Path Integral Formulation of Loop Quantum Cosmology

Adam D. Henderson

Institute for Gravitation and the Cosmos
Pennsylvania State University

International Loop Quantum Gravity Seminar
March 17 2008

Motivations

1. There have been arguments against the singularity resolution in LQC using path integrals. The canonical quantization is well understood and the singularity resolution is robust, so where do the path integral arguments break down?
2. Recent developements have allowed for the construction of new spin foam models that are more closely related to the canonical theory. It is important to make connections to spin-foams by building path integrals from the canonical theory when possible.
3. A path integral representation of Loop Quantum Cosmology will aid in studying the physics of more complicated LQC models ($k = 1$, $\Lambda \neq 0$, Bianchi). Using standard path integral techniques one can argue why the effective equations are so surprisingly accurate.

Introduction

- We construct a path integral for the exactly soluble Loop Quantum Cosmology starting with the canonical quantum theory.
- The construction defines each component of the path integral. Each has non-trivial changes from the standard path integral.
- We see the origin of singularity resolution in the path integral representation of LQC.
- The structure of the path integral features similarities to spin foam models.
- The path integral can give an argument for the surprising accuracy of the effective equations used in more complicated models.

Outline

Path Integrals - Overview

- Path Integrals: Covariant formulation of quantum theory expressed as a sum over paths
- Formally the path integral can be written as

$$\int \mathcal{D}q e^{iS[q]/\hbar} \quad \text{or} \quad \int \mathcal{D}q\mathcal{D}p e^{iS[q,p]/\hbar} \tag{1}$$

- To define a path integral directly involves defining the components of these formal expressions:
  1. What is the space of paths $q(t)$ over which we integrate?
  2. $\mathcal{D}$: What is the measure on this space of paths?
  3. $S[q]$: What is the phase associated with each path.
- Each of these components can be determined by constructing the path integral from the canonical theory (when possible).

Standard Construction

- Outline the standard construction of a path integral representation for Non-Relativistic Quantum Mechanics with a polynomial Hamiltonian, $H(q,p)$.
- We want path integral representation of the propagator

$$\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x\rangle = \langle x',t'|x,t\rangle \tag{2}$$

- Split the exponential into a product of N identical terms.

$$\langle x'|\prod_{n=1}^{N} e^{-i\hat{H}\epsilon/\hbar}|x\rangle \quad \text{where} \quad \epsilon = \frac{\Delta t}{N} \tag{3}$$

- Insert a complete basis in x between each exponential.

$$\langle x'|e^{-i\hat{H}\epsilon/\hbar}\int dx_{n-1}|x_{n-1}\rangle\langle x_{n-1}|e^{-i\hat{H}\epsilon/\hbar}...|x\rangle \tag{4}$$

Step 2 - Expand in ϵ and Evaluate

- By defining $x_N = x'$ & $x_0 = x$ this can be written in a simple form.

$$\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x\rangle = \prod_{m=1}^{N-1}\left[\int dx_m\right]\prod_{n=1}^{N}\langle x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}\rangle \tag{5}$$

- In the limit $N \to \infty$ ($\epsilon \to 0$) we can expand each term in the 2nd product of eqn (5) in ϵ.

$$\langle x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}\rangle = \langle x_n|1 - i\hat{H}\epsilon/\hbar|x_{n-1}\rangle + \mathcal{O}(\epsilon^2) \tag{6}$$

- This can be written as an integral over momentum p_n.

$$\begin{aligned}\langle x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}\rangle &= \frac{1}{2\pi\hbar}\int dp_n e^{ip_n(x_n-x_{n-1})/\hbar} \\ &\times \left[1 - i\epsilon H(p_n, x_{n-1}, x_n)/\hbar + \mathcal{O}(\epsilon^2)\right]\end{aligned} \tag{7}$$

Step 3

- Inserting the matrix elements $\langle x_n | e^{-i\hat{H}\epsilon/\hbar} | x_{n-1} \rangle$ into the expression for the full matrix element and simplifying.

