# A canonical formalism for simplicial gravity

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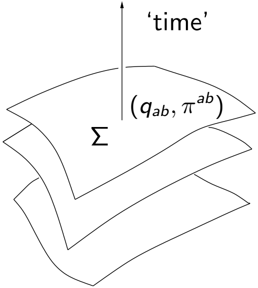
based on B. Dittrich, PH arXiv:1108.1974, 0912.1817 and *wip*
(summary PH 1110.3947)

# Outline

# Recap: canonical General Relativity

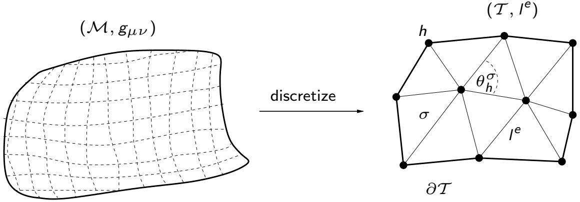
- Assume $\mathcal{M} = \mathbb{R} \times \Sigma \Rightarrow$ can do 3+1 splitting
- $q_{ab}$ ind. metric on $\Sigma$, $\pi^{ab}$ related to extr. curv.: canonical pair [Arnowitt, Deser, Misner '62]
- diffeo and Hamiltonian constraints $D_a(q,\pi) \approx 0$, $H(q,\pi) \approx 0$ form hypersurface deformation algebra: generates both (infinitesimal) symmetries <u>and</u> time evolution
- hypersurfaces evolve in 'multi–fingered' time through 4D solution
- want <u>analogous</u> structure in simplicial gravity



# Discretizing General Relativity: Recap of Regge Calculus

- Regge Calculus [Regge '61]: replace smooth $D$–dim. spacetime $(\mathcal{M}, g_{\mu\nu})$ by piecewise-linear flat metric living on triangulation $\mathcal{T}$, comprised of $D$–simplices $\sigma$

$h$: 'hinge' ($(D-2)$–subsimplex)
$\theta_h^\sigma$: interior dihedral angle at $h$ in $\sigma$
$A_h$: volume of $h$
$\epsilon_h := 2\pi - \sum_{\sigma\supset h} \theta_h^\sigma$: deficit angle
$\psi_h := \pi - \sum_{\sigma\supset h} \theta_h^\sigma$: exterior angle

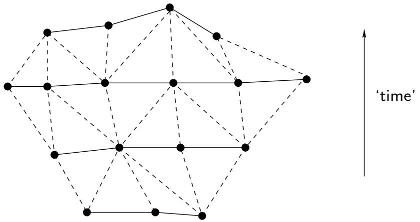



- configuration variables: edge lengths $\{l^e\}_{e\in\mathcal{T}}$, encode complete geometry
- (Euclidean) action $S_{EH} = -\frac{1}{2}\int_{\mathcal{M}} \sqrt{g} R d^4x \underset{\text{discretize}}{\longrightarrow} S_R$

$$S_R(\{l^e\}) = -\sum_{h\subset\mathcal{T}\setminus\partial\mathcal{T}} A_h\epsilon_h - \sum_{h\subset\partial\mathcal{T}} A_h\psi_h \qquad \Rightarrow \qquad S_R \text{ additive}$$

# Challenge for a canonical framework of simplicial gravity

- thus far mainly covariant formulations for simplicial gravity
  - benefited from numerical methods available for lattice gauge theories (e.g. (C)DT, Quantum Regge Calculus,...)
- general canonical framework lacking
- major obstacle: problem of foliations
  - hypersurfaces of different numbers of $\sigma$'s which carry variables $\Rightarrow$ need mapping between phase spaces of different dimension



# Previous results

- previous attempts [Piran, Williams '86; Friedman, Jack '86]: use continuum 3+1 splitting, discretize space (i.e. $\Sigma$), but keep (inf.) time evolution generated by set of constraints
- BUT: consistent canonical formulation must reproduce covariant solutions
  - in particular: discrete time evolution, not generated by constraints
  - need: well–defined set of discrete evolution moves
  - constraints: reduced to generating (inf.) transformations, under certain conditions gauge symmetries of action
- Based on earlier ideas by [Barrett, Galassi, Miller, Sorkin, Tuckey, Williams '97; Gambini, Pullin '03; Bahr, Dittrich '09] recently first consistent canonical formulation, however, spatial triangulation preserved [Dittrich, PH '09]

# Goal and Motivation

- How to treat situation where lattice evolves/changes? (as in LQG)
  - Numbers of degrees of freedom may vary

- why bother to construct general canonical formalism?
  1. classically: provides better notion of discrete dynamics, capture full space of (discrete) solutions
  2. possibly advantageous for several physical situations (e.g. expanding/contracting universe) and their numerical implementation
  3. in Quantum gravity
     - transition amplitudes between 3D geometries $\Rightarrow$ identify states as elements of Hilbert space of 3D geometries
     - comparing/connecting different approaches to quantum gravity (LQG vs. Spin Foams, etc....)
     - constraints vs. discrete time evolution

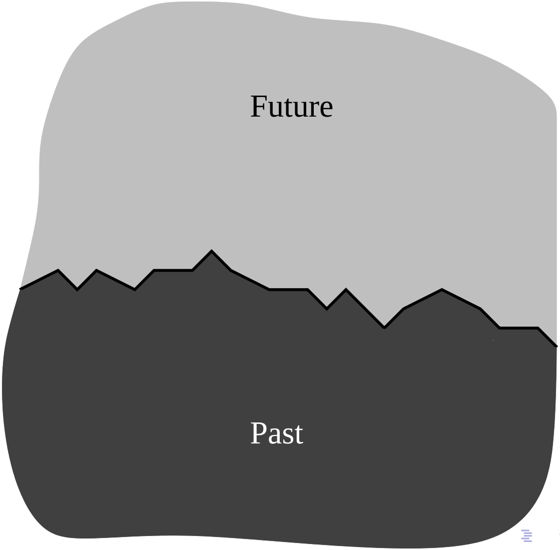
- Goal of this talk: general canonical scheme, reproducing all triangulations

