Spin Foams from the LQG point of view

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Plan

- The goal:
  a generalization of the EPRL spin foam model to all the spin-network states
- What for?
  For the compatibility with LQG: either we generalize SFs or we restrict/modify LQG to the piecewise linear spaces
- Strategy: We go on with the EPRL construction:
  - sufficiently general spin-foam: yes
  - characterization of vertex: yes
  - characterization of spin-foam: yes
  - vertex amplitude: yes
  - the scheme of the SF models of gravity: yes
  - Barrett-Crane vertex: yes
  - EPRL vertex: yes
- Technical result: injectivity of

  $$\text{SU(2) invariants} \quad \rightarrow \quad \text{EPRL SU(2)}\times\text{SU(2) invariants}$$
- The limits

Papers

- classic: Reisenberger 1994, Reisenberger-Rovelli 1997, Barrett-Crane 1998, Yetter 1998, Barrett 1998, Reisenberger 1998, Baez 2000, Perez 2003
- newer: Bianchi-Modesto-Rovelli-Speciale 2006, Alesci-Rovelli 2007, Engle-Livine-Pereira-Rovelli 2008, also Freidel-Krasnov 2008 (sorry for not considering that paper here!)
- our paper: KKL 2009

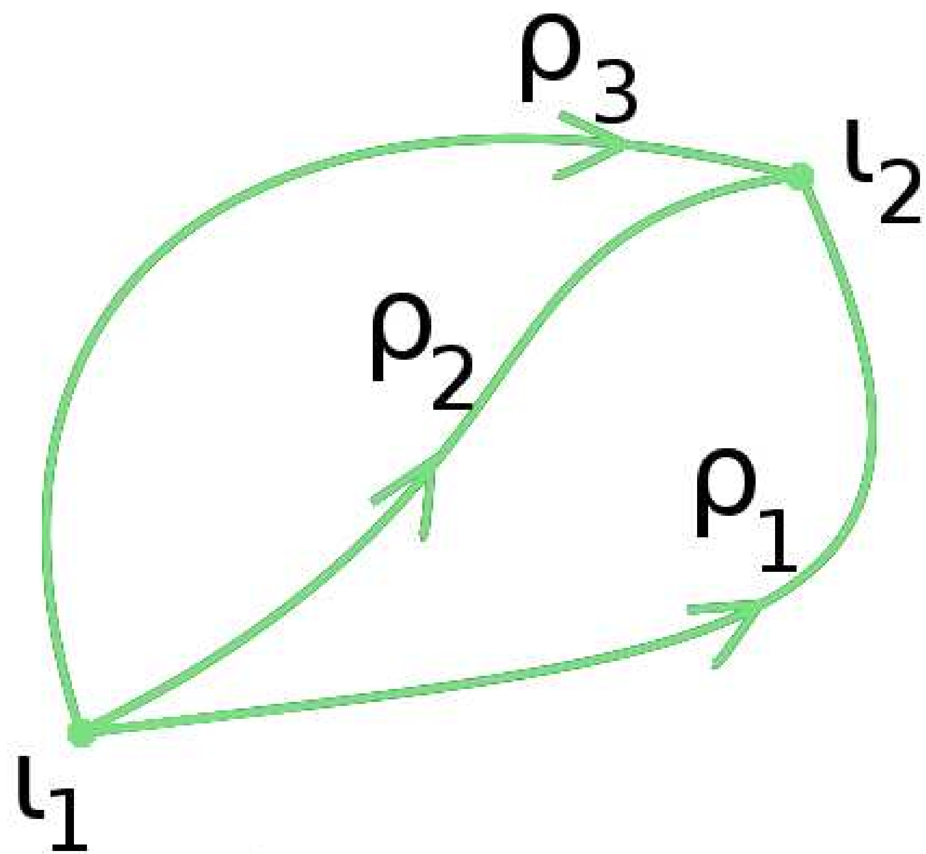
Diffeomorphism invariant theories of connections