$$\langle x' | e^{-i\hat{H}\Delta t/\hbar} | x \rangle = \underset{N\to\infty}{Lim} \prod_{m=1}^{N-1} \left[\int dx_m \right] \prod_{n=1}^{N} \left[\frac{1}{2\pi\hbar} \int dp_n \right] \tag{8}$$

$$e^{i\epsilon \sum_{n=1}^{N} (p_n(x_n - x_{n-1})/\epsilon\hbar)} \prod_{n=1}^{N} [1 - i\epsilon H(p_n, x_n, x_{n-1})/\hbar + \mathcal{O}(\epsilon^2)]$$

- The limit $N \to \infty$ defines the measure of the phase space path integral integral as the integral over the position at each time between t & t' and the integral over all momenta.
- Almost in path integral form, except the final product.

Infinite Product: Re-Exponentiation

- The infinite product has the form

$$\underset{N\to\infty}{Lim}\prod_{n=1}^{N}(1+a_n\epsilon+\mathcal{O}(\epsilon^2)) \quad \text{where} \quad \epsilon=\Delta t/N \tag{9}$$

- If $a_n = a \ \ \forall n$: Recall from intro calculus

$$\underset{N\to\infty}{Lim}(1+a\epsilon+\mathcal{O}(\epsilon^2))^N = e^{a\Delta T} \tag{10}$$

- If the terms are not the same for all n this can be generalized.

$$\underset{N\to\infty}{Lim}\prod_{n=1}^{N}(1+a_n\epsilon) = \underset{N\to\infty}{Lim}\, e^{\sum_{n=1}^{N} a_n\epsilon} \tag{11}$$

Extending to the product in the previous slide

$$\underset{N\to\infty}{Lim}\prod_{n=1}^{N}[1-i\epsilon H(p_n,x_{n-1},x_n)\hbar+\mathcal{O}(\epsilon^2)] = exp\left[-i/\hbar\,\underset{N\to\infty}{Lim}\sum_{n=1}^{N}\epsilon H(p_n,x_{n-1},x_n)\right] \tag{12}$$

Final Result

- Combining these results:

$$\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x\rangle = \underset{N\to\infty}{Lim} \prod_{m=1}^{N-1}\left[\int dx_m\right]\prod_{n=1}^{N}\left[\frac{1}{2\pi\hbar}\int dp_n\right] \exp\left(i/\hbar\, S[x,p,N]\right) \quad (13)$$

- Where we recognize S as the discretized action.

$$S[x,p,N] = \sum_{n=1}^{N}\epsilon\left(p_n(x_n - x_{n-1})/\epsilon - H[x_n, x_{n-1}, p_n]\right) \quad (14)$$

- In the limit $N \to \infty$ this is the action

$$S[x,p] = \int dt(p\dot{x} - H(x,p)) \quad (15)$$

Overview of Path Integral Construction

Strategy to construct path integral

1. Start from the propagator $\langle x', t' | x, t \rangle$
2. Split $e^{-i\Delta t/\hbar \widehat{H}}$ into N copies $e^{-i\epsilon \widehat{H}/\hbar}$
3. Insert a complete basis between each factor of $e^{-i\epsilon \widehat{H}/\hbar}$
4. Evaluate each $\langle x_n | e^{-i\hat{H}\epsilon/\hbar} | x_{n-1} \rangle$ to first order in ϵ
5. Re-exponentiate resulting expression.