# Evolution in discrete 'multi-fingered' or 'bubble' time

step $k$

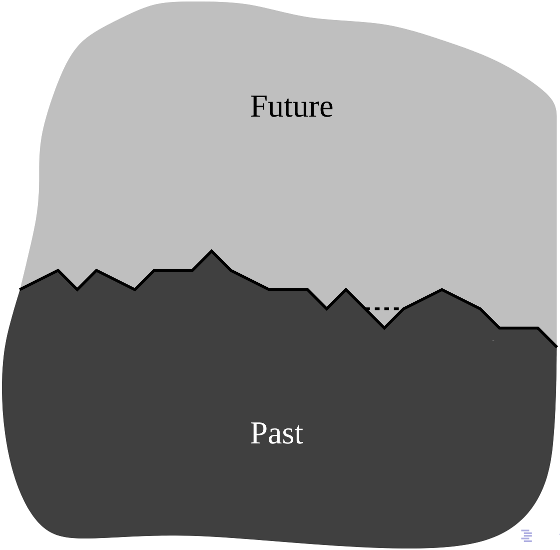
<u>Idea:</u>
glue single $D$-simplex, to $D-1$-dimensional triangulated hypersurface $\Sigma_k$ at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive



# Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+1$

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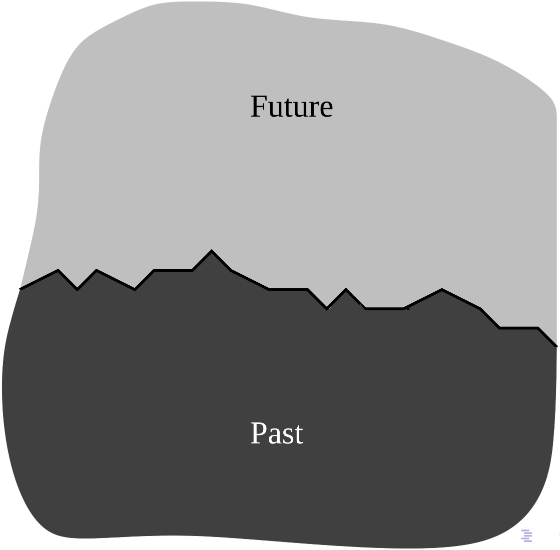


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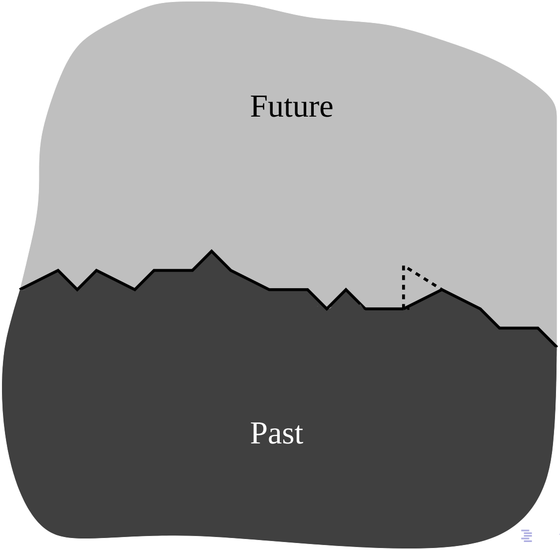


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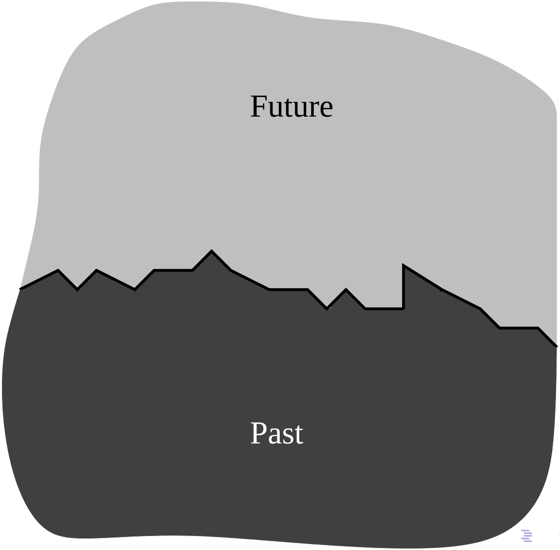
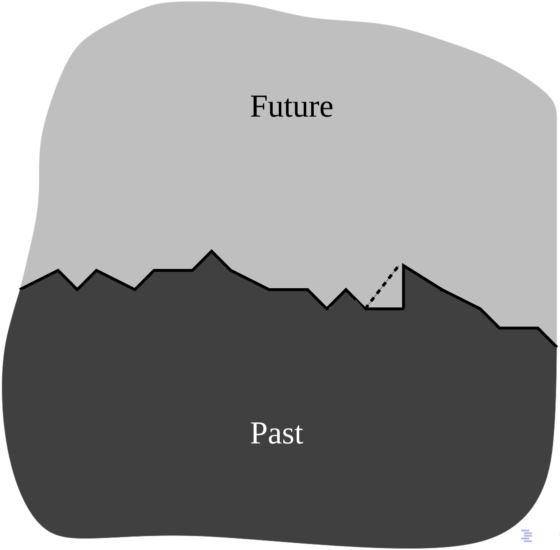


# Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k+3$

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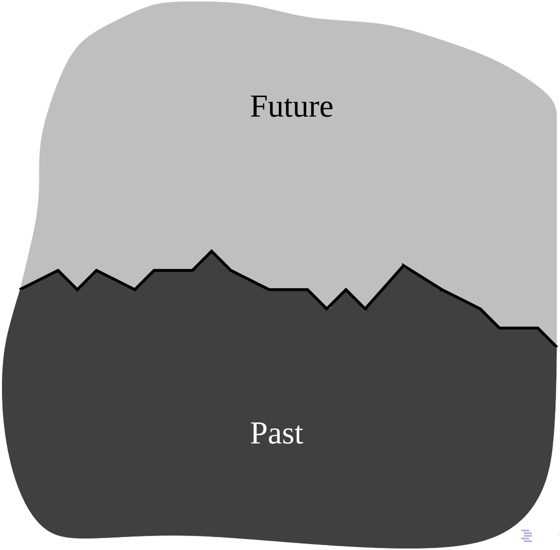

# Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k + 4$

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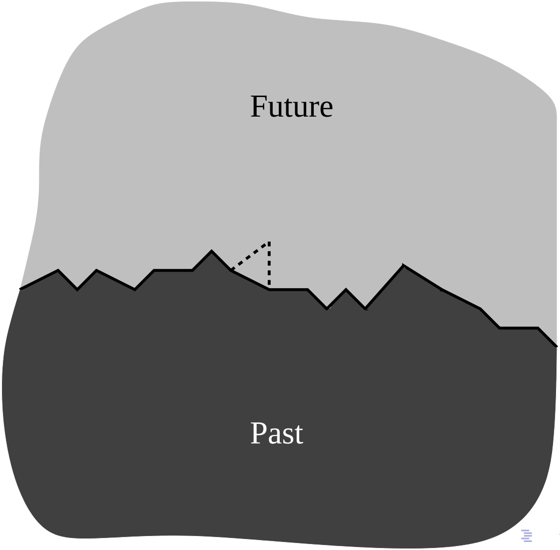

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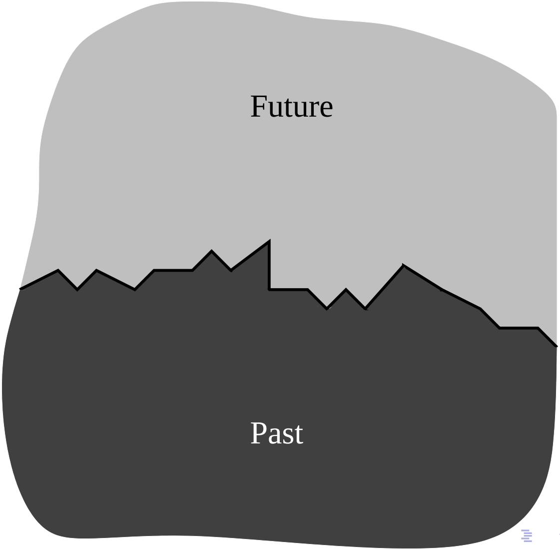


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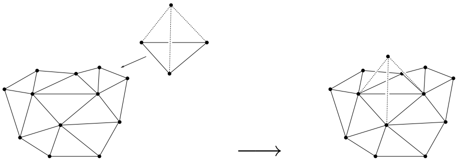
# Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

3D Example: gluing of tetrahedron onto single triangle

3D perspective:



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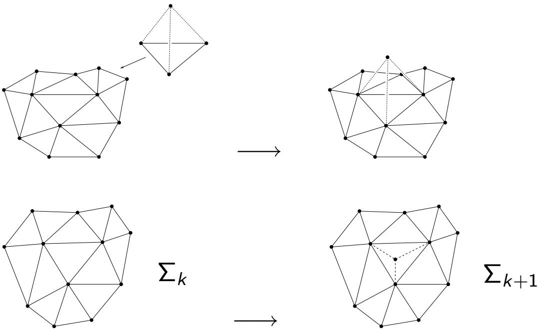
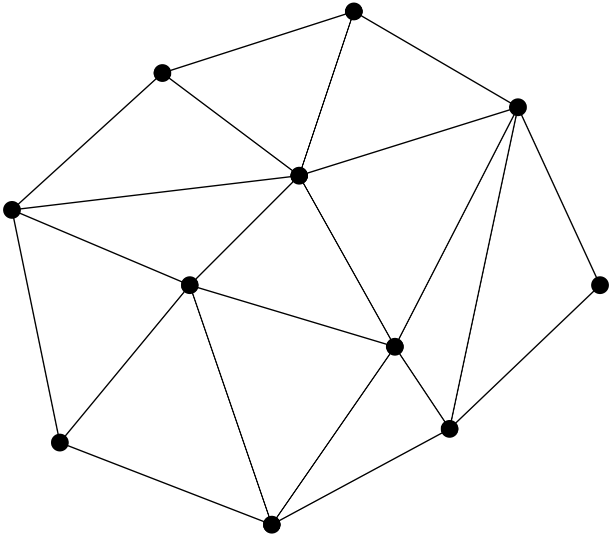
2D perspective:




$\Rightarrow$ 1–3 Pachner move (other Pachner moves in 3D and 4D similarly)
$\Rightarrow$ Pachner moves [Pachner '86] are ergodic and topology preserving
$\Rightarrow$ $\mathcal{T} = I \times \Sigma$ (as in canon. GR)

# Local evolution of the hypersurface with Pachner moves

step $k$



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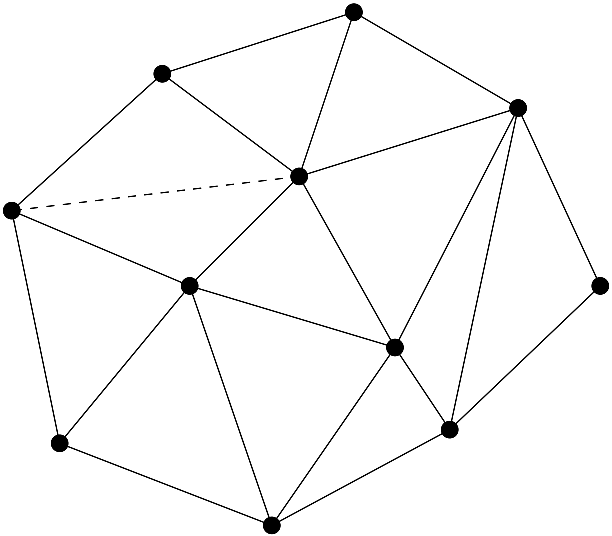