- Given: a 3-manifold Σ, a Lie group G, its Lie algebra g, and the set $\mathcal{A}$ of the g valued differential one-forms A (connections) on Σ.
- Parallel transport defined by $A \in \mathcal{A}$ along a finite curve e in Σ:
  $$A(e) \ := \ \mathrm{Pexp} \int_e -A$$
- The space $\mathrm{Cyl}(\mathcal{A})$ of the cylindrical functions: a cylindrical $\Psi : \mathcal{A} \to \mathbb{C}$, is defined by a finite set of finite, oriented curves $e_1, ..., e_n$ in Σ and by a continues function $\psi : G^n \to \mathbb{C}$,
  $$\Psi(A) \ := \ \psi(A(e_1), ..., A(e_n)).$$
- There is a natural, diffeomorphism invariant integral: $\int : \mathrm{Cyl}(\mathcal{A}) \ \to \ \mathbb{C}$
- Defines the scalar product: $(\Psi|\Psi') \ = \ \int \bar{\Psi}\Psi'$
- the kinematical Hilbert space for diffeomorphism invariant theories of connections: $(\ \mathrm{Cyl}(\mathcal{A}), \ (\ |\ )\ )$, the completion
- LQG: $G =$ SU(2), SFM: $G =$SU$(2)\times$SU(2) reduced to SU(2) by the simplicity constraints
- The cylindrical functions can be constructed from spin-networks embedded in Σ: spin-network states

The spin-network states

A graph embedded in Σ, irreducible representations ρ_I of a group G in Hilbert spaces $\mathcal{H}_I$, intertwiners: $\iota_1 \in \mathrm{Inv}(\rho_1 \otimes \rho_2 \otimes \rho_3) \subset \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$,

$$\iota_2 \in \mathrm{Inv}(\rho_1^* \otimes \rho_2^* \otimes \rho_3^*) \subset \mathcal{H}_1^* \otimes \mathcal{H}_2^* \otimes \mathcal{H}_3^*$$

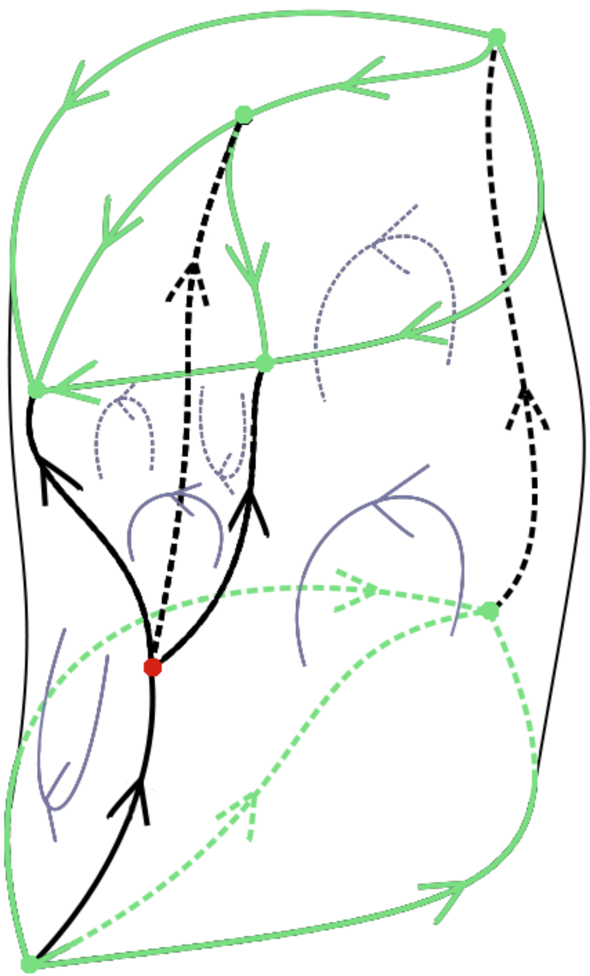


$$\Psi(A) := {\rho_1}^{C_1}_{B_1}(A(e_1)){\rho_2}^{C_2}_{B_2}(A(e_2)){\rho_3}^{C_3}_{B_3}(A(e_3)){\iota_1}^{B_1B_2B_3}{\iota_2}_{C_1C_2C_3}$$