SLQC - Classical

- We work with k=0 FRW model in the b, ν variables with a scalar field.

$$\nu = \varepsilon \frac{a^3 {}^oV}{2\pi \ell_{\rm Pl}^2 \gamma} \quad , \quad b = -\varepsilon \frac{4\pi\gamma G}{3 {}^oV} \frac{p_a}{a^2} \tag{16}$$

- With Poisson bracket

$$\{b, \nu\} = 2/\hbar \tag{17}$$

- Classically they have range $(-\infty, \infty)$.
- To simplify the constraint the lapse is chosen to be $N = a^3 N'$
- The phase space action is then

$$S = \int dt[\dot{b}\nu\frac{\hbar}{2} + p_\phi\dot{\phi} - \frac{N'}{2}(p_\phi^2 - 3\pi\hbar^2 G b^2 \nu^2)] \tag{18}$$

Argument Against Singularity Resolution

- We can try to construct a path integral directly from this classical theory.
- The path integral will be something of the form,

$$\langle \nu', t' | \nu, t \rangle = \int \mathcal{D}b\mathcal{D}\nu ... e^{iS/\hbar} \tag{19}$$

- The action evaluated along the classical path between ν and ν' is

$$S = \frac{p_\phi}{\sqrt{12\pi G}} ln(\nu/\nu') \tag{20}$$

- Quantum corrections are negligible when $S_{cl} >> \hbar$
- No quantum corrections to classical singular paths $\rightarrow$ No bounce!

SLQC - Physical Hilbert Space

- We will construct the path integral starting from the physical Hilbert space of SLQC.
- The physical states are solutions to the quantum Hamiltonian constraint.

$$\partial_\phi^2 \psi(\nu, \phi) = -\widehat{\Theta}\psi(\nu, \phi) \tag{21}$$

- The physical Hilbert space can be obtained by group averaging procedure.
- Find that physical states satisfy a Schrodinger like equation.

$$-i\partial_\phi \psi(\nu, \phi) = \widehat{\sqrt{\Theta}}\psi(\nu, \phi) \tag{22}$$

- The physical inner product is.

$$\langle \psi_1 | \psi_2 \rangle = \frac{\lambda}{\pi} \sum_{\nu = 4n\lambda} \frac{1}{|\nu|} \bar{\psi}_1(\nu, \phi_o)\psi_2(\nu, \phi_o) \tag{23}$$

Schrodinger Eqn $\rightarrow$ Path Integral Construction

- We have SLQC written in the form of a Schrodinger equation with ϕ as time, so we apply the construction above to obtain a path integral.
- Similar to the construction in non-relativistic quantum mechanics we want a path integral representation of the propagator.

$$\langle \nu' | e^{i\widehat{\sqrt{\Theta}}\Delta\phi} | \nu \rangle = \langle \nu', \phi' | \nu, \phi \rangle \tag{24}$$

- More generally we could construct the path integral from the definition of the physical inner product in terms of group averaging.

SLQC Propagator - Exact

- We have some knowledge of the exact propagator.
- One can show that

$$\langle \nu', \phi' | \nu, \phi \rangle = 0 \quad \text{if} \quad \nu' < 0 \ \& \ \nu > 0 \tag{25}$$

- This allows us to simplify the calculations by restricting to positive or negative ν
- The propagator can be written as an integral.

$$\begin{aligned}\langle \nu', \phi' | \nu, \phi \rangle &= \frac{\lambda}{2\pi\nu} \int_0^{\pi/\lambda} db \ \ e^{i\left[\frac{b\nu}{2} - \frac{1}{\lambda} \tan^{-1}(e^{\Delta\phi} \tan(\lambda b/2))\nu'\right]} \\ &\quad +(\Delta\phi \rightarrow -\Delta\phi)\end{aligned} \tag{26}$$

Step 1 - Split Exponential/Insert Complete Basis

- As before we split the exponential into N copies and insert a complete basis of ν between each.

$$1 = \frac{\pi}{\lambda} \sum_{\nu=4n\lambda} |\nu| \, |\nu\rangle\langle\nu| \tag{27}$$

- Giving

$$\langle \nu', \phi' | \nu, \phi \rangle = \prod_{n=1}^{N-1} \left[\frac{\pi}{\lambda} \sum_{\nu_n} |\nu_n| \right] \prod_{n=1}^{N} \left[\langle \nu_n | e^{i\epsilon\widehat{\sqrt{\Theta}}} | \nu_{n-1} \rangle \right] \tag{28}$$

