LQG with all the degrees of freedom

Marcin Domagała, Kristina Giesel, Wojciech Kamiński, Jerzy Lewandowski

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Quantum gravity within reach

- The recent advances in loop quantum gravity *Ashtekar - the book; Rovelli - the book; Thiemann - the book; Han, Huang, Ma Int. J. Mod. Phys. D16: 1397-1474 (2007); Ashtekar, L Class. Quant. Grav. 21: R53 (2004);* strongly suggest that the goal of constructing a candidate for quantum theory of gravity and the Standard Model is within reach. Remarkably, that goal can be addressed within the canonical formulation of the original Einstein's general relativity in four dimensional spacetime.

- Physical dynamics and spacetime emerge from general relativity where spacetime diffeomorphisms are treated as a gauge symmetry, due to the framework of the relational Dirac observables *Rovelli the book; Dittrich Class. Quantum Grav. 23, 6155, 2006, Thiemman the book*.

Deparametrization $+$ *LQG*

- The exact and most powerful example of the relational observables is the deparametrization technique *Kijowski, Smolski, Gornicka "Hamiltonian theory of self-gravitating perfect fluid and a method of effective deparametrization of the Einsteinian theory of gravitation" Phys. Rev. D41 (1990); Rovelli, Smolin Phys. Rev. Lett. 72 446 (1993); Brown, Kuchar Phys. Rev. D 51, 5600, (1995)*. This allows to map canonical General Relativity into a theory with a (true) non-vanishing Hamiltonian. All this can be achieved at the classical level.

- The framework of loop quantum gravity (LQG) itself, provides the quantum states, the Hilbert spaces, quantum operators of the geometry and fields, and well defined quantum operators for the constraints of General Relativity. This is exactly an ingredient missing in the deparametrization works of Brown and Kuchar.

New QFT needed

- Applying LQG techniques to perform the quantization step has the consequence that the quantum fields of the Standard Model have to be reintroduced within the scheme of LQG. The resulting quantum theory of gravity cannot be just coupled to the Standard Model in it's present form. The formulation of the full Standard Model within LQG will require some work. For this reason, we proceed step by step, increasing gradually the level of complexity.

The symmetry reduced models

- The first step was constructing various cosmological models by analogy with LQG by performing a symmetry reduction already at the classical level. They give rise to loop quantum cosmology (LQC) *Bojowald, Class. Quantum Grav. 17, 1489 (2000) - 18, 1071 (2001); Ashtekar, Pawowski, Singh Phys. Rev. D 74, 084003 (2006); Ashtekar, Corichi, Singh Phys. Rev. D77: 024046 (2008)*. We have learned from them a lot about qualitative properties of quantum spacetime and its quantum dynamics *Kamiński, L, Pawłowski Class. Quantum Grav. 26, 035012 (2009)*.
- There are also more advanced symmetry reduced models includding quasi-local degrees of freedom *Martn-Benito, Mena Marugn, Wilson-Ewing arXiv: 1006.2369; Gambini, Pullin, Rastgoo Class. Quant. Grav. 26: 215011 (2009)*

Gravity quantized

- The knowledge we gained by considering the symmetry reduced models is very useful in performing the second step, that is introducing quantum models with the full set of the local gravitational degrees of freedom.

- The first quantum model of the full, four dimensional theory of gravity was obtained by applying the LQG techniques to the Brown-Kuchar model of gravity coupled to dust *Giesel, Thiemann Class. Quantum Grav. 27 175009 (2010)*.

- In the current paper we apply LQG to the model introduced by Rovelli and Smolin whose classical canonical structure was studied in detail by *Kuchar, Romano Phys. Rev. D 51 5579 (1995)*. This is a model of gravity coupled to a massless scalar field. Our goal is to complete the construction of the quantum model with the tools of LQG.

Plan of the talk

- The “formal” structure of the quantum model: the space of solutions to the quantum constraints, the Dirac observables, and dynamics, all that assuming that the suitable Hilbert products and any operators we need do exist.
- Conclusion: the list of mathematical elements necessary and sufficient for the model to be constructed.
- Application of the LQG framework: what exists right away, what we need to define, how we accomplish that.
- Discussion: the advantages and the strong points of the model.

