

Conformally invariant approach to Einstein spacetimes with non-zero cosmological constant including scri

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Cartan Connections

Model example

G - a Lie group

ω_{MC} - a Maurer-Cartan form

$H \subset G$ - a Lie subgroup

G

$H \downarrow$ - the quotient PFB

G/H

$A := \omega_{\text{MC}}$

the Cartan connection

$$dA + A \wedge A = 0$$

Generalization

G, H - as before

P
 $H \downarrow$ - a PFB

M

$$\dim P = \dim G$$

A a 1-form that takes values in the Lie algebra of G and has the properties of the MC form with respect to the action of H on P

$$dA + A \wedge A =: F$$

Example: affine Cartan connection

G - the group of affine transformations

H - the group of linear transformations



given a linear connection Γ there is a Cartan affine connection A

such that $dA + A \wedge A =: F$

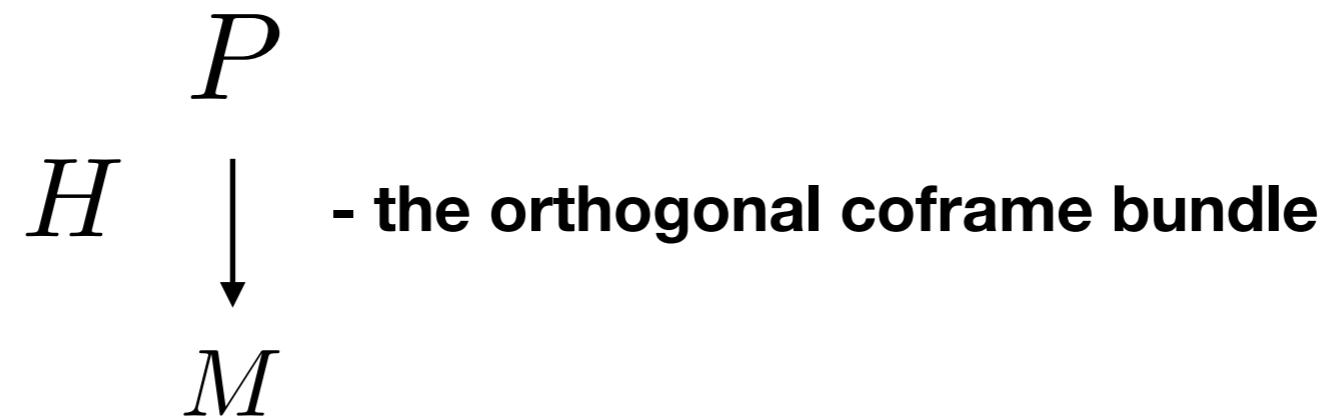
encodes $d\Gamma + \Gamma \wedge \Gamma =: R$

and the torsion of Γ

Example: the Witten connection

$$G = \text{Iso}(2,1)$$

$$H = \text{SO}(2,1)$$



Γ - the metric connection

A - the restriction of the corresponding affine connection

$$L_{\text{CS}} = \frac{k}{4\pi} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

describes the 2+1 gravity

given

Working definition

$$\begin{array}{ccc} P & & \\ H \downarrow & - \text{ a PFB} & \\ M & & \end{array}$$

$$\dim P = \dim G$$

A - a Cartan connection

$$F = dA + A \wedge A$$

we use local sections:

$$\sigma : U \rightarrow P$$

$$U \subset M$$

and: $\sigma^* A$

$$\sigma^* F = d\sigma^* A + \sigma^* A \wedge \sigma^* A$$

subject to:

$$\sigma'^* A = h^{-1} \sigma^* A h + h^{-1} dh$$

$$\sigma'^* F = h^{-1} \sigma^* F h$$

$$h : U \rightarrow H$$

$$\sigma^* A \equiv A$$

Normal conformal Cartan connection

Working definition

$$G = \mathbf{So}(2,4) = \mathbf{So}(\mathbf{Q})$$

M 4d spacetime

$$g = \eta_{ab}\theta^a \otimes \theta^b$$

$$\eta_{ab} = \text{const} \quad - + ++$$

$$\text{Vol} := \frac{1}{4!} \sqrt{|\det \eta|} \varepsilon_{abcd} \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$

$$= \sqrt{|\det \eta|} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3,$$

$$\epsilon_{abcd} = \sqrt{|\det \eta|} \varepsilon_{abcd}$$

$$Q = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \theta_b & 0 \\ P^a & \Gamma^a{}_b & \theta^a \\ 0 & P_b & 0 \end{bmatrix}$$

