Non-Metric Gravity

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Motivations

Using “polynomial” formulation(s), develop a new perspective on (non)-renormalizability of perturbative quantum gravity, as well as on “quantum geometry” at Planck scales.

Outline: Quantum Theory

We will work with Plebanski formulation of GR:

- Jerzy Plebanski, "On the separation of Einsteinian substructures", *J. Math. Phys.* **Vol. 18** 2511-2520 (1977).

Dimensional analysis: the action is incomplete, counterterms (quantum corrections) must be added.

Detailed analysis: the possible counterterms are of a very special type, can be combined into

$$\phi(\mathrm{Tr}(\Psi)^2, \mathrm{Tr}(\Psi)^3). \tag{1}$$

Makes the behavior of gravity under renormalization much more transparent than in the usual metric-based treatment: renormalization group flow is that in the space of ϕ. Complete understanding may be within reach, e.g. testing of the asymptotic safety conjecture of Weinberg.

Outline: Modified Gravity

Quantum corrected theory is a modified gravity theory of a new type. Second derivatives only!

Metric only emerges up to a conformal factor: non-metric gravity.

Modifications become important when curvatures are large.

Spherically-symmetric solution: avoidance of the spacetime singularity inside the black hole.

Part I: Polynomial formulations of gravity

Einstein formulated GR as the theory of the metric of spacetime

$$S_{EH}[g] = \frac{1}{16\pi G}\int_M d^4x\sqrt{-\det(g)}(R-2\Lambda). \qquad (2)$$

The Newton constant G is absorbed into the fluctuating part of the metric field, giving it the mass dimension 1. Coupling constant $\sqrt{G}$, mass dimension $[\sqrt{G}] = -1$: non-renormalizable! Hence, do not know its UV completion.

S-matrix is finite at one-loop (one-loop divergences become zero on-shell), but is divergent at two loops.

First order formalism

Einstein gravity as the second-order formalism: second derivatives in the action. First order formalism available. In its Einstein-Cartan form:

$$S[\theta,\omega] = \frac{1}{8\pi G}\int_M \epsilon^{IJKL}\theta^I \wedge \theta^J \wedge F_\omega^{KL} + \frac{\Lambda}{2}\epsilon^{IJKL}\theta^I \wedge \theta^J \wedge \theta^K \wedge \theta^L. \qquad (3)$$

Here θ^I are the frame field one-forms, $F_\omega = d\omega + (1/2)\omega \wedge \omega$ is the curvature of the spin connection, indices $I,\ldots,K = 0,\ldots,3$ are the internal ones, and ϵ^{IJKL} is the totally anti-symmetric tensor in the internal space.

Newton's constant

Newton's constant can be absorbed into the fields so that there is no dimensionfull coupling constant left: $\theta/\sqrt{G} = \tilde{\theta}, \Lambda G = \tilde{\Lambda}$. New mass dimensions:

$$[\tilde{\theta}] = 1, \quad [\omega] = 1, \quad [\tilde{\Lambda}] = 0. \tag{4}$$

Note: the presence of G in the usual metric-based perturbation theory (with background $\eta_{\mu\nu}$) is due to the split $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}h_{\mu\nu}$. No Newton's constant in pure gravity unless there is a background.

Quantization?

Starting point for quantization. Works in 3D, where gravity is shown to be (super) renormalizable in this formulation.

Does not work in higher D: no kinetic term.

Plebanski formulation

There is now a “kinetic term”.

Separation of metric, connection A and curvature Ψ from each other.

We will use the original self-dual version (version without the self-dual split is available).

$$S[B, A, \Psi] = \frac{1}{2\pi i G} \int_M B^a F_A^a + \frac{1}{2}(\Lambda \delta^{ab} + \Psi^{ab}) B^a B^b. \tag{5}$$

Here a, b are the $\mathfrak{su}(2)$ Lie algebra indices, Ψ^{ab} is a field that on-shell becomes the Weyl part of the curvature (it is required to be symmetric traceless), B^a is a Lie algebra valued 2-form field that on-shell becomes expressed through a tetrad.