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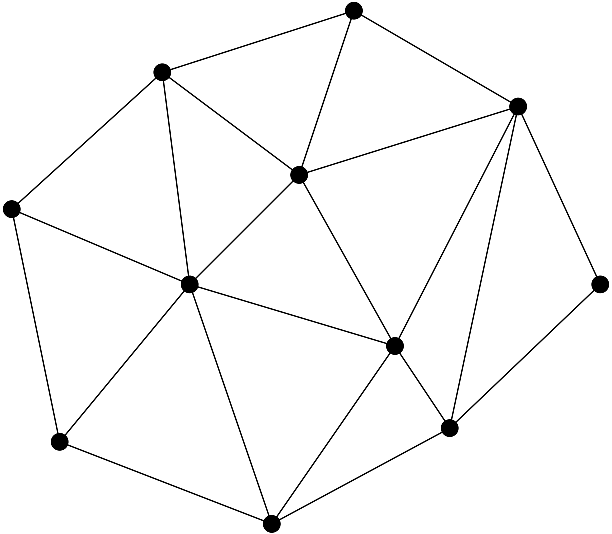
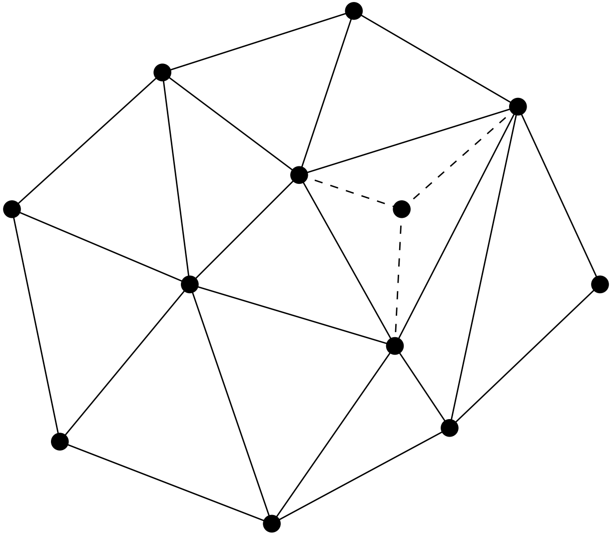


# Local evolution of the hypersurface with Pachner moves

step $k + 2$



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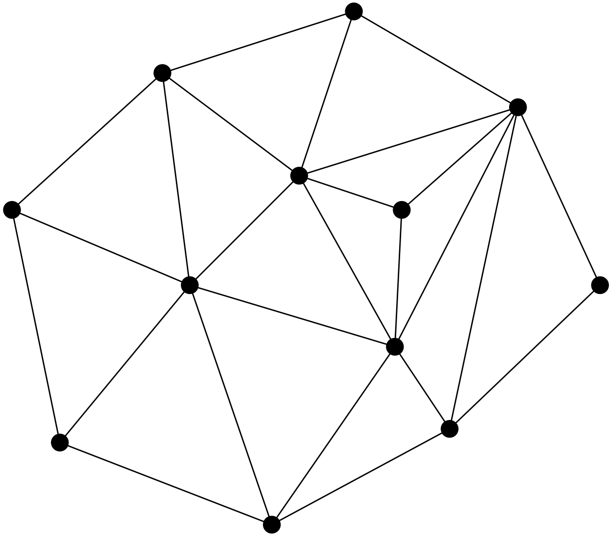

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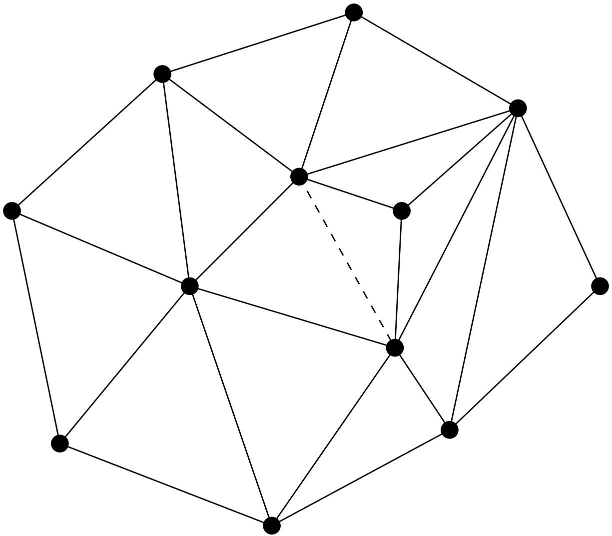

# Local evolution of the hypersurface with Pachner moves

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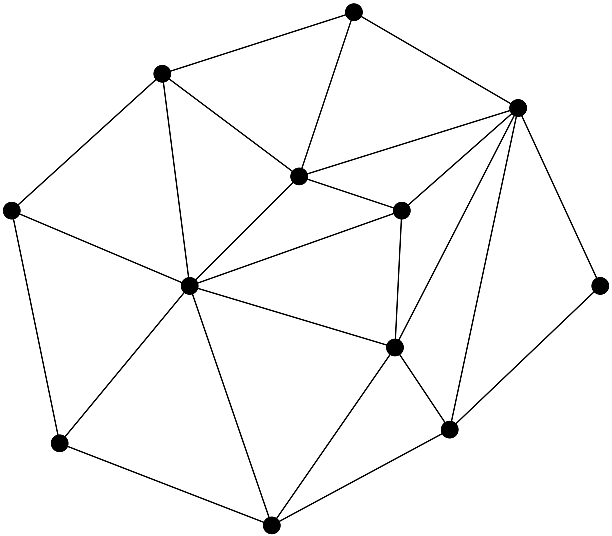
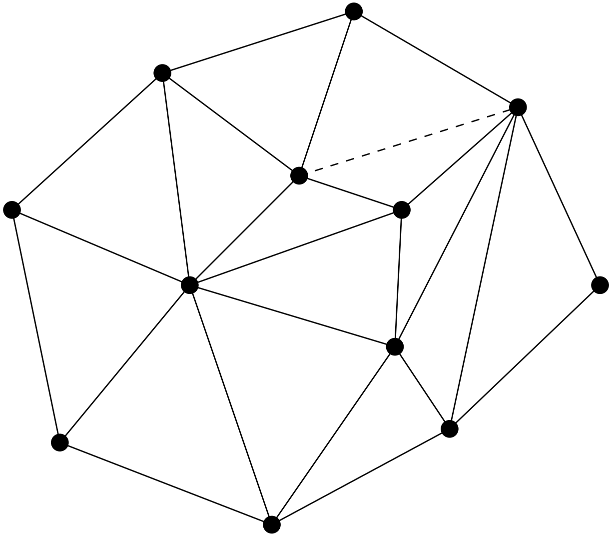


