Entanglement and holography in spin networks

based on ArXiv:2207.07625,

as well as (quant-ph/0103030,1904.08370,1808.05939,2012.12622,2302.05922,1601.01694,2105.06454)

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Overview

- **Boundary conditions, algebras and entropies**
- **Entanglement structures in spin networks**
- **Holographic information transport**

Boundary conditions, algebras and entropies

Algebraic subsystems [quant-ph/0103030]

General point of view: System described by unital *-algebra $\mathcal{A}$
Consists of "accessible operations" on system, e.g. finite resolution
Subsystems correspond directly to unital *-subalgebras
For entanglement, require notion of 'complementary subsystems'

Definition: complement via commutant

$\mathcal{A}_{L|R} \subset \mathcal{A}$ complementary $:\Leftrightarrow (\mathcal{A}_L)' = \{X \in \mathcal{A} | [X, \mathcal{A}_L] = 0\} = \mathcal{A}_R$ and vice versa

For $\mathcal{A}_L \cup \mathcal{A}_R \cong \mathcal{A}_L \otimes \mathcal{A}_R$, representations give usual form

Hilbert space subsystems for centerless algebras

$\mathbb{H}_{L|R}$ complementary in $\mathbb{H} :\Leftrightarrow \mathbb{H} \cong \mathbb{H}_L \otimes \mathbb{H}_R$.

Centers [1904.08370]

In general, complementary $\mathcal{A}_L \cup \mathcal{A}_R \neq \mathcal{A}$, and $\mathcal{Z} := \mathcal{A}_L \cap \mathcal{A}_R \neq \emptyset$

Subsystem decompositions can have centers

In particular: $\mathcal{A}_L \cup \mathcal{A}_R \ncong \mathcal{A}_L \otimes \mathcal{A}_R$

Representations can *diagonalise* $\mathcal{Z}$, so **block diagonal**:

Hilbert space of decomposition with center

$$\mathbb{H}_{L\cup R} \cong \bigoplus_{E\in\mathrm{spec}(\mathcal{Z})} \mathbb{H}_{L,E} \otimes \mathbb{H}_{R,E}$$

Density matrices *in* $\mathcal{A}_L \cup \mathcal{A}_R$ *must commute with* $\mathcal{Z}$.

$$\mathcal{A}_L \cup \mathcal{A}_R \ni \rho = \sum_E p_E \rho_E \qquad \mathrm{Tr}_E[\rho_E] = 1, \sum_E p_E = 1, p_E \geq 0 \tag{1}$$

Boundary subalgebras [1808.05939]

Typical case: Finite region R with boundary conditions

Generically, **boundary condition** $\leftrightarrow$ **boundary subalgebra** $\mathcal{A}_{\partial R}$

E.g. scalar field theory on R: $\mathcal{A}_{\partial R}$ gen. by $\phi(x), \pi(x) \forall x \in \partial R$

Dirichlet: ϕ only, *Neumann:* π only (both central!)

$$\mathbb{H}_{Dir} \cong \bigoplus_{\phi_\partial} \mathbb{H}[\phi_\partial] \qquad \rho = \sum_{\phi_\partial} p[\phi_\partial]\rho[\phi_\partial] \tag{2}$$

Example: Lattice gauge theory

Lattice gauge theories: *Gauge invariant algebra always has center* $\mathcal{Z}_{Gauss}$

Boundary algebra (open boundary conditions!) depends on shape of boundary:

- 1-valent boundary vertices: Electric fields
- boundary tangential links: Boundary magnetic fields

Center for first case: generated by casimirs $C_e := E_e^2 \; \forall e \perp \partial R$

Algebra options on R for LGT

Electric: no holonomies $h_e, e \subset \partial\gamma$

Magnetic: No electric fields $E_e, e \perp \partial\gamma$

For nonabelian casimirs, requires 'constancy'(mixedness) of ρ across the representation:

$$[\mathcal{Z}, \rho] = 0 \implies \rho = \sum_E p_E \rho_E \, , \rho_{\{s_e\}} = \bar{\rho}_{\{s_e\}} \otimes \bigotimes_e \frac{\mathbb{I}_{s_e}}{D_{s_e}} \tag{3}$$