Gravity and massless scalar field deparametrized

Fields $q_{ab}, p^{ab}, \phi, \pi$ defined on a 3-manifold M.

$$\{q_{ab}(x), p^{cd}(y)\} \;=\; \delta(x,y)\delta^c_{(a}\delta^d_{b)}, \quad \{\phi(x), \pi(y)\} \;=\; \delta(x,y). \tag{1}$$

The constraints:

$$C(x) \quad = \quad C^{\rm gr}(x) \;+\; \frac{1}{2}\frac{\pi^2(x)}{\sqrt{q(x)}} + \frac{1}{2}q^{ab}(x)\phi_{,a}(x)\phi_{,b}(x)\sqrt{q(x)} \tag{2}$$

$$C_a(x) \quad = \quad C^{\rm gr}_a(x) \;+\; \pi(x)\phi_{,a}(x). \tag{3}$$

Replace $C(x)$ by $C'(x)$:

$$C'(x) \quad = \quad \pi(x) - h(x), \tag{4}$$

$$h \quad := \quad \sqrt{-\sqrt{q}C^{\rm gr} + \sqrt{q}\sqrt{(C^{\rm gr})^2 - q^{ab}C^{\rm gr}_a C^{\rm gr}_b}}. \tag{5}$$

$$\pi > 0, \;\; C^{\rm gr} \; < \; 0. \tag{6}$$

$$\{C'(x), C'(y)\} \;=\; 0, \;\; \{h(x), h(y)\} \;=\; 0. \tag{7}$$

The Dirac observables

A Dirac observable is a restriction to the set of zeros of the constraints (the constraint surface) of a function on $\Gamma = \{(q, p, \phi, \pi) \ : \text{as above}\}$

$$f : \Gamma \to \mathbb{R} \tag{8}$$

such that

$$\{f, C_a(x)\} \ = \ \{f, C'(x)\} \ = \ 0. \tag{9}$$

- The vanishing of the first Poisson bracket means, that: f is invariant with respect to the action of the local diffeomorphisms.
- The vanishing of the second Poisson bracket reads

$$\{f, \pi(x)\} \ = \ \{f, h(x)\}. \tag{10}$$

Quantum theory: the formal structure

Quantum states:

$$(\phi, q_{ab}) \mapsto \Psi(\phi, q_{ab}), \tag{11}$$

Quantum fields:

$$\hat{\phi}(x)\Psi(\phi, q_{ab}) = \phi(x)\Psi(\phi, q_{ab}), \qquad \hat{\pi}(x)\Psi(\phi, q_{ab}) = \frac{1}{i}\frac{\delta}{\delta\phi(x)}\Psi(\phi, q_{ab})$$

$$\hat{q}_{ab}(x)\Psi(\phi, q_{ab}) = q_{ab}(x)\Psi(\phi, q_{ab}) \quad \hat{p}^{ab}(x)\Psi(\phi, q_{ab}) = \frac{1}{i}\frac{\delta}{\delta q_{ab}(x)}\Psi(\phi, q_{ab})$$

The quantum constraints:

$$\Psi(\varphi^*\phi, (\varphi^{-1})^* q_{ab}) = \Psi(\phi, q_{ab}) \qquad \text{for every } \varphi \in \text{Diff}(M)$$

$$\hat{C}'\Psi = \left(\hat{\pi}(x) - \hat{h}(x)\right)\Psi, \quad \hat{h}(x) = h(\hat{q}_{ab}, \hat{p}^{ab})(x),$$

The consistency condition

$$[\hat{h}(x), \hat{h}(y)] = 0 \tag{12}$$

General solution to the quantum constraints

The Scalar Constraint

$$\frac{\delta}{\delta\phi(x)}\Psi(\phi, q_{ab}) \;=\; i\hat{h}(x)\Psi(\phi, q_{ab}). \tag{13}$$

We write Ψ as

$$\Psi \;=\; e^{i\int d^3x\hat{\phi}(x)\hat{h}(x)}\psi,\,, \tag{14}$$

$$\Rightarrow \quad \frac{\delta}{\delta\phi(x)}\psi \;=\; 0, \quad \Psi(\phi, q_{ab}) \;=\; e^{i\int d^3x\hat{\phi}(x)\hat{h}(x)}\psi(q_{ab}) \tag{15}$$

The Vector Constraint $\Rightarrow$ the diffeo invariance

$$\Rightarrow \quad \psi(\varphi_* q_{ab}) \;=\; \psi(q_{ab}) \tag{16}$$

for every local diffeomorphism φ. This is general solution.

