LQC - unFAQ

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The Questions

1. Quantum spacetime according to LQC
2. The status of the kinematical quantum geometry in LQC
3. Where is the cell? Is there any fiducial scale dependence?
4. The discrete universe
5. What are true solutions of the constraint $\widehat{C(N)}$

Quantum spacetime (1)

Homogeneous and isotropic space-time metric in $\mathbb{R} \times \Sigma$

$$ds^2 \;=\; -N^2 dt^2 \;+\; a(t)^2\, q^{(0)}_{ab}\, dx^a dx^b \tag{1}$$

coupled with the homogeneous scalar field:

$$\phi \qquad .$$

The constraint

$$C(N) \;=\; (C_{\mathrm{gr}} + \frac{1}{2}\frac{\pi^2}{V})N \tag{2}$$

Where, $\tilde{\pi}$ is the canonically conjugate momentum to ϕ, and:

$$\pi \;:=\; \int_{\mathcal{U}_0} \tilde{\pi}, \qquad V \;:=\; \int_{\mathcal{U}_0} a^3 \sqrt{\det q^{(0)}} \tag{3}$$

and $\mathcal{U}_0$ is a fixed finite region ("cell") in Σ.

Quantum spacetime (2)

The kinematical operators in $\mathcal{H}_{\rm sc}\otimes\mathcal{H}_{\rm gr}$:

$$\hat{\pi}\psi(\phi,v) \;=\; \frac{\hbar}{i}\partial_\phi\psi(\phi,v), \qquad \hat{V}\psi(\phi,v) \;=\; |v|\psi(\phi,v) \tag{4}$$

The quantum constraint

$$-\hbar^2\partial_\phi^2\psi(\phi,v) \;=\; 2\widehat{VC_{\rm gr}}\,\psi(\phi,v) \tag{5}$$

The physical states are solutions:

$$\mathbb{R}\ni\phi \;\mapsto\; \psi_\phi\in\mathcal{H}_{\rm gr} \tag{6}$$

The the following operators are observables:

$$\hat{\pi} \;:=\; \pm\sqrt{\widehat{2VC_{\rm gr}}}, \qquad (\hat{V}_{\phi_0}\psi)_{\phi_0} \;:=\; \hat{V}\psi_{\phi_0}. \tag{7}$$

Quantum spacetime (3)

What is the corresponding space-time metric? Let us try:

$$\text{"}\widehat{ds^2} \;=\; -\widehat{Ndt}^{\,2} \;+\; |\hat{V}^{\frac{2}{3}}|\,\frac{q^{(0)}_{ab}dx^a dx^b}{\int_{\mathcal{U}_0}\sqrt{\det q^{(0)}}}\text{"} \tag{8}$$

Classically

$$\frac{d\phi}{dt} \;=\; \frac{\pi}{V}N \quad\Rightarrow\quad Ndt \;=\; \frac{V}{\pi}d\phi \tag{9}$$

Hence,

$$\widehat{Ndt} \;=\; \widehat{V/\pi}d\phi \;=\; \sqrt{\widehat{V/2C_{\rm gr}}}d\phi, \tag{10}$$

meaning:

$$T_{\phi_2,\phi_1} \;=\; \frac{1}{\sqrt{2}}\int_{\phi_0}^{\phi_1}(\,\psi_\phi\,|\,\sqrt{\widehat{V/C_{\rm gr}}}\psi_\phi\,)d\phi \tag{11}$$

Quantum spacetime (4)

In conclusion

$$ds^2 \;=\; -\widehat{V^2/\pi^2}_{\phi} d\phi^2 + \;|\hat{V}_{\phi}^{\frac{2}{3}}|\,\frac{q^{(0)}_{ab}dx^a dx^b}{\int_{\mathcal{U}_0}\sqrt{\det q^{(0)}}} \tag{12}$$

Since $\hat{\pi} \;=\; \pm\sqrt{\widehat{2VC_{\rm gr}}}$,

$$[\,\hat{V}_{\phi},\,\pi\,] \;\neq\; 0 \tag{13}$$

and we have the factoring problem However,

$$\frac{1}{2}\widehat{\pi^2/V^2}_{\phi} \;=\; \hat{\rho}. \tag{14}$$

The same formula for $\widehat{Ndt}$ is true in the homogeneous non-isotropic case (Szulc). It can be generalized to the full GR treated by the Brown-Kuchar mechanism Giesel, private communication.