Decomposition of von Neumann entropy

We can compute the Entropy as usual (P_s total weight of representation s):

$$\begin{aligned} S_{vN}(\rho) &= -\operatorname{Tr}[\rho \ln(\rho)] \\ &= \sum_E p_E S_{vN}(\bar{\rho}_E) - \sum_E p_E \ln(p_E) + \sum_s P_s \ln(D_s) \\ &= \langle S_{vN}(\bar{\rho}) \rangle_p + H(p) + G(p) \end{aligned} \tag{4}$$

Pieces of von Neumann entropy

1. $\langle S_{vN}(\bar{\rho}) \rangle_p$: **Distillable entanglement**
2. $H(p)$: Shannon entropy of mixture weights (includes abelian edge modes)
3. $G(p)$: Nonabelian edge mode piece

Can do the same thing for the Rényi entropies $e^{-(k-1)S_k(\rho)} = \frac{\operatorname{Tr}[\rho^k]}{\operatorname{Tr}[\rho]^k}$.

Interpretations

Distillable entanglement $S^{dist} = \langle S_{vN}(\bar{\rho}) \rangle_p$ **:** No edge modes, only operationally accessible entanglement. Can transfer into external Qubit reservoir.
Gauge invariant entropy $S^{gi} = S^{dist} + H(p)$ **:** Entropy calculated on gauge-invariant observables only.
Full entropy $S = S^{gi} + G(p)$**:** Includes Kabat contact terms, edge modes. Calculated on the full algebra.

Any prescription for entropy must know what it's calculating:
Replica trick and extended Hilbert space procedure calculate full entropy S**.**
Replica trick with gauge-fixed boundary conditions: S^{gi}**.**

Entanglement structures in spin networks

Notation and basics

Fix now some (compact) Lie group G and for simplicty a valence D.

Spin network vertex Hilbert space

$$\mathbb{H}_x = L^2(G^D)/G_{diag} \cong \bigoplus_{\mathbf{j}} \mathbb{H}_{\mathbf{j}}, \mathbb{H}_{\mathbf{j}} = \mathcal{I}_{\mathbf{j}} \otimes V_{\mathbf{j}} \quad (5)$$

Algebra can be understood as having center $\hat{J}^i_{diag}$ via Gauss constraint.

Graph Hilbert space for γ

$$\mathbb{H}_\gamma = \left(\bigotimes_{e \in E_\gamma} L^2(G_e) \right) / \bigotimes_{x \in V_\gamma} G_{x,diag} \quad (6)$$

Graph boundary: Here as 1-valent vertices

$\implies$ **Corner algebra** Holonomy-fluxes on boundary links (keep only fluxes)

$\implies$ **Corner center** $Z_{\partial\gamma}$ gen. by $J_e^2, e \perp \partial\gamma$

Single link [2012.12622]

Decompose link state on graph into *semilinks*:

$L^2(G_e) \cong \Pi_{G_{e;R}}(L^2(G_{s(e)} \times G_{t(e)}))$ - $\Psi(g_{s(e)}, g_{t(e)}) = \psi(g_{s(e)} g_{t(e)}^{-1})$

Projection onto *right-invariant* functions of group elements at ends

On basis $|\mathbf{j}^{s(e)} \mathbf{m}^{s(e)} \iota^{s(e)}\rangle \otimes |\mathbf{j}^{t(e)} \mathbf{m}^{t(e)} \iota^{t(e)}\rangle$ acts as projection on

$$|e; j_e\rangle = \frac{1}{\sqrt{D_{j_e}}} \sum_{m_e} (-1)^{j_e + m_e} |j_e, m_e\rangle_{s(e)} |j_e, -m_e\rangle_{t(e)} \tag{7}$$

which is **maximally entangled**

Glueing of links

Can glue $\mathbb{H}_{s(e)} \otimes \mathbb{H}_{t(e)} \mapsto \mathbb{H}_\gamma$ by projecting onto *maximally entangled link states* $|e; j_e\rangle$ or superpositions thereof, or equiv. via Projection operator Π_Γ

Here Γ refers to a to-be-glued subset of semilinks on the unconnected vertices.