# Local evolution of the hypersurface with Pachner moves

step $k + 4$



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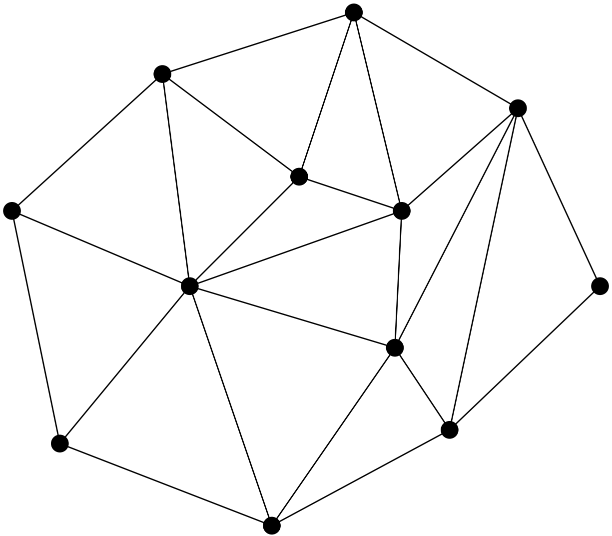

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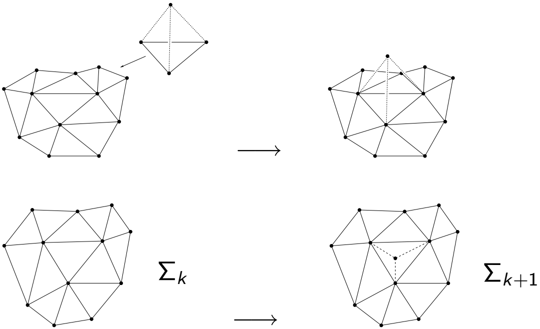



In general, face 'problems':

(a) subsets of variables coincide at different steps, i.e. $\Sigma_{k+1} \cap \Sigma_k \neq \emptyset$

(b) numbers of variables differ (phase space dim. varies) from step to step

# Canonical discrete dynamics [Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11]

- central idea: use Hamilton's principal function $S(x_{ini}, x_{fin})$ as generating fct. of 1st kind to generate time evolution
- discrete action $S = \sum_{n=1}^{N} S_n(x_{n-1}, x_n)$

  $S_n : \mathcal{Q}_{n-1} \times \mathcal{Q}_n \to \mathbb{R}$
- in cont. $L : T\mathcal{Q} \to \mathbb{R}$, $L(q, \dot{q}) \in \mathbb{R}$
- $S_n$ as generating fct.

$$ {}^{-}p^{n-1} := -\frac{\partial S_n(x_{n-1}, x_n)}{\partial x_{n-1}} \quad , \quad {}^{+}p^{n} := \frac{\partial S_n(x_{n-1}, x_n)}{\partial x_n} $$

${}^{-}p$: pre–momenta, ${}^{+}p$: post–momenta

# Momentum matching [Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11]

$$ {}^{-}p^{n-1} := -\frac{\partial S_n(x_{n-1}, x_n)}{\partial x_{n-1}} \quad , \quad {}^{+}p^{n} := \frac{\partial S_n(x_{n-1}, x_n)}{\partial x_n} \qquad (1) $$

${}^{-}p$: pre–momenta, ${}^{+}p$: post–momenta

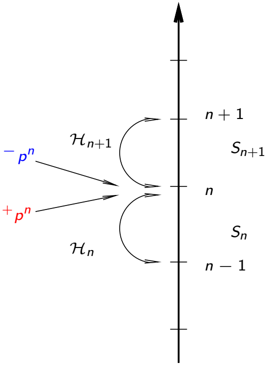
- defines time evolution map

$$ \mathcal{H}_n : (x_{n-1}, {}^{-}p^{n-1}) \mapsto (x_n, {}^{+}p^{n}) \qquad (2) $$

- similarly, use $S_{n+1}(x_n, x_{n+1})$ as gen. fct.

$$ {}^{-}p^{n} = -\frac{\partial S_{n+1}}{\partial x_n} \qquad (3) $$

- eom $\frac{\partial S_n}{\partial x_n} + \frac{\partial S_{n+1}}{\partial x_n} = 0 \Leftrightarrow$ *momentum matching* ${}^{+}p^{n} = {}^{-}p^{n}$



# Discrete Legendre transformations [Marsden, West '01; Dittrich, PH '11]

- in cont. $\mathbb{F}: T\mathcal{Q} \to T^*\mathcal{Q}, \quad (q,\dot{q}) \mapsto (q, \frac{\partial L(q,\dot{q})}{\partial \dot{q}})$
- Now in discrete: using action $S_n$, define <u>two</u> Legendre transf. at each $n$:

pre– and post–Legendre transf.

$$\begin{aligned}
\mathbb{F}^+ S_n : \mathcal{Q}_{n-1} \times \mathcal{Q}_n &\longrightarrow T^*\mathcal{Q}_n \\
(x_{n-1}, x_n) &\mapsto (x_n, {}^+p^n) = \left(x_n, \frac{\partial S_n}{\partial x_n}\right) \\
\mathbb{F}^- S_n : \mathcal{Q}_{n-1} \times \mathcal{Q}_n &\longrightarrow T^*\mathcal{Q}_{n-1} \\
(x_{n-1}, x_n) &\mapsto (x_{n-1}, {}^-p^{n-1}) = \left(x_{n-1}, -\frac{\partial S_n}{\partial x_{n-1}}\right)
\end{aligned}$$

- can show

$$\Omega_L^n = (\mathbb{F}^+)^*\omega^n = (\mathbb{F}^-)^*\omega^{n-1}$$

$\omega^n$: symplectic form on $T^*\mathcal{Q}_n$,

$\Omega_L^n = \frac{\partial^2 S_n}{\partial x_{n-1}\partial x_n} dx_{n-1} \wedge dx_n$: Lagrange 2–form on $\mathcal{Q}_{n-1} \times \mathcal{Q}_n$