Quantum Dirac observables

Dirac observable is given by an operator $\mathcal{O}$ such that:

- it is diffeomorphism invariant, and $[\mathcal{O}, \hat{C}'] = 0$

We construct a sufficiently large class.

Consider a linear operator $\hat{L}$ which maps each of the functions $q_{ab} \mapsto \psi(q_{ab})$ into $q_{ab} \mapsto \hat{L}\psi(q_{ab})$.
Define:

$$\mathcal{O}(\hat{L}) = e^{i\int d^3x \hat{\phi}(x)\hat{h}(x)} \hat{L} e^{-i\int d^3x \hat{\phi}(x)\hat{h}(x)} \tag{17}$$

Then

$$[\mathcal{O}(\hat{L}), \hat{C}'(x)] = 0. \tag{18}$$

The operator $\mathcal{O}(\hat{L})$ is diffeomorphism invariant if and only if the operator $\hat{L}$ is. The action on the solutions:

$$\mathcal{O}(\hat{L}) e^{i\int d^3x \hat{\phi}(x)\hat{h}(x)} \psi(q_{ab}) = e^{i\int d^3x \hat{\phi}(x)\hat{h}(x)} \psi'(q_{ab}), \tag{19}$$

$$\psi' = \hat{L}\psi. \tag{20}$$

Semiclassical interpretation of our Dirac observables

Suppose the operator $\hat{L}$ corresponds in the quantum theory to a classical function L defined on the gravitational phase space $\Gamma_{\rm gr}$ and its support is contained in the set on which

$$C^{\rm gr} \ < \ 0. \tag{21}$$

Consider a point $(q_{ab}, p^{ab}, \phi, \pi)$, such that the field ϕ can be, briefly speaking, gauge transformed to zero. In other words, the orbit of the transformations generated by the Deparametrised Scalar Constraints passing through $(q_{ab}, p^{ab}, \phi, \pi)$ contains a unique point $(q'_{ab}, p'^{ab}, \phi', \pi')$ such that

$$\phi' = 0. \tag{22}$$

Define a function $\mathcal{O}(L)$ by the following composition:

$$(q_{ab}, p^{ab}, \phi, \pi) \ \mapsto \ (q'_{ab}, p'^{ab}, 0, \pi') \ \mapsto \ L(q'_{ab}, p'^{ab}, 0, \pi). \tag{23}$$

Then, the quantum operator $\mathcal{O}(\hat{L})$ corresponds to $\mathcal{O}(L)$.

Dynamical evolution of the observables

The Dirac observables we have defined are often called “partial” (Rovelli, Dittrich, ...) or “evolving”. The correct name is relational.

A natural group of automorphisms labelled by the functions ϕ_0 in the algebra of the Dirac observables:

$$\mathcal{O}(\hat{L}) \ \mapsto \ \mathcal{O}_{\phi_0}(\hat{L}) \ = \ \mathcal{O}(e^{-i\int d^3x\phi_0(x)\hat{h}(x)}\hat{L}e^{i\int d^3x\phi_0\hat{h}(x)}) \tag{24}$$

$$\text{the diffeo invariance} \ \ \rightarrow \ \ \phi_0 = \text{const} \tag{25}$$

