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FIELD THEORETICAL ASPECTS OF GFT:

SYMMETRIES AND REPRESENTATIONS

GROUP FIELD THEORY

Group field theory is formulated as statistical field theory

$$Z\,[J] = \int \mathrm{D}\phi \; e^{-S[\phi]+J\cdot\phi}$$

S - non-local GFT action

ϕ - functions from $G^{\times n} \to \mathbb{C}$ usually with gauge invariance

$$\phi(g_1, \cdots, g_n) = \phi(g_1 h, \cdots, g_n h) \qquad \forall h \in G$$

n - dimension of the theory

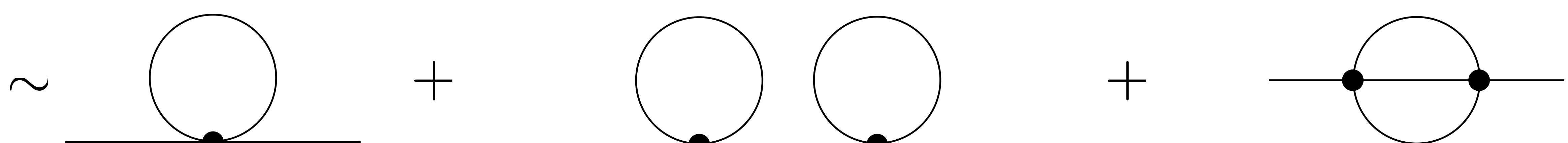
In all following examples I will assume

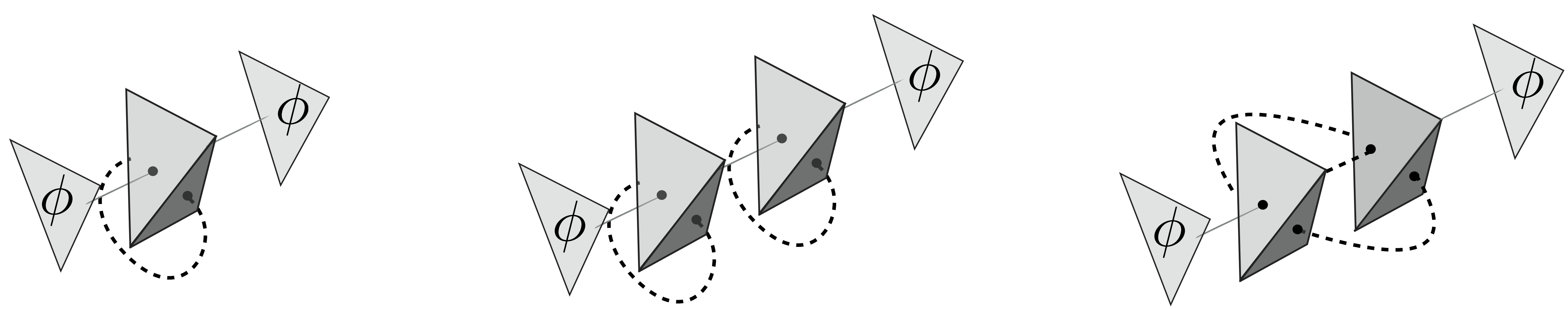
$$n = 3 \qquad\qquad \phi(g_1, g_2, g_3) = \phi(g_1 h, g_2 h, g_3 h)$$

GROUP FIELD THEORY & SPIN FOAMS

Perturbative expansion of correlation functions in GFT becomes a sum over spin foam amplitudes with boundaries

$$\langle\phi\phi\rangle = \int \mathrm{D}\phi\ e^{-(\phi K \phi + V_{\mathrm{int}})}\phi\phi = \sum_n \frac{1}{n!} \int \mathrm{D}\phi\ e^{-(\phi K \phi)}\ V_{\mathrm{int}}^n \phi\phi$$





GROUP FIELD THEORY & SPIN FOAMS

Group field theory provides a prescription for summing over two complexes of spin foam models.