The status of the kinematical quantum geometry (1)

We wrote the constraint as

$$(\hat{\pi}^2 + 2\widehat{VC_{\rm gr}})\psi \;=\; 0 \tag{15}$$

and assumed the second term is defined as a self-adjoint (at least symmetric) operator in $\mathcal{H}_{\rm gr}$. That can be achieved by defining

$$\widehat{VC_{\rm gr}} \;=\; \sqrt{V}C_{\rm gr}\sqrt{V} \tag{16}$$

(a subtlety: in fact, in LQC one uses $\widehat{V^{-1}}^{-1}$ instead of $\hat{V}$ in this formula.)

Then, *every* self-adjoint operator $\hat{A}$ defined in $\mathcal{H}_{\rm gr}$ - that is every quantum geometry operator - passes to a self-adjoint $\hat{A}_{\phi_1}$ defined on the physical solutions at an instant $\phi = \phi_1$ (*Kamiski, L, Szulc*).

This observation can be generalized to to the full gravity with the Brow-Kuchar dust fields (*Giesel, Thiemann).*

The status of the kinematical quantum geometry (2)

In the Ashtekar-Pawlowski-Singh model, one defines non-symmetrically

$$\widehat{VC_{\rm gr}} \;=\; \widehat{V^{-1}}^{-1}\widehat{C_{\rm gr}} \tag{17}$$

and modifies the scalar product

$$(\cdot|\cdot)_{\rm APS} \;=\; (\cdot|\widehat{V^{-1}}\cdot)_{\rm gr} \tag{18}$$

Now, however, only the operators $f(\hat{V})$ pass to the space of the physical solutions, and somehow $\widehat{V^{-1}}^{-1}\widehat{C_{\rm gr}}$ does. The extrinsic quantum geometry operators - the holonomy operators - do not pass (unless their definitions are changed.) May be it is OK, but it is good to be aware.

Where is the cell? (1)

The LQC open models are constructed by using a cell $\mathcal{U}_0 \subset \Sigma$. (The closed model can also be mistaken for an open one and quantized with a cell). Suppose we are living now in the Universe described by our model. How do we find the cell?

For every region $\mathcal{U} \subset \Sigma$, the corresponding volume operator $\hat{V}_{\mathcal{U},\phi}$ has the following spectrum

$$\{\, \lambda|\epsilon\ell_{\rm Pl}^3 + a_0\sqrt{\Delta}n\ell_{\rm Pl}^2| \;:\; n \in \mathbb{Z}\,\} \tag{19}$$

where ϵ labels our super-selected physical Hilbert space, Δ is the smallest non-zero area operator eigenvalue in the full LQG, and a_0 is some constant calculated from LQC. The constant λ depends on $\mathcal{U}$. Any region $\mathcal{U}_0$ corresponding to $\lambda = 1$ is the cell used by the creators of our quantum Universe. If Δ was not known to us, we could read it from the $\hat{\rho}_\phi$ operator.

Where is the cell? (2)

Now, in the flat case, the operator $\hat{\rho}_\phi$ is defined uniquely at any instant of ϕ, by the operator

$$\hat{\pi}^2 \;=\; 2\widehat{\hat{V}C_{\rm gr}} \;=\; \theta_0 \tag{20}$$

uniquely defined in the model.

However, in the closed $k = 1$ Universe case when $\Sigma = S^3$

$$2\widehat{\hat{V}C_{\rm gr}} \;=\; \theta_0 + W_{k=1,\ell_0} \tag{21}$$

where ℓ_0 is a fixed background radius of S^3. It is fixed to be $\ell_0 = 2$, however that value is *a priori* arbitrary.

In conclusion: there is a unique flat FRW LQC model, however there are other non-equivalent closed FRW LQC models given by changing ℓ_0.