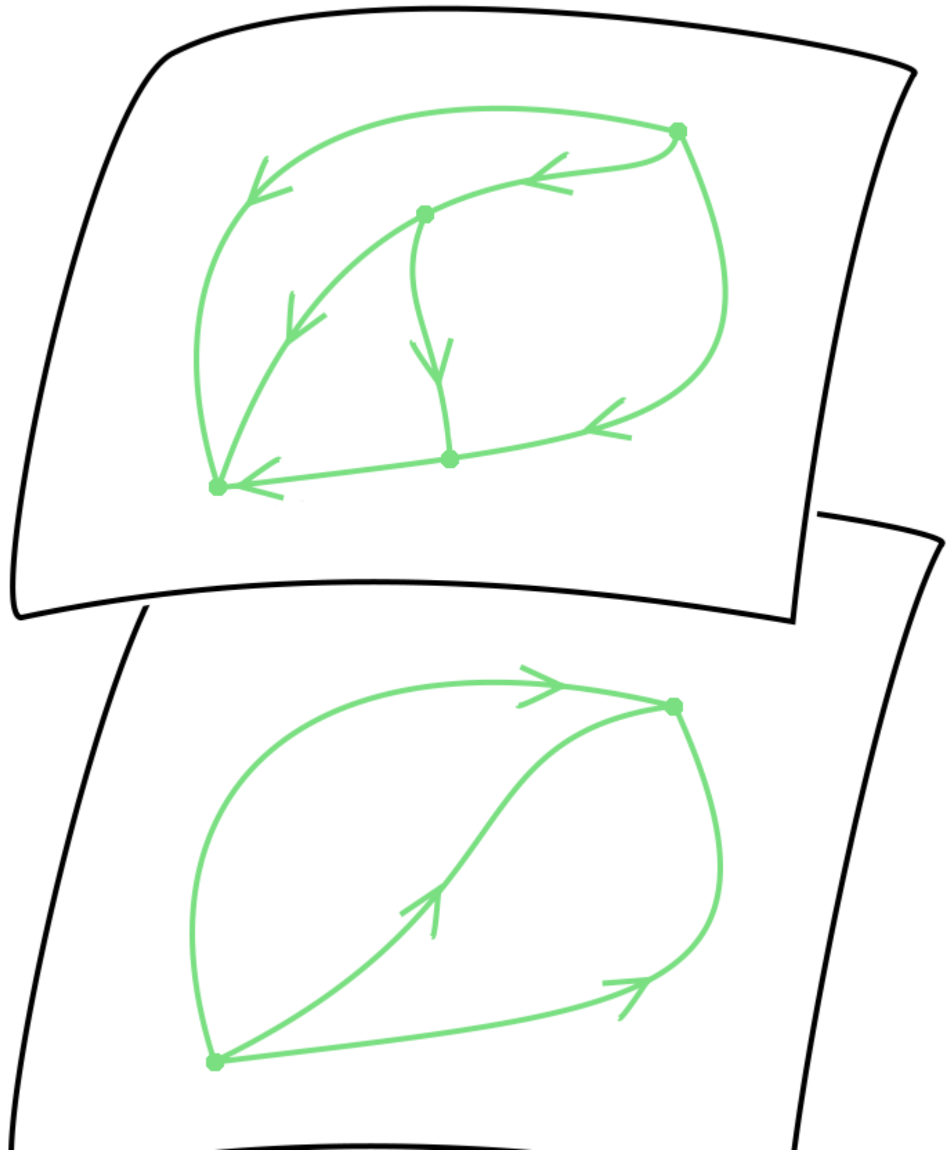
spin-network trace: $\Psi(0) = {\iota_1}^{C_1C_2C_3}{\iota_2}_{C_1C_2C_3}$