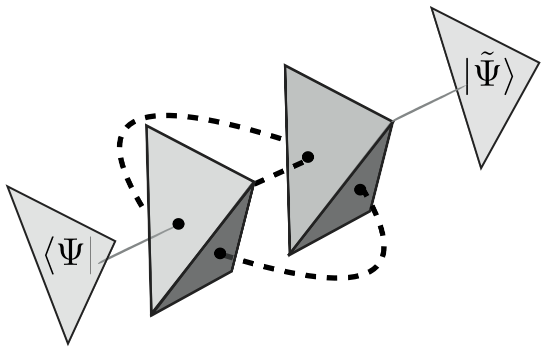
$$Z = \int \mathrm{D}\phi \; e^{-S[\phi]} = \sum_{\Gamma} \frac{\lambda_\Gamma}{n!} \mathcal{A}_\Gamma$$

$\mathcal{A}_\Gamma$- spin foam amplitude on the two complex Γ

λ_Γ - symmetry factor of the Feynman graph

GROUP FIELD THEORY & LQG

Boundaries of the spin foam amplitudes can be interpreted as states on a Hilbert space crated by creation and annihilation operators acting on the Fock vacuum



With field operators satisfying the usual CCR

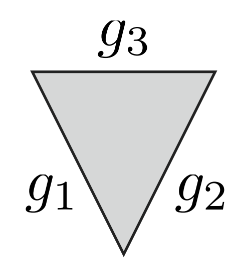
$$[\phi(g), \phi^\dagger(\tilde{g})] = \delta(g\tilde{g}^{-1})$$

The Fock vacuum

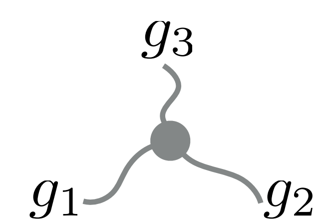
$$\phi|\emptyset\rangle = 0$$

and creation operators creating a triangle labelled by group elements

$$\phi^\dagger(g_1, g_2, g_3)|\emptyset\rangle = |g_1, g_2, g_3\rangle \sim$$

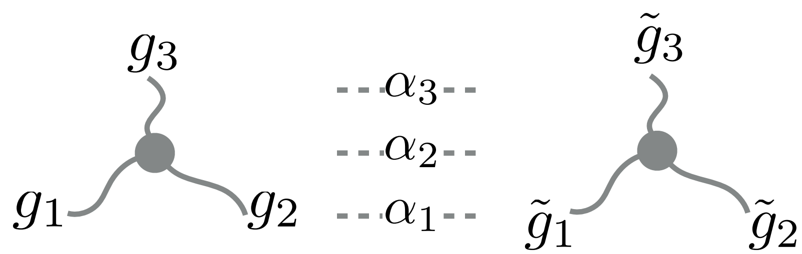


$\sim$

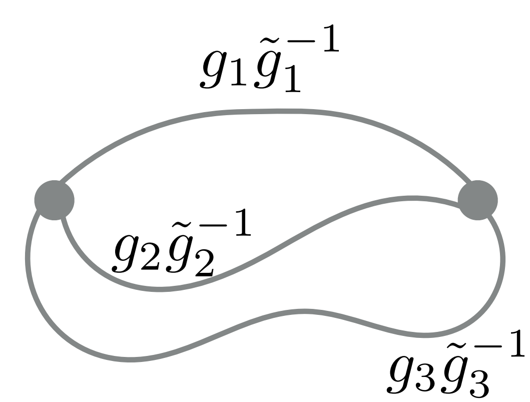


We can glue vertices together to obtain spin networks [Oriti, 16]

$$\int_G \mathrm{d}\alpha$$



$\Rightarrow$



GROUP FIELD THEORY

Group field theory can be seen as:

1. A field theory of space time
2. A field theoretical formulation of spin foam models
3. A many body formulation of Loop Quantum Gravity

Investigating the field theoretical structure of the frame work is beneficial for any of the above formalisms.

Part I: Symmetries of GFT

Part II: Representations of GFT

SYMMETRIES OF GFT [AK, Oriti, 15, 16]

