetry, at most 2 solutions \( b_{t\tau}^\pm \) corresponding to the selfdual and anti-selfdual sectors of a 4-simplex \( \sigma \) with boundary geometry matching that defined by the \( j_t, n_{t\tau} \).

If there are two solutions then:

\[
(b_{t\tau}, c_\gamma b'_{t\tau}) \in \{(b_{t\tau}^+, b_{t\tau}^+), (b_{t\tau}^+, b_{t\tau}^-), (b_{t\tau}^-, b_{t\tau}^+), (b_{t\tau}^-, b_{t\tau}^-)\}
\] (22)
New models asymptotics (3):

\[
\sum_{t \in \tau} b_{t\tau} = 0 \quad \sum_{t \in \tau} b'_{t\tau} = 0
\]

\[
b_{t\tau} = -b'_{t\tau}, \quad b'_{t\tau} = -b'_{t\tau}.
\]

If \(j_t, n_{t\tau}\) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions \(b^\pm\) corresponding to selfdual and antiselfdual parts of a 4-simplex \(\sigma\) then with the geometric phase choice we have:

\[
S^{EPR, \gamma<1}_{\pm, \pm} = S^{BF}_{b^\pm} + c_\gamma S^{BF}_{b^\pm} = i2 (\pm S_{Regge} + c_\gamma (\pm S_{Regge})) \quad (24)
\]

\[
S^{FK, \gamma>1}_{\pm, \pm} = S^{BF}_{b^\pm} + c_\gamma \overline{S^{BF}}_{b^\pm} = i2 (\pm S_{Regge}(\sigma) - c_\gamma (\pm S_{Regge}(\sigma))) \quad (25)
\]
New models asymptotics (4):

\[ \sum_{t \in \tau} b_{t\tau} = 0 \quad \sum_{t \in \tau} b'_{t\tau} = 0 \]

\[ b_{t\tau} = -b_{t\tau}', \quad b'_{t\tau} = -b'_{t\tau}' \] \hspace{1cm} (26)

If \( j_t, n_{t\tau} \) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions \( b^\pm \) corresponding to selfdual and antselfdual parts of a 4-simplex \( \sigma \) then with the geometric phase choice we have:

\[ S^{EPR, \gamma < 1}_{\pm, \pm} = \frac{i2}{1 - \gamma} (((\pm + \pm) S_{\text{Regge}}(\sigma) - \gamma (\pm - \pm) S_{\text{Regge}}(\sigma))) \] \hspace{1cm} (27)

\[ S^{FK, \gamma > 1}_{\pm, \pm} = \frac{i2}{1 - \gamma} (((\pm + \pm) S_{\text{Regge}}(\sigma) - \gamma (\pm - \pm) S_{\text{Regge}}(\sigma))) \] \hspace{1cm} (28)

\[ S^{FK}_{\pm, \pm} = i2 (((\pm - \pm) S_{\text{Regge}}(\sigma))) \] \hspace{1cm} (29)
New models asymptotics (5):

\[ \sum_{t \in \tau} b_{t\tau} = 0 \quad \sum_{t \in \tau} b'_{t\tau} = 0 \]
\[ b_{t\tau} = -b_{t\tau}' \quad b'_{t\tau} = -b'_{t\tau}' \]  

If \( j_t, n_{t\tau} \) satisfy 3d non-degeneracy and adjacent triangle geometries and orientations agree and we have two solutions \( b^\pm \) corresponding to selfdual and antiselfdual parts of a 4-simplex \( \sigma \) then with the geometric phase choice we have:

\[ Z^{EPR, \gamma<1}(\lambda\sigma) = \lambda^{-12} \sum_{\pm\pm} N_\pm N_\pm \exp(S^{EPR, \gamma<1}|_{\pm}) + O(\lambda^{-13}) \]
\[ Z^{FK, \gamma>1}(\lambda\sigma) = \lambda^{-12} \sum_{\pm\pm} N_\pm N_\pm \exp(S^{FK, \gamma>1}|_{\pm}) + O(\lambda^{-13}) \]  

(31)
Conclusions

- Diagonal SU(2) BF solutions are general solutions of the simplicity constraints.
- Heuristically also for the continuum action: \( S = bF^+ + c_\gamma bF^- \) if \( F^+ = F^- \) that’s just BF theory.
- “Degenerate” sector in BC model finally beginning to be understood.
- Go back and reinterpret old results in this light: There is a SU(2) BF theory floating around!

To Do: Classify SU(2) BF solutions in the non geometric sector.
To Do: Study whole statesums. \( k, n \) variation. Does geometricity propagate? Pathintegral reformulation ala Conrady and Freidel seems to indicate promise as well as problems (Bonzom).

Clarify with complementary analysis of classical discrete EoMs. (currently ongoing)