$$d\theta^a + \Gamma^a{}_b \wedge \theta^b = 0$$

$$\Gamma_{ab} = -\Gamma_{ba}$$

$$P_a := \left(\frac{1}{12} R \eta_{ab} - \frac{1}{2} R_{ab} \right) \theta^b$$

$$\frac{1}{2} R^a{}_{bcd} \theta^a \wedge \theta^d = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b.$$

gauge transformations

nice property:

$$A' = h^{-1} A h + h^{-1} dh$$

$$\theta'^a = f \theta^a, \quad f \in C^\infty(M),$$

$$h = \begin{bmatrix} \frac{1}{f} & 0 & 0 \\ \frac{f_a}{f^2} & \delta^a_b & 0 \\ \frac{f_{,c} f_c}{2f^3} & \frac{f_{,b}}{f} & f \end{bmatrix}$$

$$\theta'^a = \Lambda^a_b \theta^b,$$

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \Lambda^a_b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

there are also:

$$h = \begin{pmatrix} 1 & 0 & 0 \\ b^\mu & \delta^\mu_\nu & 0 \\ \frac{1}{2} b_\sigma b^\sigma & b_\nu & 1 \end{pmatrix}$$

however we gauge fixe them:

$$A = \begin{bmatrix} 0 & \theta_b & 0 \\ P^a & \Gamma^a_b & \theta^a \\ 0 & P_b & 0 \end{bmatrix}$$

The curvature:

$$A' = h^{-1}Ah + h^{-1}dh$$

$$F' = h^{-1}Fh$$

$$F = dA + A \wedge A$$

The Bianchi identity:

$$D_A F := dF + A \wedge F - F \wedge A = 0$$

encodes the differential identities satisfied by the Weyl and the Schouten tensors

$$F = \begin{bmatrix} 0 & 0 & 0 \\ DP^a & C^a_b & 0 \\ 0 & DP_b & 0 \end{bmatrix}$$

$$DP^a = dP^a + \Gamma^a_b \wedge P^b$$

$$C^a_b = \frac{1}{2} C^a_{bcd} \theta^c \wedge \theta^d,$$

the Weyl tensor

$$\begin{pmatrix} \psi & \theta^4 & \text{Re}\omega_1 & \text{Im}\omega_1 & -\omega_0 & 0 & \dots \\ -\theta^4 & \psi & -\text{Im}\omega_1 & \text{Re}\omega_1 & 0 & -\omega_0 & \dots \\ \text{Re}\omega_3 & \text{Im}\omega_3 & 0 & -2\theta^4 & \text{Im}\omega_1 & \text{Re}\omega_1 & \dots \\ -\text{Im}\omega_3 & \text{Re}\omega_3 & 2\theta^4 & 0 & -\text{Re}\omega_1 & \text{Im}\omega_1 & \dots \\ \omega_4 & 0 & -\text{Im}\omega_3 & -\text{Re}\omega_3 & -\psi & \theta^4 & \dots \\ 0 & \omega_4 & \text{Re}\omega_3 & -\text{Im}\omega_3 & -\theta^4 & -\psi & \dots \end{pmatrix}$$

The Bach tensor:

$$D_A \star F := d\star F + A \wedge \star F - \star F \wedge A = \begin{bmatrix} 0 & 0 & 0 \\ B^{ac} \star \theta_c & 0 & 0 \\ 0 & B_b{}^c \star \theta_c & 0 \end{bmatrix}$$

$$B_{ab} = 2\nabla^c \nabla_{[b} P_{c]a} - 2P^{cd} C_{cadb}.$$

$$R_{ab} = \Lambda \eta_{ab} \implies B_{ab} = 0 \implies D\star F = 0.$$

Examples of reduced holonomy:

The spinor representation of the NCCC is the local twistor connection of Penrose.
Iff the NCCC can be gauge transformed to the following non-generic form:

$$A = \begin{pmatrix} \psi & \theta^4 & \text{Re}\omega_1 & \text{Im}\omega_1 & -\omega_0 & 0 \\ -\theta^4 & \psi & -\text{Im}\omega_1 & \text{Re}\omega_1 & 0 & -\omega_0 \\ \text{Re}\omega_3 & \text{Im}\omega_3 & 0 & -2\theta^4 & \text{Im}\omega_1 & \text{Re}\omega_1 \\ -\text{Im}\omega_3 & \text{Re}\omega_3 & 2\theta^4 & 0 & -\text{Re}\omega_1 & \text{Im}\omega_1 \\ \omega_4 & 0 & -\text{Im}\omega_3 & -\text{Re}\omega_3 & -\psi & \theta^4 \\ 0 & \omega_4 & \text{Re}\omega_3 & -\text{Im}\omega_3 & -\theta^4 & -\psi \end{pmatrix}$$

then the spacetime conformal geometry admits solutions to the twistor equation:

$$\nabla^{(A}{}_{A'}\omega^{B)} = 0$$

This is the Fefferman family of spacetime metric tensors. Among them are known examples of the Bach flat metric tensors that are not conformal to Einstein.

Symplectic potential densities

The Cartan-Yang-Mills Lagrangian:

$$L_{\text{CYM}}(\theta) = \frac{1}{2} F^I{}_J \wedge \star F^J{}_I$$

$$L_{\text{CYM}}(\theta) = L_{\text{CYM}}(f\theta),$$

$$L_{\text{CYM}}(\theta) = \frac{1}{4} C_{abcd} C^{abcd} \text{Vol},$$

$$\delta = \delta_\theta$$

$$\delta L_{\text{CYM}}(\theta) = \delta A^I{}_J \wedge D_A \star F^J{}_I + \frac{1}{2} F^I{}_J \wedge (\delta \star) F^J{}_I + d(\delta A^I{}_J \wedge \star F^J{}_I)$$

$$\star C^a{}_b = \frac{1}{2} \epsilon^a{}_{bc}{}^d C^c{}_d \quad \Rightarrow \delta(\star) F = 0$$

The symplectic potential density:

$$L_{\text{CYM}}(\theta) = \frac{1}{2} F^I_J \wedge \star F^J_I$$

$$\delta L_{\text{CYM}}(\theta) = 2\delta\theta^a \wedge B_{ab}\star\theta^b + d(\delta A^I_J \wedge \star F^J_I)$$

$$\Theta_{\text{CYM}}(\theta; \delta\theta) = \delta A^I_J \wedge \star F^J_I$$

$$\begin{aligned} (\delta A^I_J \wedge \star F^J_I)(f\theta; f\delta\theta) &= \left((h^{-1}\delta Ah)^I_J \wedge \star (h^{-1}Fh)^J_I \right)(\theta; \delta\theta) \\ &= (\delta A^I_J \wedge \star F^J_I)(\theta; \delta\theta) \end{aligned}$$

$$\Theta_{\text{CYM}}(\theta; \delta\theta) = 2\delta\theta^a \wedge \star DP_a + \delta\Gamma^a_b \wedge \star C^b_a$$

Useful decomposition

$$L_{\text{CYM}} = \frac{1}{4}\mathcal{E} + L_1$$

$$\mathcal{E}(\theta) := \epsilon^{abcd} \mathcal{R}_{ab} \wedge \mathcal{R}_{cd} \quad \quad \mathcal{R}^a{}_b := \frac{1}{2} R^a{}_{bcd} \theta^a \wedge \theta^d = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b.$$

$$\delta \mathcal{E}(\theta) = d\left(2\epsilon^{abcd} \delta \Gamma_{ab} \wedge \mathcal{R}_{cd}\right)$$

$$\Theta_{\mathcal{E}}(\theta; \delta \theta) := 2\epsilon^{abcd} \delta \Gamma_{ab} \wedge \mathcal{R}_{cd}$$

$$L_1(\theta) := -4P_a^{[a} P_b^{b]} \text{Vol} \quad \quad \delta L_1(\theta) = 2\delta \theta^a \wedge B_{ab} \star \theta^b + d\Theta_1$$

$$\Theta_1(\theta; \delta \theta) = 2\delta \theta^a \wedge \star D P_a + \epsilon^{abcd} \delta \Gamma_{ab} \wedge \theta_c \wedge P_d$$

$$\Theta_{\text{CYM}} = \frac{1}{4}\Theta_{\mathcal{E}} + \Theta_1$$

Einstein \longrightarrow Bach flat

$$R_{ab} = \Lambda \eta_{ab} \quad P_a = P_{ab} \theta^b = -\frac{\Lambda}{6} \theta_a$$

$$\Theta_{\text{CYM}}(\theta; \delta\theta) = \delta\Gamma^a{}_b \wedge \star C^b{}_a \quad \Theta_1(\theta; \delta\theta) = -\frac{\Lambda}{6} \epsilon^{abcd} \theta_a \wedge \theta_b \wedge \delta\Gamma_{cd}$$