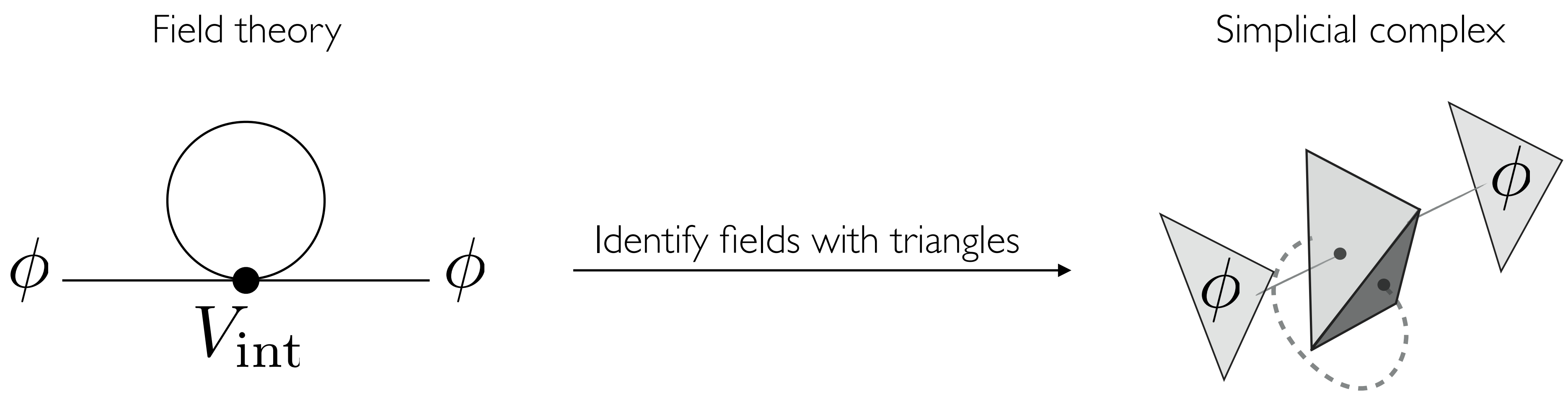
SYMMETRIES OF GFT

Classical symmetries in quantum field theory provide a powerful tool for the analysis of non-perturbative effects. In particular:

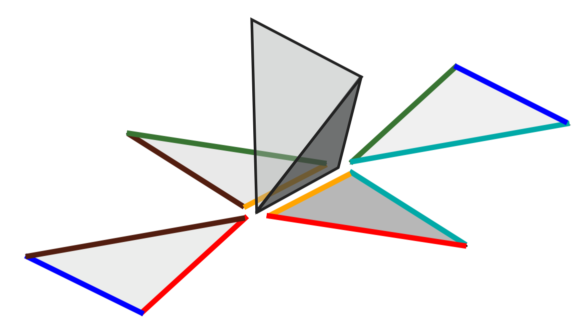
1. give mathematical tools to solve equations of motion
2. explain and describe conserved quantities
3. become non-trivial relations between correlation function in the absence of anomalies

Symmetries are very well understood for local field theories. But in GFT the action is **non-local.**

NON-LOCALITY IN GFT



Fields of GFT are associated with triangles. In order to produce a tetrahedron with four triangles we need to glue them in a non-local way.
Where non-locality means that the fields entering the interaction are evaluated at different points of the group.



A particular example is the interaction for the Ponzano-Regge spin foam model:

$$V_{\mathrm{int}} = \int \mathrm{d}g_{\mathrm{All}}\ \phi(g_1, g_2, g_3)\phi(g_1, g_4, g_5)\phi(g_6, g_2, g_5)\phi(g_6, g_4, g_3)$$

NON-LOCALITY IN GFT

For non-local actions we need to understand:

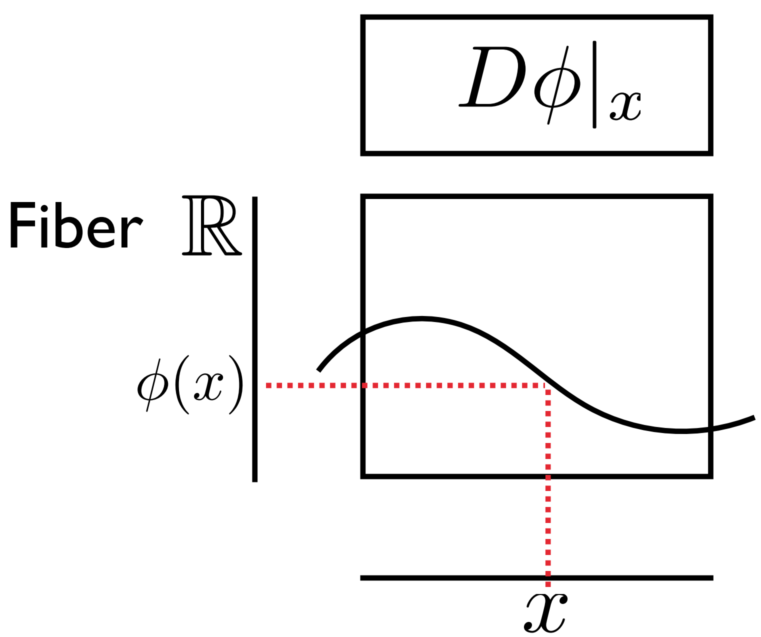
1. How to define a continuous symmetry ?
2. What are the consequences of continuous symmetries - Noether Theorem ?
3. Can we characterise continuous symmetries for a given action ?