# Constraints [Dittrich, PH '11]

- in cont. $\mathbb{F}$ not isomor. $\Leftrightarrow \det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0 \quad \Rightarrow$ get constraints
- Now in discrete have two Leg. transf.:

  $\mathbb{F}^{\pm} S_n$ fail to be isomorphisms $\Leftrightarrow \det\left(\frac{\partial^2 S_n}{\partial x^i_{n-1} \partial x^j_n}\right) = 0$
- Obtain two constraint surfaces:

  $\mathcal{C}^+_n := \mathrm{Im}(\mathbb{F}^+ S_n) \subset T^*\mathcal{Q}_n$: post–constraint surface

  $\mathcal{C}^-_{n-1} := \mathrm{Im}(\mathbb{F}^- S_n) \subset T^*\mathcal{Q}_{n-1}$: pre–constraint surface
- clearly, $\dim \mathcal{C}^+_n = \dim \mathcal{C}^-_{n-1}$

# Constraints and free parameters [Dittrich, PH '11]

- time evol. map only defined between constraint surfaces $\mathcal{H}_n : \mathcal{C}^-_{n-1} \to \mathcal{C}^+_n$

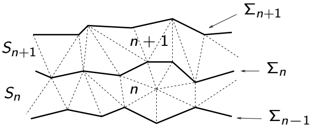
$$ {}^-p^{n-1} := -\frac{\partial S_n(x_{n-1}, x_n)}{\partial x_{n-1}} \quad , \quad {}^+p^n := \frac{\partial S_n(x_{n-1}, x_n)}{\partial x_n} \tag{4} $$

- Theorem: sympl. form restricted to constraint surfaces preserved under $\mathcal{H}_n$
- impl. fct. thm.: $x_n = x_n(x_{n-1}, p^{n-1}, \lambda^m_{n-1})$,

  $\lambda^m$: *a priori* free parameter $\Rightarrow$ arbitrariness in evol. as in continuum

  ($\lambda^m$ correspond to degenerate directions of sympl. form restricted to constraint surface)

# Canonical momenta in Regge Calculus




choose fat slicing, count fat slices by $n$, elementary moves by $k$

$l_n^e$: lengths of edges in $\Sigma_n$, $l_n^i$: lengths of edges between $\Sigma_n$ and $\Sigma_{n-1}$

- $S_n(l_n^e, l_n^i, l_{n-1}^{e'})$ as 'generating function' [Gambini, Pullin '03; Dittrich, Bahr '09; Dittrich, PH '09, '11] for evolution by fat slices

$$^{+}p_e^n := \frac{\partial S_n}{\partial l_n^e} \qquad {}^{-}p_e^{n-1} := -\frac{\partial S_n}{\partial l_{n-1}^e} \tag{5}$$

$$^{+}p_i^n := \frac{\partial S_n}{\partial l_n^i} \qquad {}^{-}p_i^{n-1} := -\frac{\partial S_n}{\partial l_{n-1}^i} = 0 \tag{6}$$

- $p_e \propto \psi_h$: "extrinsic curvature of $\Sigma_n$", $\{l_n^e\}$: "metric on $\Sigma_n$"

# Canonical momenta in Regge Calculus

$$
\begin{aligned}
{}^{+}p^{n}_{e} &:= \frac{\partial S_n}{\partial l^{e}_{n}} & {}^{-}p^{n-1}_{e} &:= -\frac{\partial S_n}{\partial l^{e}_{n-1}} \qquad (7)\\
{}^{+}p^{n}_{i} &:= \frac{\partial S_n}{\partial l^{i}_{n}} & {}^{-}p^{n-1}_{i} &:= -\frac{\partial S_n}{\partial l^{i}_{n-1}} = 0 \qquad (8)
\end{aligned}
$$

- similarly, ${}^{-}p^{n}_{e} = -\frac{\partial S_{n+1}}{\partial l^{e}_{n}}$
- eom $\frac{\partial S_n}{\partial l^{e}_{n}} + \frac{\partial S_{n+1}}{\partial l^{e}_{n}} = 0 \Leftrightarrow$ *momentum matching* ${}^{+}p^{n}_{e} = {}^{-}p^{n}_{e}$
- likewise, for internal variables $l^i$, eom $\frac{\partial S}{\partial l^i} = 0 \Leftrightarrow p_i = 0$

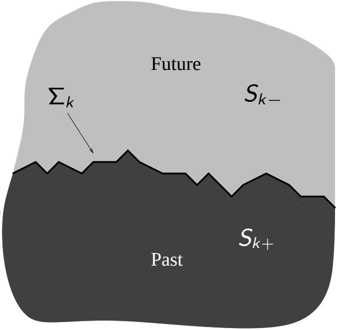
  constraints as equations of motion

# Momentum updating [Dittrich, PH '11]

- return to Pachner moves, counted by $k$ and assume $\Sigma_k = \Sigma_n$
- define 'past' action $S_{k+} = \sum_{k'=1}^{k} S_{\sigma_{k'}}$ ($S_{\sigma_{k'}}$ action of $k'$-th simplex)

  $\Rightarrow$ at each $k$ previous momenta translate into

$$p_e^k = \frac{\partial S_{k+}}{\partial l_k^e} \tag{9}$$



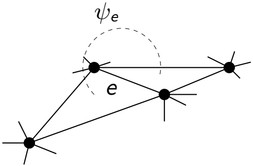

- action additive $\Rightarrow$ Pachner moves require *momentum updating*

$$p_e^{k+1} = p_e^k + \frac{\partial S_{\sigma_{k+1}}}{\partial l_k^e} \tag{10}$$

# Momentum updating in 3D

- in 3D on solutions $S_{Regge} = \sum_{e \subset \partial T} l_e \psi_e$, hence $p_e = \psi_e$

  $\Rightarrow$ "mom. up. = angle up."