- Important difference: Instead of continuous integrals there are discrete sums over ν at each ϕ.
- The next step is to compute each term of the product:

$$\langle \nu_n | e^{i\epsilon\widehat{\sqrt{\Theta}}} | \nu_{n-1} \rangle \tag{29}$$

Step 2 - Evaluate Each Term in Product

- Want to evaluate each term to first order in ϵ

$$\langle \nu_n | e^{i\epsilon\widehat{\sqrt{\Theta}}} | \nu_{n-1} \rangle \tag{30}$$

- Problem: Even computing to first order in epsilon requires knowing the spectrum of $\widehat{\Theta}$
- The resolution is to rewrite each term as

$$\langle \nu_n | e^{i\epsilon\widehat{\sqrt{\Theta}}} | \nu_{n-1} \rangle = \int_{-\infty}^{\infty} dp_{\phi_n} |p_{\phi_n}| \Theta(p_{\phi_n}) \int_{-\infty}^{\infty} dN_n \frac{\epsilon}{2\pi\hbar} \tag{31}$$
$$\text{x} \;\; e^{ip_{\phi_n}\epsilon/\hbar} \langle \nu_n | e^{-i\epsilon\frac{N_n}{2\hbar}({p_{\phi_n}}^2 - \hbar^2\widehat{\Theta})} | \nu_{n-1} \rangle$$

- We then only need to evaluate the the following to first order in epsilon.

$$\langle \nu_n | e^{i\epsilon\frac{\hbar N_n}{2}\widehat{\Theta})} | \nu_{n-1} \rangle \tag{32}$$

Step 2b - Try Again

- Evaluating this term is simple given the action of $\widehat{\Theta}$

$$\langle\nu_n|e^{i\epsilon\frac{\hbar N_n}{2}\widehat{\Theta}}|\nu_{n-1}\rangle = \frac{\lambda}{\pi\nu_{n-1}}\Big(\delta_{\nu_n,\nu_{n-1}} - i\epsilon\frac{\hbar N_n}{2}\frac{3\pi G}{4\lambda^2}\nu_n\frac{\nu_n+\nu_{n-1}}{2} \times [\delta_{\nu_n,\nu_{n-1}+4\lambda} + \delta_{\nu_n,\nu_{n-1}-4\lambda} - 2\delta_{\nu_n,\nu_{n-1}}] + \mathcal{O}(\epsilon^2)\Big) \tag{33}$$

- Expressed as an integral by writing the delta functions as integrals over b.

$$\frac{1}{\nu_{n-1}}\frac{\lambda^2}{\pi^2}\int_0^{\pi/\lambda} db_n e^{-i(\nu_n-\nu_{n-1})b_n/2}[1+i\epsilon\frac{\hbar N_n}{2}\frac{3\pi G}{\lambda^2}\nu_n\frac{\nu_n+\nu_{n-1}}{2}sin^2(\lambda b_n)] \tag{34}$$

Step 3 - Re-exponentiate

- Combining together the results from the previous slides and re-exponentiating the product we arrive at the path integral:

$$\langle \nu', \phi' | \nu, \phi \rangle = \underset{N\to\infty}{Lim}$$
$$\frac{1}{\nu_0} \prod_{n=1}^{N-1} \left[\sum_{\nu_n} \right] \prod_{n=1}^{N} \left[\frac{\lambda}{\pi} \int_0^{\pi/\lambda} db_n \int_{-\infty}^{\infty} dp_{\phi_n} |p_{\phi_n}| \Theta(p_{\phi_n}) \frac{\epsilon}{2\pi\hbar} \int_{-\infty}^{\infty} dN_n \right]$$
$$\exp \frac{i}{\hbar} \sum_{n=1}^{N} \epsilon \left[p_{\phi_n} - \frac{\hbar}{2} \frac{(\nu_n - \nu_{n-1})}{\epsilon} b_n - \frac{N_n}{2} (p_{\phi_n}^2 - \frac{3\pi G \hbar^2}{\lambda^2} \nu_{n-1} \frac{\nu_n + \nu_{n-1}}{2} sin^2(\lambda b_n) \tag{3$$