The discrete universe with $\Lambda < 0$ (1)

More specifically, the constraint equation has the form

$$-\partial_\phi^2 \psi \;=\; (\theta_0 \;-\; b_0 \Lambda \widehat{V^{-1}}^{\,-1} \hat{V})\psi \;=:\; \theta\psi \tag{22}$$

where θ_0 is positive and symmetric (and essentially self adjoint - *Kamiski, L*). If

$$\Lambda \;<\; 0$$

we have

$$0 \;\leq \theta_0 \;\leq \theta_0 - \; b_0 \Lambda \widehat{V^{-1}}^{\,-1} \hat{V}. \tag{23}$$

$\Rightarrow \theta$ is self-adjoint, its spectrum is discrete, and more over

$$\dim \mathcal{H}_{\theta<\omega^2} \;\leq\; \dim \mathcal{H}_{-b_0 \Lambda \widehat{V^{-1}}^{\,-1} \hat{V} < \omega^2} \;< \infty \tag{24}$$

(*Kamiński, L*).

The discrete universe with $\Lambda < 0$ (2)

In particular:

- there is a grand state of the Universe, $\omega_0 > 0$
- there are excited states labeled by ω_n
- $\omega_{n+1} - \omega_n \simeq 3.87|\Lambda G|^{0.489}$ for $n \to \infty$ (*Pawłowski*).

What are the physical consequences? Analogy with quantum Hydrogen atom?

Remark. In the case $\Lambda > 0$ our arguments do not work, and in fact the operator θ is proven to have inequivalent self-adjoint extensions (*Kamiński, Pawłowski*). However, the spectra continue to be discrete.

Universal self-adjointness of $\widehat{C(N)}$ (1)

Let us go back to the scalar constraint in the original form,

$$\widehat{C(N)} \;=\; N(\widehat{C_{\mathrm{gr}}} + \frac{1}{2}\hat{\pi}^2\widehat{V^{-1}}). \tag{25}$$

$$[\,\widehat{C(N)},\, \hat{\pi}\,] \;=\; 0, \tag{26}$$

the assumption

$$\hat{\pi}\psi \;=\; \omega\psi \tag{27}$$

reduces the operator to

$$\hat{C}_\omega \;=\; \widehat{C_{\mathrm{gr}}} \;+\; \frac{1}{2}\omega^2\widehat{V^{-1}} \tag{28}$$

Universal self-adjointness of $\widehat{C(N)}$ (2)

The dependence on k and Λ (*Kamiński, L, Szulc, Ashtekar, Pawłowski, Vandersloot*):

$$\widehat{C_{\rm gr}} \;=\; \widehat{C_{\rm gr,flat}} \;+\; W_k(\hat{V}) \;-\; b_0\Lambda\hat{V} \qquad (29)$$

Remarkably, the operator $\widehat{C_{\rm gr}}$ is essentially self adjoint for *every* $k = -1, 0, 1$ and every value of Λ.

Universal self-adjointness of $\widehat{C(N)}$ (3)

The technical reason is the following general fact: In the Hilbert space

$$\overline{\text{Span}(..., |\epsilon - 1\rangle, |\epsilon\rangle, |\epsilon + 1\rangle, ...)}, \quad \langle v|v'\rangle \;=\; \delta_{v,v'}$$

consider operators

$$\hat{V}|v\rangle \;=\; |v||v\rangle, \qquad h|v\rangle \;=\; |v+1\rangle$$

and an operator (in the LQG case, $A(v) \sim |v|$)

$$hA(\hat{V})h + h^{-1}A(\hat{V})h^{-1} - A(\hat{V}+1) - A(\hat{V}-1) + W(\hat{V}).$$

The operator is essentially self-adjoint for every function W and every nowhere-vanishing function A such that (Kaminski)

$$\sum_{n\in\mathbb{Z}_\pm} \frac{1}{|A(\epsilon+n)|} \;=\; \infty$$

Universal self-adjointness of $\widehat{C(N)}$ (4)

Comparison between two constructions: writing the constraint as

$$\hat{\pi}^2 + 2\widehat{V C_{\rm gr}}$$

we encounter ambiguity in the self-adjoint extensions, whereas writing

$$\hat{\pi}^2 \widehat{V^{-1}} + 2\widehat{C_{\rm gr}}$$

we find the self-adjoint extension is unique.

The conjecture (Kaminski, Pawlowski) is that *every* solution given by any possible extensions of the first operator gives rise to a distinct solution of the second constraint.