HOW TO DEFINE A SYMMETRY

In local field theories the action is an integral of a Lagrangian

$$S[\phi] = \int_\Omega \mathrm{d}x \; L(x, \phi(x), D\phi|_x) = \int_\Omega L(j)$$

where the Lagrangian is a function on a jet bundle.



J Jet space - derivatives of fields at a point

E Vector bundle - locally $E \simeq M \times \mathbb{R}$

ϕ Fields - sections in the vector bundle

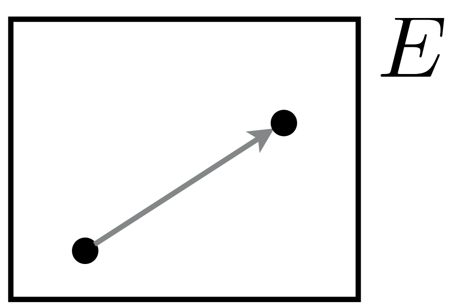
M Base manifold

A point of a jet bundle is specified in coordinates as follows

$$j = (x, \phi(x), D\phi|_x)$$

HOW TO DEFINE A SYMMETRY

The simplest notion of a symmetry in local field theory is the **point** symmetry.



A point symmetry is a diffeomorphism on the vector bundle that leaves the action invariant for any integral domain Ω *and any field configuration* ϕ .

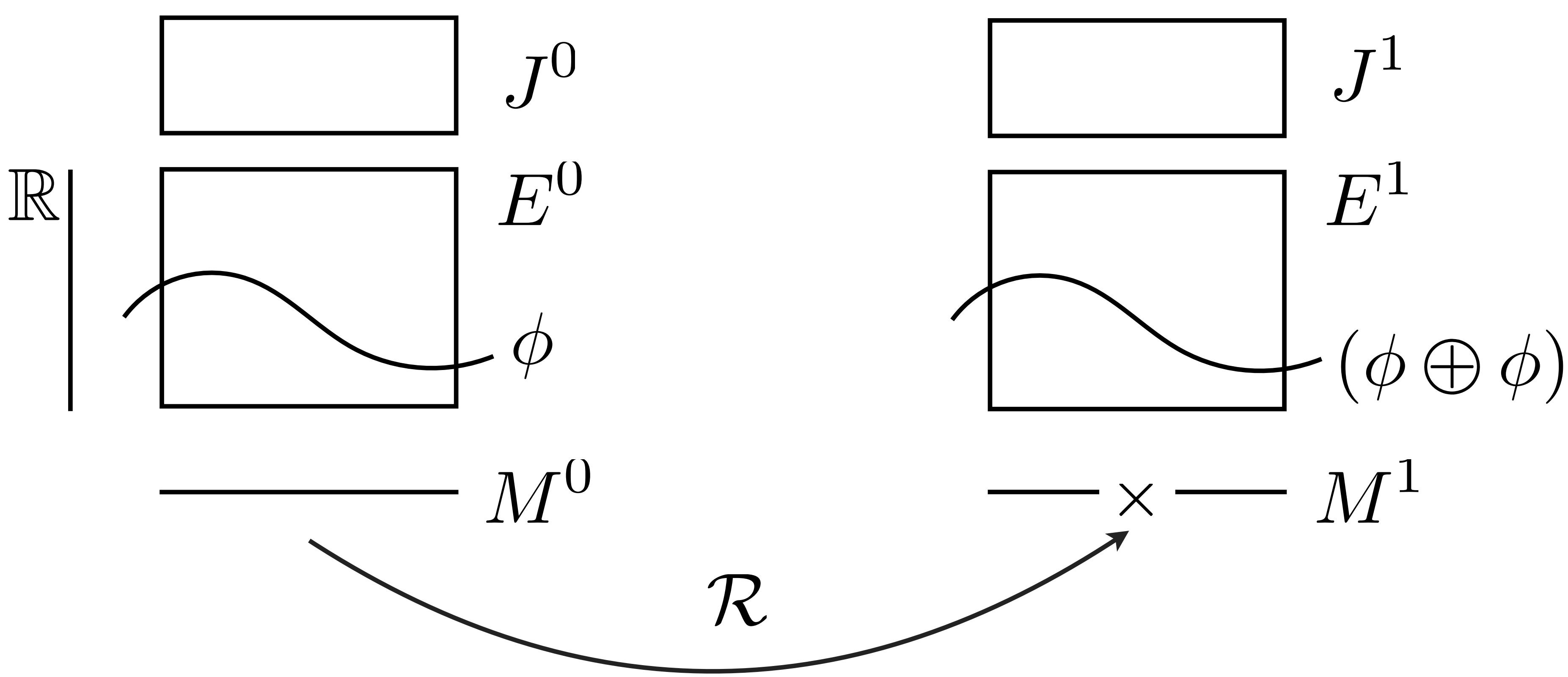
$$S[\phi] = \int_{\Omega} L(j) = \int_{\tilde{\Omega}} L(\tilde{j}) = \tilde{S}[\tilde{\phi}]$$