Euler-Lagrange equations:

$$B^a B^b = \frac{1}{3}\delta^{ab}\delta^{cd}B^c B^d, \quad F_A^a = -(\Lambda\delta^{ab} + \Psi^{ab})B^b, \quad d_A B^a = 0. \tag{6}$$

First implies that B^a is the self-dual part of the two form $B^{IJ} := (1/2)\epsilon^{IJKL}\theta^{[K}\theta^{L]}$ for some tetrad θ^I, second and third identifies Ψ^{ab} as the self-dual part of the Weyl curvature tensor.

Again, no dimensionfull coupling after field rescalings. There is now a "kinetic term" for the fields, except for Ψ. Let us treat Ψ as an external field (i.e. postpone integration over Ψ).

Compare e.g. QED: can start with fermions in an external electromagnetic field

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu(\partial_\mu - ieA_\mu) + m)\psi. \tag{7}$$

Quantum corrections will generate the kinetic term for A: the usual F^2 Lagrangian. Let us see if anything like this happens in Plebanski theory.

Part II: Non-metric quantum gravity

The first step in analyzing the UV behavior is to produce a list of possible divergent terms (of mass dimension four compatible with symmetries).

After G is absorbed into the fields, the mass dimensions are:

$$[A] = 1, \quad [B] = 2, \quad [\Psi] = 0. \tag{8}$$

It is thus obvious that all powers of Ψ will appear.

Thus, in addition to the term $\Psi^{ab}B^aB^b$ need to add the terms of the form

$$\frac{1}{2}(\Psi^{k_1})^{ab}(\mathrm{Tr}(\Psi^2))^{k_2}\ldots(\mathrm{Tr}(\Psi^n))^{k_n}B^aB^b \tag{9}$$

The theory does seem to be as non-renormalizable as in the usual perturbative quantum gravity. Usual case: dimensionfull Newton's constant; our case - a field Ψ of mass dimension zero.

Other terms

Clear that all powers of Ψ will get generated also in front of BF and FF terms. Detailed analysis shows that no derivatives of Ψ appears in counterterms (in one loop). The most general renormalized action is:

$$i\mathcal{L} = \frac{1}{2}\bar{X}(\Psi)^{ab}F^a_A F^b_A + \bar{Y}(\Psi)^{ab}B^a F^b_A + \frac{1}{2}\bar{Z}(\Psi)^{ab}B^a B^b, \tag{10}$$

where $\bar{X}(\Psi), \bar{Y}(\Psi), \bar{Z}(\Psi)$ are all tensors, polynomials in Ψ and its traces. The coefficients of these polynomials are undetermined. Infinite number of them, seemingly no predictive power. Usual non-renormalizability? Not quite.

Field B redefinition

Can redefine the field $B \rightarrow B + H(\Psi)F(A)$ to get rid of the $F^a F^b$ term. Can then "rescale" the field B to map the $B^a F^b$ term into its canonical form. After this B field redefinitions one gets

$$i\mathcal{L} = \tilde{B}^a F_A^a + \frac{1}{2}\tilde{\Psi}(\Psi)^{ab}\tilde{B}^a\tilde{B}^b, \tag{11}$$

where

$$\tilde{\Psi}(\Psi) = (Y(\Psi)^T Z(\Psi)^{-1} Y(\Psi) - X(\Psi))^{-1}. \tag{12}$$

Field Ψ redefinition

The whole effect of the counterterms is to replace the curvature field Ψ^{ab} by a non-trivial, depending on many new parameters (coupling constants) functional $\tilde{\Psi}^{ab}(\Psi)$.

Rewrite:

$$\tilde{\Psi}(\Psi)^{ab} = \Phi^{ab}(\Psi) + \delta^{ab}\phi(\Psi), \tag{13}$$

where $\Phi^{ab}(\Psi)$ is the traceless part of $\tilde{\Psi}$. The field Φ^{ab} just replaces the original field Ψ^{ab} after the renormalization!