Full graph, fixed representations

For fixed-representation spin networks, proceed the same way, collecting labels into $\vec{\mathbf{j}} = \{\mathbf{j}^v | v \in V_\gamma\}$:

$$\mathbb{H}_{\gamma,\vec{\mathbf{j}}} \cong \bigotimes_{x \in V_\gamma} \mathcal{I}_{\mathbf{j}^x} \otimes \bigotimes_{e \in E_\gamma} V_{j_e} \cong \Pi_\Gamma\Big(\bigotimes_{x \in V_\gamma} \mathbb{H}_{x,\mathbf{j}^x} \Big) \tag{8}$$

Spin network basis as a **Projected Entangled Pair State**(PEPS):

$$|\gamma; \vec{\mathbf{j}}, \{m_e\}_{e \in \partial\gamma}, \vec{\iota}\rangle = \bigotimes_{e \in \Gamma} \langle e; j_e | \bigotimes_{x \in V_\gamma} |\mathbf{j}^x \mathbf{m}^x \iota^x\rangle \tag{9}$$

This is typical for many bases and has a very clean entanglement structure

Spin tensor networks [2207.07625]

Preferred 'corner' of Hilbert space $\mathbb{H}_\gamma$ (now with no restriction on representations): **PEPS-like Spin networks**
Weigh maximally entangled states by coefficients $|e;g\rangle = \sum_{j_e} g_{j_e} |e;j_e\rangle$

Spin tensor networks from vertex states

$$|\gamma;\psi\rangle = \bigotimes_{e\in\Gamma} \langle e;g| \bigotimes_{x\in V_\gamma} |\psi_x\rangle \tag{10}$$

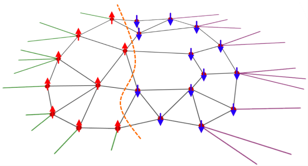
- Clean entanglement structure
- Can expect area-law-like behaviour
- Strict projection only for $g = 1$, but this way more states reached

Types of entanglement [2302.05922]

Can reduce state to various subsystems $\mathcal{A}_R \subset \mathcal{A}_\gamma$ and get different types:

- **Link entanglement:** $R = e$ single link, $\mathcal{A}_e = \mathbb{B}(L^2(G_e))$
- **Intertwiner entanglement:** $R = x, y$ 2 vertices, $\mathcal{A}_R = \mathbb{B}(\mathcal{I}_x \otimes \mathcal{I}_y)$
- **Boundary semilink entanglement:** $R = e_1, e_2 \perp \partial\gamma$,$\mathcal{A}_R = \mathbb{B}(V_{e_1} \otimes V_{e_2})$

Algebra of subregion



Now consider bipartition of $\gamma = \gamma_L \cup_S \gamma_R$ along set of links S

$\implies$ New center $\mathcal{Z}_S$ in $\mathcal{A}_{L|R}$ gen. by casimirs J_e^2 on S

Decomposition of subregion density matrix

$$\mathcal{A}_{L|R} \ni \rho = \sum_{E_{\partial,L|R},E_S} p[E_{\partial,L|R}, E_S](\bar{\rho}[E_{\partial,L|R}, E_S] \otimes \bigotimes_{e\in\partial\gamma_{L|R}} \frac{\mathbb{I}_{j_e}}{D_{j_e}}) \tag{11}$$

Entropy of subregion

Without any details about the state already:

Localisation on entangling surface

$$G(p) = \sum_{e \in \partial\gamma_{L|R}} P(j_e) \ln(D_{j_e}) \qquad P(j_e) = \sum_{\{j_{e'} | e' \in \partial\gamma_{L|R}, e \neq e\}} p[\{j_{e'} | e' \in \partial\gamma_{L|R}\}] \tag{12}$$

So, even when only one sector is active, $p[\{j_e\}] = \delta(\{j_e\}, \{\bar{j}_e\})$, we get a contribution localised on the boundary of the region

$$G(\delta_{j,\bar{j}}) = \sum_{e \in \partial\gamma_{L|R}} \ln(D_{\bar{j}_e}) \tag{13}$$

For this case it also becomes clear this accounts for all Link entanglement

Suspicion: **Only intertwiner entanglement is distillable**

Random unitaries [1601.01694]

Make statements about large classes of typical states:

Haar random states

$$\mathbb{H} \ni |\psi\rangle = U\,|\psi_0\rangle\,,\ U \in \mathcal{U}(\mathbb{H}) \qquad \langle f\rangle_U = \int d\mu_{Haar}(U) f(U) \tag{14}$$