We want to compare it with the Einstein theory symplectic potential dencity

$$L_{\text{EH}} = \frac{1}{16\pi G} \left(\frac{1}{2} \epsilon^{abcd} \theta_a \wedge \theta_b \wedge \mathcal{R}_{ab} - 2\Lambda \text{Vol} \right)$$

$$16\pi G \delta L_{\text{EH}}(\theta) = \delta\theta_a \wedge \left(\epsilon^{abcd} \theta_b \wedge \mathcal{R}_{cd} - 2\Lambda \star \theta^a \right) + d \left(\frac{1}{2} \epsilon^{abcd} \theta_a \wedge \theta_b \wedge \delta\Gamma_{cd} \right)$$

$$\Theta_{\text{EH}}(\theta; \delta\theta) = \frac{1}{32\pi G} \epsilon^{abcd} \theta_a \wedge \theta_b \wedge \delta\Gamma_{cd}$$

$$\Theta_{\text{CYM}} = \frac{1}{4} \Theta_{\mathcal{E}} - \frac{16\pi G \Lambda}{3} \Theta_{\text{EH}}$$

**At the scri of asymptotically
(A) de Sitter spacetime**

The Fefferman-Graham coordinates

$$R_{ab} = \Lambda \eta_{ab} \quad \Lambda > 0.$$

$$g = \frac{\ell^2}{\rho^2} \left(-d\rho^2 + \sum_{n=0}^{\infty} \rho^n g_{ij}^{(n)}(\rho, x^1, x^2, x^3) dx^i dx^j \right)$$

$$\mathcal{I} : \rho = 0$$

$$\hat{g} := \frac{\rho^2}{\ell^2} g = -d\rho^2 + \sum_{n=0}^{\infty} \rho^n g_{ij}^{(n)} dx^i dx^j$$

$$\Theta_{\text{CYM}}(\theta, \delta\theta) = \Theta_{\text{CYM}}(\rho\theta, \rho\delta\theta)$$

\Rightarrow finiteness at $\rho = 0$

$$\textbf{The pullback of} \quad \Theta_{\text{CYM}}(\theta^a,\delta\theta^a) \quad \textbf{on the scri}$$

$$g=\frac{\ell^2}{\rho^2}\left(-\mathrm{d}\rho^2+\sum_{n=0}^\infty\rho^ng^{(n)}_{ij}(\rho,x^1,x^2,x^3)\,\mathrm{d}x^i\,\mathrm{d}x^j\right)$$

$$g_{ij}^{(1)} = 0, \qquad g_{ij}^{(2)} = \mathring{R}_{ij} - \frac{1}{4}\mathring{g}_{ij}\mathring{R} =: \mathring{S}_{ij} \qquad \mathring{T}_{ij} := g_{ij}^{(3)}$$

$$\Theta_{\text{CYM}}(\theta^a,\delta\theta^a)=\delta\Gamma^b{}_c\wedge *C^c{}_b$$

$${\Gamma^a}_b = 2\rho^{-1}\eta^{ac}\hat e^\rho_{[c}\hat e^\beta_{b]}\hat g_{\beta\gamma}\,\mathrm{d} x^\gamma + \mathcal O(1)$$

$$\textbf{Pullback on} \quad \mathscr{I} \qquad \qquad \mathring{\mathrm{Vol}}:=\tfrac{1}{3!}\mathring{\epsilon}_{ijk}\,\mathrm{d} x^i\wedge\mathrm{d} x^j\wedge\mathrm{d} x^k.$$

$$\hat{\epsilon}_{\rho ijk}=\sqrt{|\det\hat{g}|}\varepsilon_{\rho ijk}=\sqrt{\det\mathring{g}}\varepsilon_{ijk}+\mathcal O(\rho^2)=\mathring{\epsilon}_{ijk}+\mathcal O(\rho^2),$$

$$\tilde{\Theta}_{\text{CYM}}=\lim_{\rho\rightarrow 0}\delta\Gamma^a{}_{bi}\hat{e}^\alpha_a\hat{\theta}^b_\beta\,\mathrm{d} x^i\wedge\left(\frac{1}{2}\hat{C}^\beta{}_\alpha{}^{\gamma\delta}\hat{\epsilon}_{\gamma\delta jk}\,\mathrm{d} x^j\wedge\mathrm{d} x^k\right)\quad=\frac{3}{2}\delta\mathring{g}_{ij}\mathring{T}^{ij}\mathring{\mathrm{Vol}}.$$