The idea of spin-network state evolution

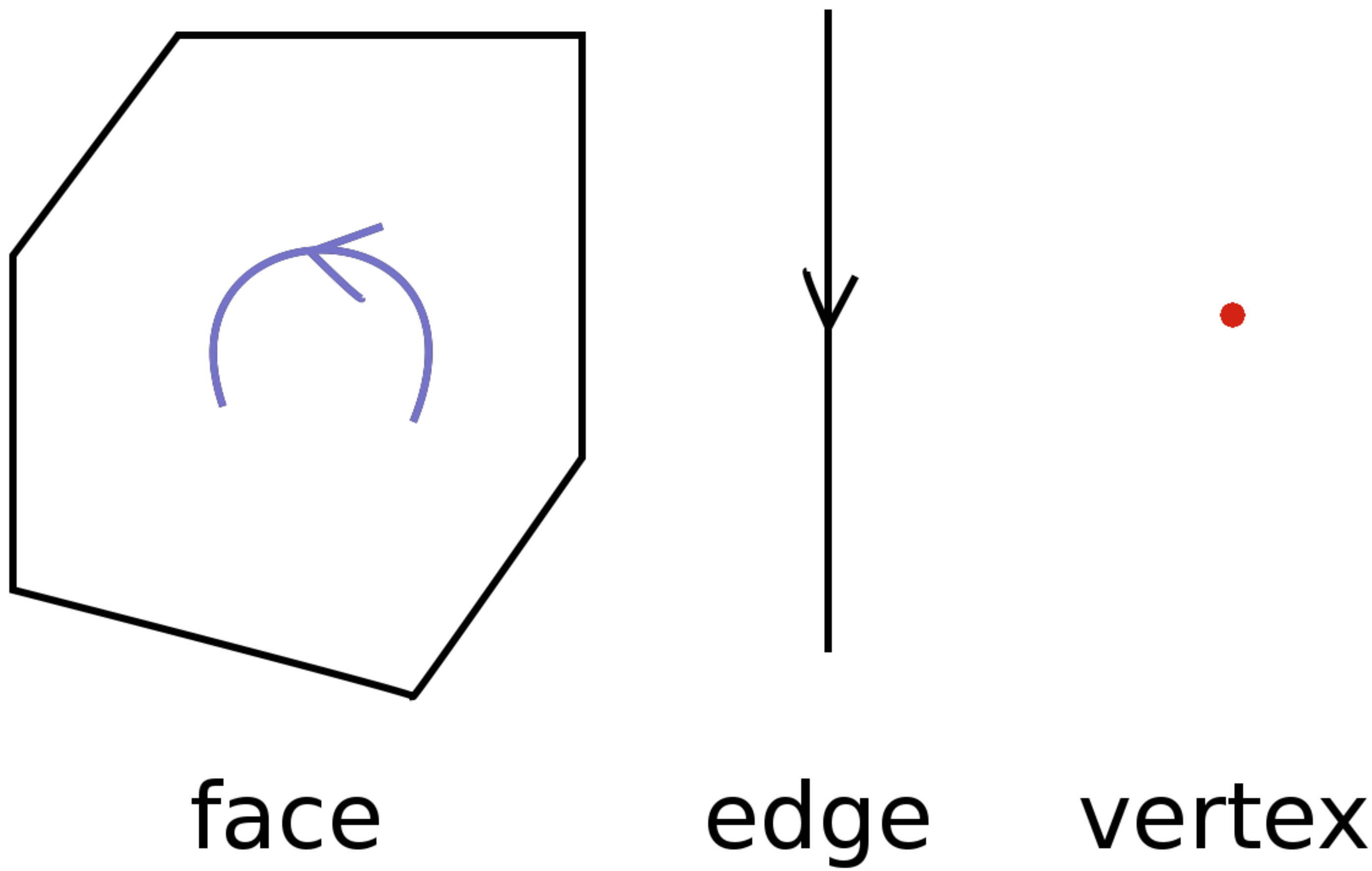
a history of a graph



the initial and final graphs



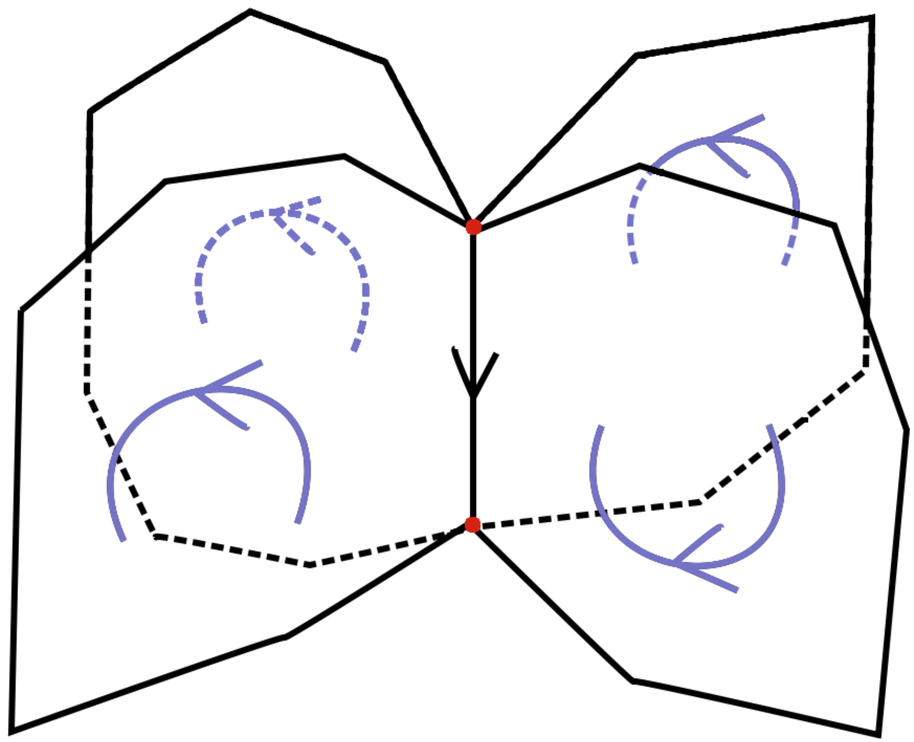
Foam: elements



The circle shows the orientation