This 1-dimensional group of automorphisms encodes the dynamics whose generator is the physical hamiltonian

$$\frac{d}{d\phi_0}\mathcal{O}_{\phi_0}(\hat{L}) \ = \ -i[\hat{h}_{\rm phys}, \mathcal{O}_{\phi_0}(\hat{L})] \tag{26}$$

$$\hat{h}_{\rm phys} \ := \ \int d^3x\hat{h}(x) \tag{27}$$

The physical hamiltonian

The physical Hamiltonian should be an exact implementation of the heuristic formula

$$\hat{h}_{\rm phys} \;=\; \int d^3x \sqrt{-\sqrt{\hat{q}}\hat{C}^{GR} + \sqrt{\hat{q}}\sqrt{(\hat{C}^{GR})^2 - \hat{q}^{ab}\hat{C}^{GR}_a\hat{C}^{GR}_b}}. \qquad (28)$$

We remember however, that the operator will be applied to diffeomorphism invariant states whereas the operator $\hat{C}_a^{\rm GR}$ should generate the diffeomorphisms. Therefore, assuming the suitable choice of the ordering, the physical Hamiltonian is

$$\hat{h}_{\rm phys} \;=\; \int d^3x \sqrt{-2\sqrt{\hat{q}}\hat{C}^{GR}}, \qquad (29)$$

where we also took into account,

$$\hat{C}^{GR} \;<\; 0. \qquad (30)$$

This result coincides with that of Rovelli-Smolin 1993.

The Hilbert product: $\mathcal{H}_{\rm phys}$

Suppose we have a sesquilinear scalar product for diffeomorphism invariant functions $q_{ab} \mapsto \psi(q_{ab})$:

$$(\psi|\psi')_{\rm gr,diff}, \tag{31}$$

and the corresponding Hilbert space by $\mathcal{H}_{\rm gr,diff}$.
We use it to define the "physical" Hilbert product in the space of solutions:

$$\left(e^{i\int\hat{\phi}\hat{h}}\psi \,|\, e^{i\int\hat{\phi}\hat{h}}\psi'\right)_{\rm phys} \;:=\; (\psi|\psi)_{\rm gr,diff}. \tag{32}$$

The resulting Hilbert space $\mathcal{H}_{\rm phys}$ is "physical", and its elements are the physical states.

An exact structure we need

In summary, to construct the quantum model we will need:

- A Hilbert space $\mathcal{H}_{\rm gr,diff}$ of diffeomorphism invariant functions (distributions) defined on the space of the 3-metric tensors
- Operators in $\mathcal{H}_{\rm gr,diff}$ which have some geometric interpretation
- A physical Hamiltonian operator $\hat{h}_{\rm phys} = \int d^3x \hat{h}(x)$ defined in some domain in $\mathcal{H}_{\rm gr,diff}$
- The operators $\hat{h}(x) = \sqrt{-2\widehat{\sqrt{q}C^{\rm gr}}}$, such that:
  - on the one hand $\hat{h}(x)$ can not preserve $\mathcal{H}_{\rm gr,diff}$
  - on the other hand should be self-adjoint to define the square roots, so it should be defined on its own in another Hilbert space $\mathcal{H}_{{\rm gr,diff},x}$
  - we also need to ensure $[\hat{h}(x), \hat{h}(y)] \ = \ 0$

Remarkably, all those structures can be constructed within the LQG.

The Ashtekar-Barbero variables

On the 3-manifold M,

$$A = A_a^i(x)\tau_i \otimes dx^a, \quad P = P_i^a(x)\tau^i \otimes \frac{\partial}{\partial x^a} \tag{33}$$

x^a are local coordinates in M, $\tau_1, \tau_2, \tau_3 \in$ su(2) such that

$$\eta(\tau_i, \tau_j) := -2\mathrm{Tr}(\tau_i \tau_j) = \delta_{ij}, \tag{34}$$

and τ^1, τ^2, τ^3 is the dual basis. The relation with the intrinsic/extrinsic geometry e_a^i / K_a^i

$$A_a^i = \Gamma_a^i + \gamma K_a^i, \quad P_a^i = \frac{1}{16\pi G\gamma} e_a^j e_b^k \epsilon^{abc} \epsilon_{ijk}, \tag{35}$$

where $\epsilon^{123} = 1 = \epsilon_{123}$ and ϵ^{abc}, ϵ_{abc} are completely antisymmetric. The Poisson bracket

$$\{A_a^i(x),\ P_j^b(y)\} = \delta(x,y)\delta_a^b \delta_j^i, \ \ \{A, A\} = \{P, P\} = 0. \tag{36}$$