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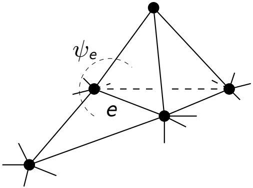
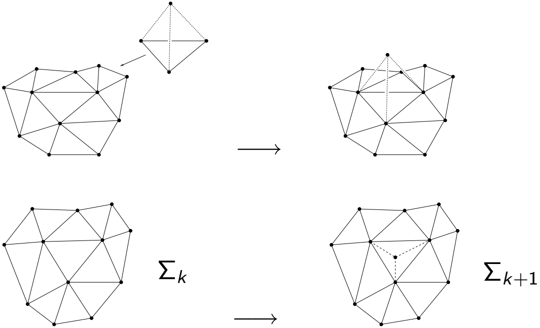


# Recall ‘problems’

3D perspective:

2D perspective:

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(a) subsets of variables coincide at different steps, i.e. $\Sigma_{k+1} \cap \Sigma_k \neq \emptyset$

(b) numbers of variables differ (phase space dim. varies) from step to step

# Pachner moves as ‘canonical transformations’

Solve ‘problems’

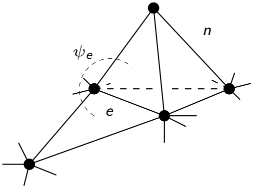
- solve ‘problem’ (a) by *momentum updating*: for all edges occurring in both $\Sigma_k$ and $\Sigma_{k+1}$

$$l^e_{k+1} = l^e_k \qquad p^{k+1}_e = p^k_e + \frac{\partial S_\sigma}{\partial l^e_k}$$

- solve ‘problem’ (b) by phase space extension: ‘add’ variables $l^n_k, l^o_k$ of edges occurring only ‘to the future’ or only ‘to the past’ of hypersurface $\Sigma_k \Rightarrow$ eoms require constraints $p^k_n = 0 = p^k_o$

- then Pachner moves implemented as ‘canonical transformation’ [Dittrich, PH ‘11]

# Example in 3D: the 1–3 Pachner move

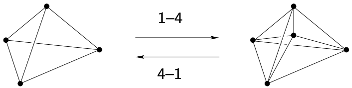



- solve 'problem' (b): extend phase space at step $k$, add $(l_k^n, p_n^k)$
- use $S_\tau(l_{k+1}^n, ...)$ as type 1 generating function (trivial dependence on $l_k^n$)

$$p_n^k = -\frac{\partial S_\tau}{\partial l_k^n} = 0 \quad , \qquad p_n^{k+1} = \frac{\partial S_\tau}{\partial l_{k+1}^n} = p_n^k + \frac{\partial S_\tau}{\partial l_{k+1}^n} \tag{11}$$

- want generating function for evolution by Pachner moves

# Example in 4D: the 1–4 Pachner move




1–4 move introduces new vertex and 4 new edges $n$, generating function

$$G_{1-4}(l^b_{k+1}, p^k_b;\ l^e_{k+1}, p^k_e;\ l^n_k, l^n_{k+1}) = \sum_b l^b_{k+1} p^k_b + \sum_e l^e_{k+1} p^k_e \ + \ S_\sigma(l^e_{k+1}, l^n_{k+1}) \quad (12)$$

yields evol. eqns. ($b$ 'passive edges' in $\Sigma_k$, $e$ edges in $\Sigma_k \cap \Sigma_{k+1}$)

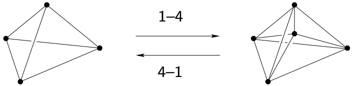
$$l^b_k = \frac{\partial G_{1-4}}{\partial p^k_b} = l^b_{k+1} \quad , \qquad p^{k+1}_b = \frac{\partial G_{1-4}}{\partial l^b_{k+1}} = p^k_b \quad , \qquad (13)$$

$$l^e_k = \frac{\partial G_{1-4}}{\partial p^k_e} = l^e_{k+1} \quad , \qquad p^{k+1}_e = \frac{\partial G_{1-4}}{\partial l^e_{k+1}} = p^k_e + \frac{\partial S_\sigma}{\partial l^e_{k+1}} \qquad (14)$$

$$p^k_n = -\frac{\partial G_{1-4}}{\partial l^n_k} = 0 \quad , \qquad p^{k+1}_n = \frac{\partial G_{1-4}}{\partial l^n_{k+1}} = \frac{\partial S_\sigma}{\partial l^n_{k+1}} \qquad (15)$$

- post–constraint $p^{k+1}_n = \frac{\partial S_\sigma}{\partial l^n_{k+1}}$, no eom, *a priori* free parameters $l^n_k, l^n_{k+1}$, i.e. here $\lambda^n = l^n_{k+1}$, 'gauge choice' $l^n_k = l^n_{k+1}$ and $l^n_k$ as additional initial data

# Synopsis: Pachner moves for 4D simplicial gravity




- 1–4 move: introduces 4 new edges, momenta satisfy $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraints)

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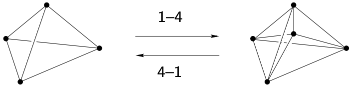


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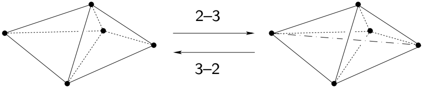


- 2–3 move: introduces 1 edge, renders 1 triangle internal, no new internal edge $\Rightarrow$ freely choosable curvature generated, new momentum $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraint)
  - all new edges can be *a priori* freely chosen, but conjugate momenta constrained by post-constraints

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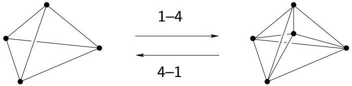
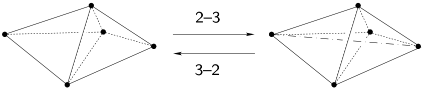



- 1–4 move: introduces 4 new edges, momenta satisfy $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraints)
- 4–1 move: removes 4 old edges, $C_o^k = p_o^k + \frac{\partial S_\sigma}{\partial l_k^o} = 0$ (pre-constraints), $C_o^k = 0$ equiv. to eom for $l_k^o$