- Where the discretized action is

$$S_N = \sum_{n=1}^{N} \epsilon \left[p_{\phi_n} - \frac{\hbar}{2} \frac{(\nu_n - \nu_{n-1})}{\epsilon} b_n - \frac{N_n}{2} (p_{\phi_n}^2 - \frac{3\pi G \hbar^2}{\lambda^2} \nu_{n-1} \frac{\nu_n + \nu_{n-1}}{2} sin^2(\lambda b_n) \right] \tag{36}$$

- There are non-trivial changes to the space of paths, measure, and action.

Allowed Paths in b, ν

- Space of Paths: Defined by the range of integration at each time.
- Paths $\nu(\phi)$ are discrete : $\nu(\phi) \, \epsilon \, (4\lambda, 8\lambda, 12\lambda, ...)$
- Paths $b(\phi)$ are continuous, but bounded: $b(\phi) \, \epsilon \, [0, \pi/\lambda]$
- We are integrating only over discrete quantum geometries
- Similar situation to that of spin foam models
  - Would try to define path integral over continuous fields ${}^4e$ and ${}^4A$
  - Instead integrate over discrete geometries.
- The space of paths has been modified due to the kinematical structure of LQC.
- The measure has changed - but is a natural measure on this space of paths.

Phase $\sim$ Effective Action

- The phase associated to each path is not the classical action.

$$
\begin{aligned}
S_N \quad = \quad & \sum_{n=1}^{N} \epsilon \left[p_{\phi_n} - \frac{\hbar}{2} \frac{(\nu_n - \nu_{n-1})}{\epsilon} b_n \right. \\
& \left. - \frac{N_n}{2} (p_{\phi_n}^2 - \frac{3\pi G}{\lambda^2} \nu_{n-1} \frac{\nu_n + \nu_{n-1}}{2} sin^2(\lambda b_n) \right]
\end{aligned} \tag{37}
$$

- This is a discretized version of an effective action which contains non-perturbative quantum corrections.

$$
S = \int_{\phi}^{\phi'} d\phi \left[p_\phi - \frac{\hbar}{2} \dot{\nu} b - \frac{N}{2} \left(p_\phi^2 - \frac{3\pi G \hbar^2}{\lambda^2} \nu^2 sin^2(\lambda b) \right) \right] \tag{38}
$$

- This is the effective action that well approximates the quantum dynamics.

Simplify Path Integral - $\int \mathcal{D}\nu \mathcal{D}b \mathcal{D}N \mathcal{D}p_\phi \rightarrow \int \mathcal{D}\nu \mathcal{D}b$

- Possible to integrate out variables to obtain a simpler expression?
- Want configuration space path integral.
- We can integrate out N and p_ϕ to obtain a path integral over b and ν only.

$$\langle \nu', \phi' | \nu, \phi \rangle = \underset{N\rightarrow\infty}{Lim} \frac{1}{\nu_0} \prod_{n=1}^{N-1} \left[\sum_{\nu_n} \right] \prod_{n=1}^{N} \left[\frac{\lambda}{\pi} \int_0^{\pi/\lambda} db_n \right] \tag{39}$$
$$\exp \frac{i}{\hbar} \sum_{n=1}^{N} \epsilon \left[\sqrt{\frac{3\pi G\hbar^2}{\lambda^2} \nu_{n-1} \frac{\nu_n + \nu_{n-1}}{2}} sin(\lambda b_n) - \frac{\hbar}{2} \frac{(\nu_n - \nu_{n-1})}{\epsilon} b_n \right]$$

- Equivalent to solving the constraint $\rightarrow$ Path integral on constraint surface.