Renormalized action

The effect of counterterms is in replacing the bare curvature field Ψ by the renormalized one, and in appearance in the action of a new "trace" term:

$$\frac{1}{i}\int_M B^a F_A^a + \frac{1}{2}\left(\Lambda\delta^{ab} + \Psi^{ab} + \delta^{ab}\phi(\Psi)\right) B^a B^b, \tag{14}$$

Still non-renormalizable in the strict sense of the word (as still an infinite number of undetermined constants). No derivatives of Ψ, even after integration over Ψ.

More on the function ϕ

Being a scalar function, ϕ can only be a function of eigenvalues of Ψ^{ab}, of which there are two (traceless!)

$$\phi = \phi(\lambda_1, \lambda_2). \tag{15}$$

Equivalently, $\phi = \phi(\mathrm{Tr}(\Psi)^2, \mathrm{Tr}(\Psi)^3)$. Renormalization group flow is that in the space of such functions!

Part III: Non-metric gravity as a modified gravity theory

The metricity equations are modified to

$$B^a B^b + \frac{d\phi(\Psi)}{d\Psi_{ab}}(B^c B_c) = \frac{1}{3}\delta^{ab}(B^c B_c). \tag{16}$$

No longer implies that the two-form field B^a is metric (comes from tetrad). Non-metricity is unavoidable whenever there is non-zero "curvature" Ψ.

Non-metric gravity

As a detailed analysis shows, certain metric g can still be introduced (up to conformal rescaling). Equations of motion one gets are second order in derivatives relating g and derivatives of g with components of Ψ and derivatives of Ψ. Schematically

$$D^2 g + f(g,\Psi)\tilde{D}^2\Psi = \Psi + \phi(\Psi). \tag{17}$$

Can be solved for Ψ, generating an infinite series in derivatives of g.

Quantum modified Plebanski theory (="non-metric" gravity) equivalent to the usual quantum modified GR with an infinite number of higher derivative terms added to the action.

Comparing the two expansions

Expansion in powers of Ψ is not the same as the expansion in powers of $R(g)$! Even $\phi = \mathrm{Tr}(\Psi)^2$ gives an infinite expansion when interpreted in metric terms!

Good, because the theory modified by a finite number of $R(g)^n$ corrections is higher derivative. Plebanski formulation gives a way to keep equations second order.

Hamiltonian formulation

Ingemar Bengtsson, private communication. Hamiltonian constraint takes the form:

$$\mathcal{H}(EEE, EEF, EFF, FFF) \approx 0, \tag{18}$$

where $\mathcal{H}$ is a homogeneous function of order 1. Note, for comparison, that in the usual GR case $\mathcal{H} = EEF$ (Ashtekar).

Spherically-symmetric solution

Joint work with Yuri Shtanov.

Start with a spherically-symmetric ansatz for the two-form B^a. One finds that $\Psi^{ab} \sim \mathrm{diag}(1,1,-2)$ (like in the usual GR case), or

$$\Psi^{ABCD} = \beta i^{(A} i^{B} o^{C} o^{D)}. \tag{19}$$

Then $\phi = \phi(\beta)$.

Modified metricity equations

A solution to the metricity equations reads:

$$B = c\, i^A i^B m \wedge l + c\, o^A o^B n \wedge \bar{m} + i^{(A} o^{B)} (l \wedge n - m \wedge \bar{m}), \tag{20}$$

where

$$l = \frac{1}{\sqrt{2}}(f\,dt - g dr), \quad n = \frac{1}{\sqrt{2}}(f\,dt + g dr), \quad m, \bar{m} = \frac{1}{\sqrt{2}} r (d\theta \pm i \sin(\theta) d\phi) \tag{21}$$

and

$$c^2 = \frac{1 - \phi_\beta/2}{1 + \phi_\beta}. \tag{22}$$

Metric

A metric can now be introduced as

$$ds^2 = l \otimes n - m \otimes \bar{m} = -f^2 dt^2 + g^2 dr^2 - r^2 d\Omega^2. \tag{23}$$

The usual GR (metric) case corresponds to $\phi = 0$ or $c = 1$. It is a matter of choice which of the terms in B get modified. Equivalently, the tetrad $l, n, m, \bar{m}$ above is defined up to a conformal factor.