Focus on Rényi entropy S_2 for states with fixed graph pattern γ

$\implies$ Spin tensor networks with random vertex $|\psi_x\rangle$

For high representation labels: Distribution localises on average

$$\langle e^{-S_2(\rho)}\rangle_U = \langle \frac{\mathrm{Tr}[\rho^2]}{\mathrm{Tr}[\rho]^2}\rangle_U \approx \frac{\langle \mathrm{Tr}[\rho^2]\rangle_U}{\langle \mathrm{Tr}[\rho]^2\rangle_U} =: \frac{Z_1}{Z_0} \tag{15}$$

Fixed spin [2105.06454]

Can rewrite $Z_{1|0}$ as Ising partition sum

- Replica trick to convert multiplication ρ^2 into swap operators $\mathcal{S}_{\gamma_L}$
- $\langle(|\psi\rangle\langle\psi|)^{\otimes 2}\rangle_U$ expressed through swap operators on vertices $\bigotimes_x(\mathbb{I}_x + \mathcal{S}_x)$
- Expand sum, label swap terms by Z_2 *Ising spin* σ_x

Average entanglement entropy

$$Z_{1|0} = \sum_{\vec{\sigma}} K_{\vec{\mathbf{j}}}^2 e^{-\beta H_{1|0}(\vec{\mathbf{j}};\vec{\sigma})} \qquad \beta = \ln(D_{j_{\text{mean}}}) \tag{16}$$

In high spin regime, **leading term from domain wall close to S**:

Universal edge term and distillable intertwiner entanglement

$$\langle S_2(\rho)\rangle_U \approx \sum_{e\in\partial\gamma_L} \ln(D_{j_e}) + S_2((\rho)_{\mathcal{I}}) \tag{17}$$

Condition for max entropy: **Small intertwiner dimensions**

Spin tensor networks

Cannot ignore center $\mathcal{Z}_S$ anymore when spins not fixed / $p(E)$ general

- Additional sum in $Z_{1|0}$ over $\vec{\mathbf{j}}$ for each replica copy
- Factors $K_{\vec{\mathbf{j}}} K_{\vec{\mathbf{k}}}$ become pendent of $p[E]$
- Some pairs of sectors $(\vec{\mathbf{j}}, \vec{\mathbf{k}})$ forbidden by trace $\implies \Delta_{1|0}(\vec{\mathbf{j}}, \vec{\mathbf{k}}; \vec{\sigma})$

Entropy for free representation labels

$$\left\langle e^{-S_2(\rho)} \right\rangle_U \approx \sum_{\vec{\mathbf{j}},\vec{\mathbf{k}}} p_{\vec{\mathbf{j}}} p_{\vec{\mathbf{j}}} Z_1^{\vec{\mathbf{j}},\vec{\mathbf{k}}} \qquad Z_1^{\vec{\mathbf{j}},\vec{\mathbf{k}}} = \sum_{\vec{\sigma}} \Delta_1(\vec{\mathbf{j}}, \vec{\mathbf{k}}; \vec{\sigma}) e^{-H_1(\vec{\mathbf{j}},\vec{\mathbf{k}};\vec{\sigma})} \tag{18}$$

No longer a notion of 'temperature' present

Can still perform *cumulant expansion* over $p \otimes p$ distribution

Cumulant expansion of entropy

$$\langle S_2(\rho) \rangle \approx \langle X \rangle_{p\otimes p} + \frac{1}{2}(\langle X^2 \rangle_{p\otimes p} - \langle X \rangle_{p\otimes p}^2) + \ldots \qquad X_{\vec{\mathbf{j}},\vec{\mathbf{k}}} = -\ln(Z_1^{\vec{\mathbf{j}},\vec{\mathbf{k}}}) \tag{19}$$

Holographic information transport

Minimal setup: Fixed spins

Application: Entanglement between intertwiners (bulk) and boundary links

Definition: Holography in fixed-representation spin networks

$|\psi\rangle \in \mathbb{H}_{\gamma,\vec{\mathbf{j}}}$ holographic: Induced $\Phi_\psi : \mathcal{I}_{\vec{\mathbf{j}}} \to V_{\partial\vec{\mathbf{j}}}$ *isometric*, $\Phi_\psi(|\zeta\rangle) = \langle\zeta\,|\psi\rangle$