HOW TO DEFINE A SYMMETRY

In the non-local theory we define the action as a sum of local integrals on different vector bundles, and relations between the bundles.

$$\int_{\mathbb{R}} \phi(x)\Delta\phi(x) \qquad\qquad \int_{\mathbb{R}\times\mathbb{R}} \phi(x)V(x-y)\phi(y)$$

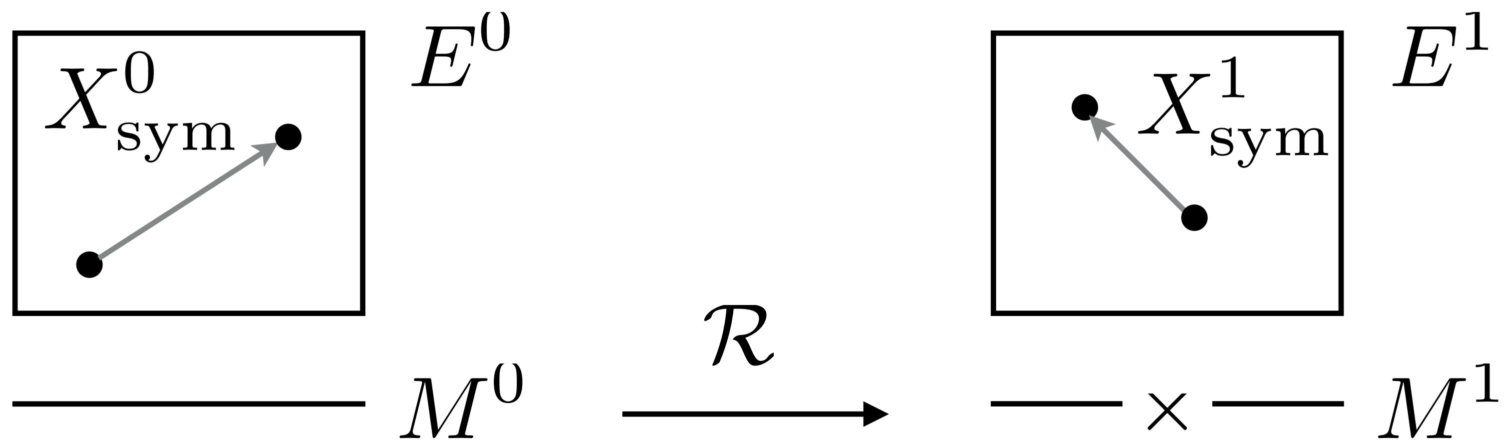
$$S[\phi] = S^0[\phi] + S^1[\phi] + \cdots$$



HOW TO DEFINE A SYMMETRY

We define a point symmetry for a non-local action as a set of compatible vector fields on each vector bundle, such that every integral in the sum is invariant in the local sense under the flow of the corresponding vector field.

$$S[\phi] = \quad S^0[\phi] \quad + \quad S^1[\phi] \quad + \quad \cdots$$



With the compatibility condition

$$\mathcal{R}\left(X^0_{\mathrm{sym}}\right) = X^1_{\mathrm{sym}}$$

NOETHER THEOREM

Every point symmetry of a non-local action gives rise to a continuity equation with „sources". *[AK, Oriti, 15]*

On solutions of equations of motion we obtain generalised Noether currents J with „sources"

$$\mathrm{div}(J) = \Delta_{\mathrm{NL}}$$

With

$$J(g) = J_{\mathrm{N}}(g) + J_{\mathrm{NL}}(g) \qquad\qquad \Delta_{\mathrm{NL}}(g) \sim \int \frac{\delta L_{NL}}{\delta \phi(g)}\, X^0$$

For local theories J becomes the usual Noether current and the „source" vanishes.