Configuration Space P.I. $\sim$ Spin foam

- Can further integrate out b to obtain the following.

$$\langle \nu', \phi' | \nu, \phi \rangle = \underset{N \to \infty}{Lim} \sum_{\sigma_N} A(\sigma) \tag{40}$$

- Where $\sigma_N = (\nu_N, \nu_{N-1}, ..., \nu_1, \nu_0)$ is a sequence of ν
- $A(\sigma)$ is the amplitude associated to each sequence, which is a product of "vertex amplitudes"

$$A(\sigma) = \prod_n A(\nu_n, \nu_{n-1}) \tag{41}$$

- The vertex amplitudes are roughly

$$A(\nu_n, \nu_{n-1}) \approx J_{\frac{\nu_n - \nu_{n-1}}{2\lambda}} \left(\epsilon \sqrt{\frac{3\pi G}{2\lambda^2}} \sqrt{\nu_{n-1}(\nu_n + \nu_{n-1})} \right) \tag{42}$$

Singularity Resolution

- Use path integral over ν, b, p_ϕ, N.
- The path integral is not dominated by the classical singular solutions.
- The classical solutions are not in the space of paths we integrate over.
  - Classically $b \to \infty$ at singularity.
  - Paths $b(\phi)$ in path integral are bounded.
  - Minimum $\nu = 4\lambda$
- The action appearing in the path integral is not the classical action, so the there are different "classical paths" that contribute to the path integral.
- The singularity resolution seems to arise due to both the restriction to discrete geometries and to the effective action.

$\sum_\nu \int_0^{\pi/\lambda} \rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}$

- We can see that the singularity resolution is due primarily to the effective action.
- The discrete sums over ν and the integrals over b can be transformed into integrals over the whole real line of ν and b

$$\sum_{\nu_n} \int_0^{\pi/\lambda} db \rightarrow \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} db \tag{43}$$

- This transforms the space of paths and the measure to those of the standard path integral.
- The only change from the standard path integral is the action.

$$S[\nu, b, p_\phi, N] = \int d\phi [p_\phi - \frac{\hbar}{2} b\dot{v} - \frac{N}{2}(p_\phi^2 - \frac{3\pi G\hbar^2}{\lambda^2}\nu^2 \sin^2(\lambda b))] \tag{44}$$

Singularity Resolution

- The path integral is then dominated by the extrema of the effective action.

$$S[\nu, b, p_\phi, N] = \int d\phi [p_\phi - \frac{\hbar}{2} b\dot{v} - \frac{N}{2}(p_\phi^2 - \frac{3\pi G\hbar^2}{\lambda^2}\nu^2 \sin^2(\lambda b))] \quad (45)$$

- The "classical solutions", x_{eff} to this action are the bouncing solutions.
- The action can be computed along these bouncing solutions between ν and ν'.
- $S_{eff}(x_{eff})$ is large in units of $\hbar \rightarrow$ loop corrections are negligible.
- Provides an additional explanation for the accuracy of the effective equations.

Physics of More Complex LQC Models

- Many of the results here can be extended to more complicated LQC models including the k=1 case and $\Lambda \neq 0$.
- For these models there are also effective equations which approximate the quantum dynamics with surprising accuracy.
- The accuracy is not well understood.
- Assumptions used to derive them from Geometric QM (Willis, Taveras) fail in high curvature region.
- Extend path integral argument from k=0 case.
- Expect that the action appearing in the path integral is the effective action which gives effective equations.
- This action evaluated along its "classical solutions" is large in units of $\hbar$, so loop corrections are negligible.

Conclusions

- We construct a path integral representation for exactly soluble LQC starting from the canonical quantization.
- Due to the kinematical structure of LQC the paths integrated over are discrete quantum geometries. Similar to spin foam models.
- The path integral can be written as a sum over sequences of ν of an amplitude for each sequence which is given by a product of "vertex amplitudes".
- The modified action appearing in the path integral has bouncing solutions as its "classical solutions" which provides an argument for singularity resolution from the path integral representation.
- The path integral representation of more complicated LQC models may provide an argument for the surprising accuracy of the effective equations.