Can convert into statement about maximal entropy:

$$\Phi^\dagger\Phi = \frac{\mathbb{I}_\mathcal{I}}{D_\mathcal{I}} \Leftrightarrow S((|\psi\rangle\langle\psi|)_\mathcal{I}) = \ln(D_\mathcal{I}) \tag{20}$$

Can again map to similar Ising model calculation (with 'magnetic fields')
$\implies$ **Generic spin networks with small intertwiners are holographic**

Extended setup for spin tensor networks

Can again not ignore nontrivial center $\mathcal{Z}_{\partial\gamma}$, so:
Replace Hilbert space mapping by algebra mapping $\mathcal{T} : \mathcal{A}_I \to \mathcal{A}_O$
Relevant sector label $E = \{j_e : e \perp \partial\gamma\}$ (contrast with bulk spins j_B)

Definition: bulk and boundary algebras

$$\mathcal{A}_I = \bigoplus_{E\in\mathcal{W}} \mathbb{B}(\mathcal{I}_E) \qquad \mathcal{A}_O = \bigoplus_{E\in\mathcal{W}} \mathbb{B}(V_E)$$
$$\mathcal{I}_E = \bigoplus_{j_B} \mathcal{I}_{\vec{\mathbf{j}}=E\cup j_B} \qquad V_E = \bigotimes_{e\perp\partial\gamma} V_{j_e} \tag{21}$$

Largest consistent choice of bulk/boundary subsystems

Transport superoperator

Use partial trace and extension maps to relate subsystems to full one

$$P\,\mathrm{Tr}_O[X] = \bigoplus_E \mathrm{Tr}_{\mathcal{I}_E}[X_E] \qquad i_I(X) = \bigoplus_E X_E \otimes \mathbb{I}_{V_E} \tag{22}$$

Can use this to define *Choi 'transport superoperator'*

Definition: state-induced transport superoperator

$$\mathcal{T}_\rho(X) = D_{\mathcal{I}}\, P\,\mathrm{Tr}_O[i_I(X)\rho^{t_I}] \tag{23}$$

For trivial center + pure ρ: **Reduces to Hilbert space formulation**

Definition: Holography in general

ρ holographic: Induced $\mathcal{T}_\rho$ isometric *in Hilbert-Schmidt* $\langle X, Y\rangle = \mathrm{Tr}[X^\dagger Y]$

Isometry condition

Explicit calculations difficult $\rightarrow$ Look for necessary criteria

Needed: $\dim(V_E) =$ **const across all** E

Reintroduces notion of scale to system $\implies$ *Low temperature limit*

Isometry condition bulk$\rightarrow$ boundary

- Restrict to fixed *total area* ($\hat{=} \dim(V_E)$)
- Require isometry in each *boundary sector* E seperately

Bulk reconstruction

Interpretation:

Parts of bulk can be reconstructed in finite regions if boundary has lax conditions (fixed total area) if not many intertwiner degrees of freedom

All this data is reconstructed from *pure gauge edge modes*

Edge modes $\rightleftarrows$ nonconstant corner gauge degrees of freedom (e.g. V_e)

Global constraint is directly related to global gauge group Gauss constraint

Possible conclusion 1

Found preferred corner of Hilbert space for holography

Possible conclusion 2

Intertwiner data is rarely reconstructible *from edge modes alone*

Final note: Boundary spin entanglement

Also possible: Fix bulk intertwiner data $|\zeta\rangle \in \mathcal{I}$
Produce induced state $\Phi_\psi(|\zeta\rangle)$
Boundary Hilbert space factorises

$$\mathbb{H}_{\partial\gamma,E} = \bigotimes_{e\perp\partial\gamma} V_{j_e} = \mathbb{H}_A \otimes \mathbb{H}_{\bar{A}} \tag{24}$$

Studying this perhaps closer to AdS/CFT "geometry from entanglement"
Would need better understanding of entropy and observables to compare though

Wrap-up

Take-home messages:

1. Entropy depends on chosen algebra, different types mean different things
2. Different types of algebras can be assigned to regions depending on Boundary conditions
3. Spin network states are superselected by boundary casimirs
4. Edge contribution to entropy in spin networks universal and from edge modes
5. Average spin network states support holographic reconstruction

Thank you for your attention!