SYMMETRIES OF GFT

Can we characterise continuous point symmetries for a given action ? **Yes**. [AK, Oriti, 16]

Model	Symmetry Group	Action
$\phi^1_{1,2,3}\phi^2_{1,4,5}\phi^3_{6,2,5}\phi^4_{6,4,3}$	$G^{\times 3} \times U(1)^{\times 3}$	$\vec{g} \mapsto L_C(\vec{g})$ $\phi^c \mapsto e^{\imath\theta_c}\phi^c \quad \sum_c \theta^c = 0$ with $C = \left(c^1, c^2, c^3\right)$
$\phi_{1,2,3}\,\phi_{3,4,5}\,\phi_{5,2,6}\,\phi_{6,4,1}$	$G^{\times 2}$	$\vec{g} \mapsto L_C(\vec{g})$ with $C = \left(c^1, c^2, c^1\right)$
$\phi^P_{1,2,3}\,\phi^P_{1,4,5}\,\phi^P_{2,5,6}\,\phi^P_{3,6,4}$	G	$\vec{g} \mapsto L_C(\vec{g})$ with $C = (c, c, c)$
$\phi_{1,2,3}\,\bar{\phi}_{3,4,5}\,\phi_{5,2,6}\,\bar{\phi}_{6,4,1}$	$G^{\times 2} \times U(1)$	$\vec{g} \mapsto L_C(\vec{g})$ $\phi \mapsto e^{\imath\theta}\phi$ with $C = \left(c^1, c^2, c^1\right)$
$\phi_{1,2,3,4}\,\phi_{4,5,6,7}\,\phi_{7,3,8,9}\,\phi_{9,6,2,10}\,\phi_{10,8,5,1}$	$SU(2)^{\times 2}$	$\vec{g} \mapsto L_C(\vec{g})$ with $C = \left(c^1, c^2, c^2, c^1\right)$
Barrett-Crane	$Spin(4)^{\times 2} \times SU(2)$	$\vec{g} \mapsto L_S(\vec{g})$ $\vec{g} \mapsto c \cdot (\vec{g}; k) \cdot c^{-1}$ with $S = \left(s^1, s^2, s^2, s^1\right)$

APPLICATIONS TO GFT

1. We defined the notion of a continuous symmetry.
2. We generalised the Noether theorem and obtain relations between symmetries and continuity equations.
3. We provided a prescription to characterise continuous symmetries for a given action.

Possible application and future work:

1. Apply the symmetries of GFT models to correlation function and spin foam amplitudes
2. Reduce and understand recurrence relations of spin foam models and recoupling theory from the field theoretical symmetries [Baratin, Girelli, Oriti, 11]
3. Better understand four dimensional spin foam models with Immirzi parameter
4. Understand the implications of the continuity equation for the cosmological calculations of GFT models [Gielen, Sindoni, Oriti, 13; Wilson-Ewing, Sindoni, Oriti, 16]

REPRESENTATIONS OF GFT [AK, Oriti, Tomlin, t.a.]

ALGEBRA OF OBSERVABLES

Goal: To construct the algebra of observables of GFT and to find its irreducible representations as bounded linear operators on a Hilbert space.

1. A concrete realisations of the algebra as operators on a Hilbert space is the usual canonical description of QFT.
2. Inequivalent representations of QFT are closely related to non-perturbative effects of QFT such as phase transitions and symmetry breaking.

ALGEBRA OF OBSERVABLES

We construct the Weyl algebra for GFT from the CCR algebra

$$[\phi(g), \phi^\dagger(\tilde{g})] = \delta(g\tilde{g}^{-1})$$

We smear the operators with smooth, square integrable functions on $G^{\times 3}$

$$\phi(f) := \int \mathrm{d}\mu\ \phi(g)\bar{f}(g) \qquad \phi^\dagger(f) := \int \mathrm{d}\mu\ \phi^\dagger(g) f(g)$$