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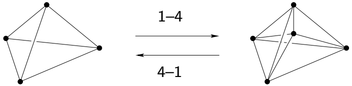


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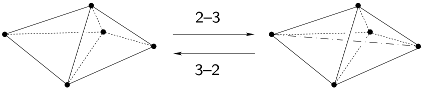



- 2–3 move: introduces 1 edge, renders 1 triangle internal, no new internal edge ⇒ freely choosable curvature generated, new momentum $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraint)
- 3–2 move: removes 1 old edge, $C_o^k = p_o^k + \frac{\partial S_\sigma}{\partial l_k^o} = 0$ (pre-constraint)

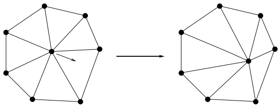
# Constraint business

- key difference 3D vs. 4D:

  in 3D constraints preserved under Pachner moves and symmetries unbroken [Dittrich, PH '11]

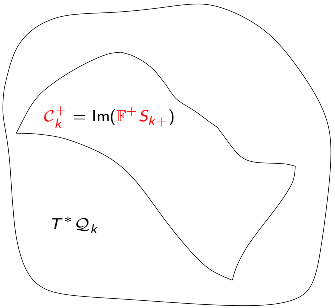
  in 4D constraints not necessarily preserved

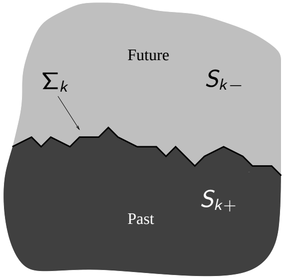
- post-constraints *a priori* do *not* generate gauge transformations of the action (vertex displacement), despite *a priori* forming an abelian Poisson-algebra $\{C_n^+, C_{n'}^+\} = 0$



- reflect lack of information in hypersurface about full 4D-Regge triangulation $\Rightarrow$ non-uniqueness of solutions given initial data

# Constraint (un-)matching




- *a posteriori* 1st or 2nd class nature depends on pre-constraints of 3–2 and 4–1 moves:
  if no complete *constraint matching*, i.e. $\mathcal{C}_k^+ \neq \mathcal{C}_k^-$, pre-constraints may *a posteriori* fix free lengths of 1–4 and 2–3 moves [Dittrich, PH '11 and to appear]
- if some lengths remain free, obtain proper gauge transformations (in general, gauge symmetry broken in presence of curvature [Rocek, Williams '84; Bahr, Dittrich '09; Dittrich, PH '09; etc.])

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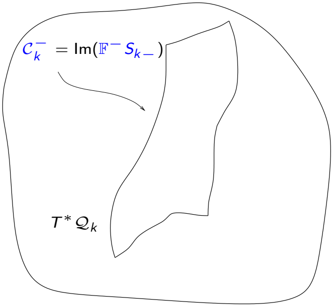


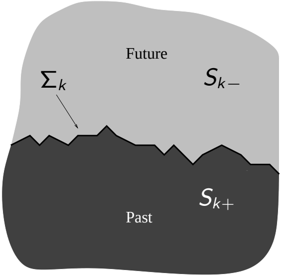


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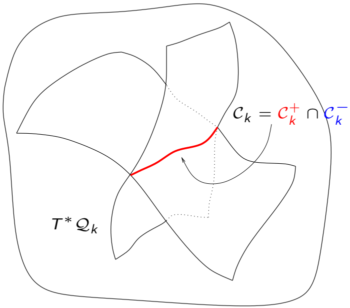


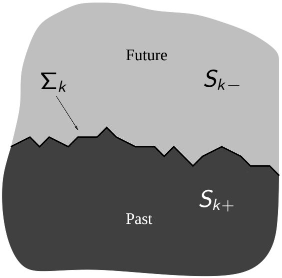


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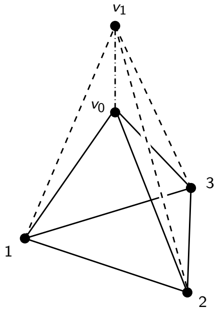
# Example: constraint (un-)matching

- Example: $\Sigma_0$ bdry surface of single 4–simplex, perform

1. 1–4 move: introduces 4 *a priori* free lengths
2. 2–3 move: introduces 1 *a priori* free length/curvature ⇒ so far 5 free parameters
3. 3–2 move: annihilates 1 edge, 1 pre–constraint
4. 4–1 move: annihilates 4 edges and $v_0$, 4 pre–constraints




- after 3–2 move: triangulation of *stacked sphere* ⇒ must be flat ⇒ 3–2 move imposes flatness ⇒ 4 free parameters remain
- after 4–1 move: still *stacked sphere* ⇒ 4 pre–constraints automatically satisfied
- 4 free parameters survive ⇒ fourfold gauge freedom at vertex $v_0$

# Conclusions

- devised general canonical framework for discrete systems (not only gravity), can cope with varying phase space dim.
- equivalent to covariant formalism
- can apply to simplicial gravity: implement general discrete time evolution scheme not generated by constraints
- most elementary evolution steps are Pachner moves
- need complete constraint classification to identify evolving DoFs [to appear]
- application to 4D linearized theory (graviton dynamics) [to appear]

# Quantization

- action as generating function $\Rightarrow$ direct connection between canonical framework and path integral
- heuristic idea: given wave function at $k$ $\psi(l_k)$, obtain wave function at $(k+1)$ by

$$“\quad \psi(l_{k+1}) = \int dl_o Exp(iS_\sigma)\psi(l_k) \quad ”$$

$S_\sigma$: action of glued simplex, $l_o$: edges going into the bulk, $dl_o$: some integration measure

- quantum version of momentum updating
- but now implement evolv. Hilbert spaces
- in Regge sector connect to recent developments on linking cov. and can. quantizations [Alesci,Bonzom,Freidel,Livine,Thiemann, Zipfel,...]

# Outlook: Hamiltonian dynamics

- Alesci–Rovelli Hamiltonian generates 1–4 move [Alesci, Rovelli '10]
- however, consider also 2–3, 3–2 and 4–1 moves in order to get interesting dynamics
- discrete time evolution or is the Hamiltonian really a constraint?