The above choice is geometrically preferred because the area of spheres of symmetry as computed using B^a is equal to that computed using the metric.

Compatibility equations

Having solved for B^{AB} in terms of a tetrad, one can solve the equation $\mathcal{D}B^{AB} = 0$ for A^{AB}. The solution is $A^{AB} = i^A i^B A^- + o^A o^B A^+ + i^{(A} o^{B)} A$, where

$$A^- = -\frac{m}{r g_* \sqrt{2}}, \quad A^+ = -\frac{\bar{m}}{r g_* \sqrt{2}}, \quad A = \frac{l+n}{g_* \sqrt{2}} \left(\frac{c^2 - 1}{c^2 r} + \frac{f'_*}{f_*} \right) - i \cos(\theta) d\phi, \tag{24}$$

and

$$g_* = cg, \qquad f_* = cf. \tag{25}$$

As in GR case, spherical symmetry implies staticity, so there are no time derivatives involved.

“Einstein” equations

Equations $F + (\Psi + \mathrm{Id}\,\phi)B = 0$ imply:

$$g_*^{-2} = 1 - r^2(2\beta - \phi), \qquad f_* g_* = \exp \int \frac{3\phi_\beta/2}{1 - \phi_\beta/2}\frac{dr}{r}, \tag{26}$$

plus an analog of Bianchi identity that gives an equation on β:

$$(1 - \phi_\beta/2)\frac{\beta'}{\beta} = -\frac{3}{r}. \tag{27}$$

Analysis

To understand the new effects due to non-metricity can choose $\phi(\beta) = l^2\beta^2$, where l is the new scale (scale where quantum corrections become important $=$ Planck scale). The "Bianchi identity" becomes:

$$(1 - l^2\beta)\frac{\beta'}{\beta} = -\frac{3}{r}, \tag{28}$$

with the solution being

$$\beta e^{-l^2\beta} = \frac{r_+}{2r^3}, \tag{29}$$

where the integration constant was chosen to agree with Schwarzschild solution when $l = 0$.

Large r corrections

For large r

$$\beta = \frac{r_+}{2r^3} + O(1/r^6), \tag{30}$$

which means the corrections $O(l^2\beta)$ for the metric functions f, g:

$$f^2, g^{-2} = 1 - \frac{r_+}{r} + O\left(\frac{l^2 r_+}{r^3}\right). \tag{31}$$

If $l \sim l_p$, these corrections to GR are absolutely tiny.

Avoidance of singularity

The largest possible value of β is $\beta_{cr} = 1/l^2$, which corresponds to

$$r_{cr} = (l^2 r_+ e/2)^{1/3}. \tag{32}$$

There is no real solution for β for $r < r_{cr}$. Near this critical radius the metric is singular, but the two-form field B^{AB} is not. Thus, the fundamental fields B^{AB}, Ψ^{ABCD} of the theory remain non-singular!

Smallest black hole

A horizon only exists in the metric if the condition $r_{cr} > l$ is satisfied. Equivalently:

$$r_+ > 2l/e. \tag{33}$$

Only black holes of size larger than Planckian exist.

Summary: Quantum theory

New approach to perturbative quantum gravity, replaces the expansion in powers of $R(g)$ by expansion in powers of the “Lagrange multiplier” field Ψ.

Non-trivial relation between the two. The quantum corrected theory is second order in derivatives, not possible with any finite number of $R^n(g)$ terms.

The beta-function is a functional of $\phi(\lambda_1, \lambda_2)$. Its computation is within reach. Asymptotic safety conjecture of Weinberg correct? I.e. is there a non-trivial limit $\phi^* = \lim_{\mu\to\infty} \phi_\mu$?

Summary: Quantum geometry

Spacetime singularity (inside a BH) is avoided. The largest curvature is $1/l^2$. The new term in the action acts as a "cosmological constant" (negative pressure) that prevents singularity from forming.

The smallest BH is of size l.