And define the Weyl operators as

$$W(f) = e^{\imath\left(\phi(f)+\phi^\dagger(f)\right)}$$

The Weyl algebra is a $C^\star$-algebra with identity labeled by the smearing functions

$$W(f)W(g) = W(f+g)\ e^{-\imath\Im(f,g)}$$
$$W^\star(f) = W(-f)$$
$$W(0) = 1$$

ALGEBRA OF OBSERVABLES

By construction one possible representation is the Fock representation

$$W_F(f) = e^{\imath\left(\phi_F(f)+\phi_F^\dagger(f)\right)} \qquad\qquad \phi_F(f)|\varnothing\rangle = 0 \qquad \forall f \in \mathcal{S}$$

In this representation we have a notion of a particle created by $\phi^\dagger(f)$ and a notion of the number of particles on the dense domain of the Hilbert space.

$$N = \sum_i \phi_F^\dagger(f_i)\phi_F(f_i) \qquad\qquad \langle\Psi|N|\Psi\rangle < \infty$$

This is the representation in which we represented the boundaries of the spin foam.

Are there different representations obtained as limits from the Fock representation ?

ALGEBRA OF OBSERVABLES

We begin with a Fock representation of the Weyl algebra in the finite (regularised) volume $V \subset G^{\times n}$.

And construct an *N*-particle state by creation operators for $f, \alpha \in L^2(V)$ as

$$|N, f\rangle := \frac{1}{\sqrt{N!}}\ \phi_F^\dagger(f)^N |\emptyset\rangle$$

or

$$|\alpha\rangle := W_F(\alpha)|\emptyset\rangle$$

$$\langle N, f|\phi_F(g)|N, f\rangle = 0$$

$$\phi_F(f)|\alpha\rangle = (f|\alpha)_{L^2}\ |\alpha\rangle$$

$$\langle\alpha|\phi_F(f)|\alpha\rangle = (f|\alpha)_{L^2}$$

With particle numbers

$$\langle N\rangle = N$$

and

$$\langle N\rangle = \int_V |\alpha(g)|^2$$

We then remove the regulator by taking the limits

$$\lim_{N\to\infty} \qquad \lim_{V\to\infty} \qquad \lim_{\substack{N\to\infty\\ V\to\infty\\ \frac{N}{V}=const}}$$

REPRESENTATIONS OF GFT

On a non-compact group G we obtain

Translation invariant

	$\lim\limits_{N\to\infty}$	$\lim\limits_{V\to\infty}$	$\lim\limits_{\substack{N\to\infty \\ V\to\infty \\ \frac{N}{V}=const}}$
$\|N, V^{-1}\rangle$	Mixture of Fock	Fock	Mixture of **in**equivalent condensate reps.
$\|\alpha = const.\rangle$	-	Fock	Condensate rep.
$\|N, f \in L^2(G)\rangle$	Mixture of Fock	Fock	Mixture of Fock
$\|\alpha \in L^2(G)\rangle$	-	Fock	Fock

SPONTANEOUS SYMMETRY BREAKING

Mixture of inequivalent representation is a necessary condition for spontaneous symmetry breaking.

If the representations are connected by a symmetry, then this symmetry is broken due to the in-equivalence of representations

$$\mathcal{H} \xrightarrow[\substack{N, V \to \infty \\ T \to \infty}]{\text{Physical process}} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_n \end{matrix} \quad \text{symmetry}$$

Ising model
Mexican hat
BEC condensation
$\vdots$

A. In order to get inequivalent representations we need translation invariance. This is **not** the symmetry which is broken [Wreszinski, Strocchi].

B. The broken symmetry is the one that connects the inequivalent representations.

SUMMARY

1. We defined a notion of symmetries of a non-local action
2. We derived generalised conservation laws from classical symmetries
3. In this frame work we were able to characterise symmetries of non-local GFT actions

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4. Constructed inequivalent representations in GFT by limits from the Fock representation
5. Obtained inequivalent representations

OUTLOOK

1. Generalise the notion of the classical symmetries to obtain more, non-standard symmetries in GFT [AK,Oriti, Thieme]

2. Characterise necessary conditions for spontaneous symmetry breaking [AK,Oriti]

3. Investigate the anomalies of classical symmetries

4. Characterise the ground state of GFT by its symmetries in the KMS sense [Kotecha, Oriti]

5. Perform the same limit for a more complicated type of states with connected vertices

6. Construct the thermodynamic limit on the compact manifold [AK, Oriti]

7. Can a non-local interaction be represented on a Fock space - Haags theorem ?

8. Can we adopt the Spin-Statistics theorem to GFT and its symmetries

THANK YOU !