Comparison with the standard

$$g = \frac{\ell^2}{\rho^2} \left(-d\rho^2 + \sum_{n=0}^{\infty} \rho^n g_{ij}^{(n)}(\rho, x^1, x^2, x^3) dx^i dx^j \right)$$

The pullback on \mathcal{I}

$$\tilde{\Theta}_{\text{CYM}} = \lim_{\rho \rightarrow 0} \delta \Gamma^a{}_{bi} \hat{e}_a^\alpha \hat{\theta}_\beta^b dx^i \wedge \left(\frac{1}{2} \hat{C}^\beta{}_\alpha{}^{\gamma\delta} \hat{\epsilon}_{\gamma\delta jk} dx^j \wedge dx^k \right) = \frac{3}{2} \delta \mathring{g}_{ij} \mathring{T}^{ij} \text{Vol.}$$

The standard definitions: $T_{ij} = \frac{3\ell^2}{16\pi G} \mathring{T}_{ij} = \frac{8\pi G}{\ell^2} \delta \mathring{g}_{ij} T^{ij} \text{Vol.}$

$$S_{\text{GR}} = \frac{1}{16\pi G} \int_M (R - 2\Lambda) \text{Vol} + \frac{1}{16\pi G} \int_{\mathcal{I}} \left(2K + \frac{4}{\ell} - \mathring{R} \right) \text{Vol.}$$

$$\tilde{\Theta}_{\text{GR}} = \frac{1}{2} \delta \mathring{g}_{ij} T^{ij} \text{Vol.}$$

$$\tilde{\Theta}_{\text{CYM}} = \frac{16\pi G \Lambda}{3} \tilde{\Theta}_{\text{GR}}.$$

**At the scri of asymptotically
flat spacetime**

The Bondi coordinates

$$g_{rr} = g_{rA} = 0, \quad g_{uu} = -1 + \frac{2M}{r} + \mathcal{O}(r^{-2}), \quad g_{AB} = r^2 \gamma_{AB} + r C_{AB} + \mathcal{O}(1),$$

$$g_{ur} = -1 + \frac{1}{16r^2} C^{AB} C_{AB} + \mathcal{O}(r^{-3}), \quad g_{uA} = \frac{1}{2} D^B C_{BA} + \mathcal{O}(r^{-1}),$$

$$\det(g_{AB}) = r^4 \det(\gamma_{AB})$$

The pullback at the scri:

$$\hat{\epsilon}_{\Omega ijk} = \sqrt{|\det \hat{g}|} \varepsilon_{\Omega ijk} = \sqrt{\det \gamma} \varepsilon_{ijk} + \mathcal{O}(\Omega) = \mathring{\epsilon}_{ijk} + \mathcal{O}(\Omega)$$

$$\mathring{\text{Vol}} := \frac{1}{3!} \mathring{\epsilon}_{ijk} dx^i \wedge dx^j \wedge dx^k := \frac{1}{3!} \sqrt{\det \gamma} \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$$

$$\tilde{\Theta}_{\text{CYM}} = \left[\delta \mathring{g}_{uu} \left(\frac{1}{4} C^{AB} N_{AB} + 2M \right) + \frac{1}{2} \delta \mathring{g}_{AB} \partial_u N^{AB} \right] \mathring{\text{Vol.}}$$

$$\Theta_{\text{CYM}} = \frac{1}{4} \Theta_{\mathcal{E}} - \frac{16\pi G \Lambda}{3} \Theta_{\text{EH}}$$

The Noether currents

Diffeomorphisms

ξ - vector field tangent to the spacetime

$$J_\xi(\theta) = \mathcal{L}_\xi A^I_J \wedge \star F^J_I - \xi \lrcorner \left(\frac{1}{2} F^I_J \wedge \star F^J_I \right)$$

$$J_\xi(\theta) = d((\xi \lrcorner A^I_J) \star F^J_I) - (\xi \lrcorner A^I_J) D_A \star F^J_I$$

$$Q_\xi(\theta) = (\xi \lrcorner A^I_J) \star F^J_I$$

l - generator of the Lorentz rotations or conformal rescalings

$$J_l = d(l^I_J \wedge \star F^J_I) - l^I_J \wedge D_A \star F^I_J$$

$$= d(l^a_b * C^b_a) \quad \text{for the Lorentz}$$

Summary

Thank you!