The holonomy-flux algebra

Parallel transport functionals of A and fluxes of P

$$A_e = \text{Pexp}\int_e -A, \quad P_{S,f} = \frac{1}{2}\int_S f^i P_i^a \epsilon_{abc} dx^b \wedge dx^c \tag{37}$$

Define a Lie algebra with the Poisson bracket. Via

$$\{\cdot,\cdot\} \mapsto \frac{1}{i\hbar}[\cdot,\cdot]$$

give rise to the quantum holonomy-flux *-algebra **HF**.
On **HF** there is a unique Diff(M) invariant state

$$\omega : \mathbf{HF} \to \mathbb{C}.$$

Using GNS it defines the quantum representation space for the parallel transport functionals and the flux functionals: operators $\hat{A}_e{}^A_B,\ \hat{P}_{S,f}$ in the Hilbert space $\mathcal{H}_{\text{gr}}$.

Skech of the construction

To define the operators $\hat{h}(x)$ we introduce for each $x \in M$, a Hilbert space $\mathcal{H}_{\mathrm{gr,diff},x}$. We use the rigging map $\eta_{\mathrm{diff},x} \; = \; \text{``}\int_{\mathrm{Diff}(M,x)}\text{''}$ available in LQG

$$\mathrm{H}_{\mathrm{gr}} \ni \psi \mapsto \eta_{\mathrm{diff},x}(\psi) \tag{38}$$

where $\eta_{\mathrm{diff},x}(\psi)$ is a linear functional on (some domain) in $\mathcal{H}_{\mathrm{gr}}$, and the new scalar product

$$\big(\,\eta_{\mathrm{diff},x}(\psi)|\eta_{\mathrm{diff},x}(\psi')\,\big)_{\mathrm{gr,diff},x} \;:=\; \langle \eta_{\mathrm{diff},x}(\psi), \psi' \rangle \,. \tag{39}$$

In that space the square root $\sqrt{-2\sqrt{q}(x)C^{\mathrm{gr}}(x)}$ can be defined using the LQG framework, the (symmetrised) operators

$$\widehat{C^{\mathrm{gr}}}(x) \;=\; \sum_{y\in M} \delta(x,y)\hat{C}^{\mathrm{gr}}_y, \quad \widehat{\sqrt{q}}(x) \;=\; \sum_{y\in M} \delta(x,y)\widehat{\sqrt{q}}_y, \tag{40}$$

$$\hat{h}(x) = \widehat{\sqrt{-2\sqrt{q}(x)C^{\mathrm{gr}}(x)}} \;=\; \sum_{y\in M} \delta(x,y)\sqrt{-2\sqrt{\widehat{\sqrt{q}}_y}\hat{C}^{\mathrm{gr}}_y\sqrt{\widehat{\sqrt{q}}_y}}. \tag{41}$$

Bonus: simplifications

- There exists even simpler construction of the physical hamiltonian. In the Euclidean (++++) case, we have

$$-2\sqrt{q}(x)C^{\mathrm{gr}}(x) \;=\; -8\pi G\epsilon_{ijk}P^a_i(x)P^b_j(x)F^k_{ab}(x), \tag{42}$$

  the original Ashtekar form of the scalar constraint. The corresponding quantum operator is defined without the Thiemann trick. And takes much simpler form.

- Moreover, to that operator we can apply the results of *L, Marolf, Int. J. Mod. Phys. D7, 299 (1998)* to conclude that the corresponding operators $\hat{h}(x)$ satisfy the required

$$[\hat{h}(x), \hat{h}(y)] \;=\; 0. \tag{43}$$

We do have quantum gravity

- ...“but there is no quantum gravity”... Wrong!
- Within the model presented in this talk we can consider
  - the Big-Bang
  - the Hawking evaporation conjecture

  with all the degrees of freedom as exactly as in the toy models of LQC.
- In this model the graphs of LQG are not preserved by the dynamics, therefore it does not predict we all live on a fixed lattice.

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