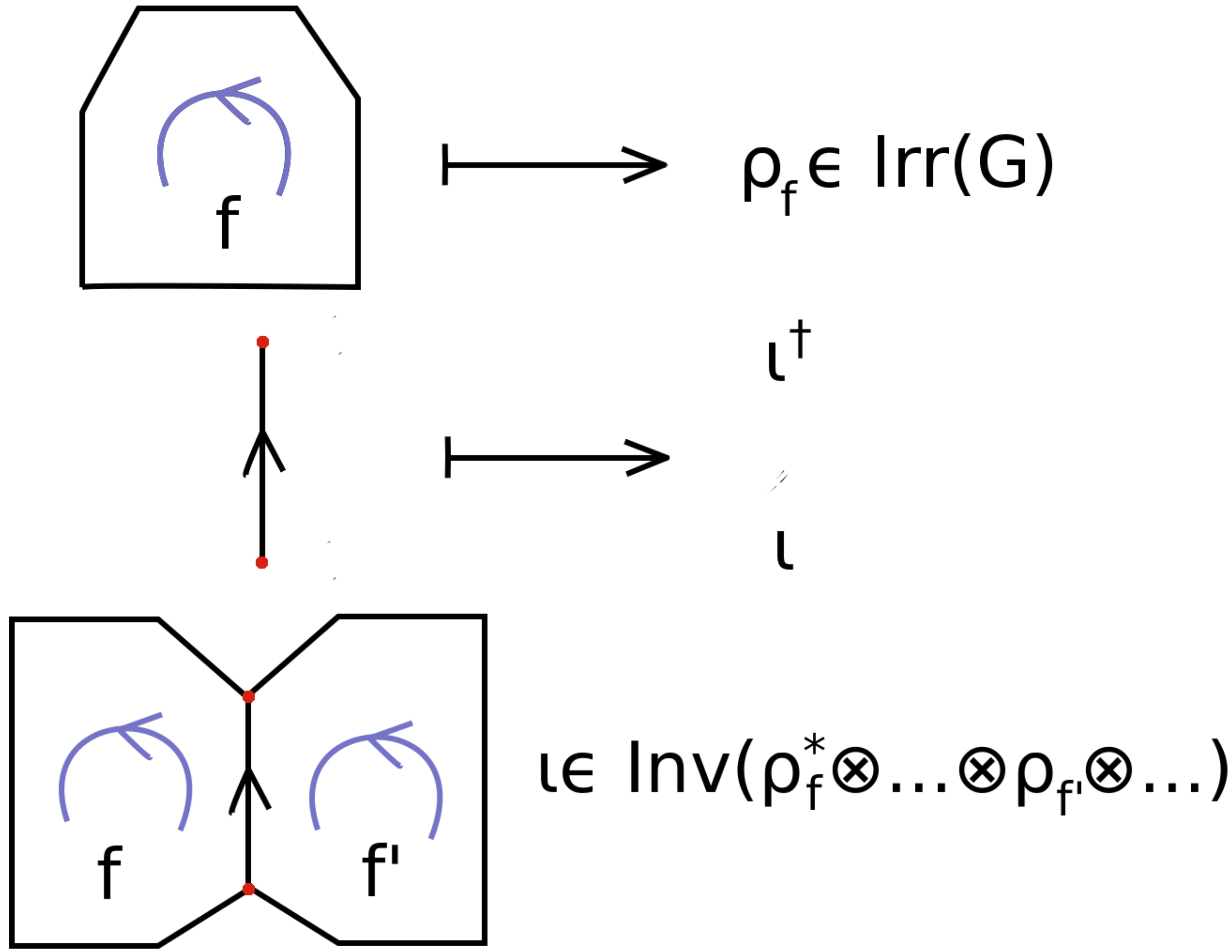
Foam: glueing

Faces are glued with other faces along the edges.



Mathematically, a foam is a linear 2-cell complex with boundary

Spin-foam: coloured foam



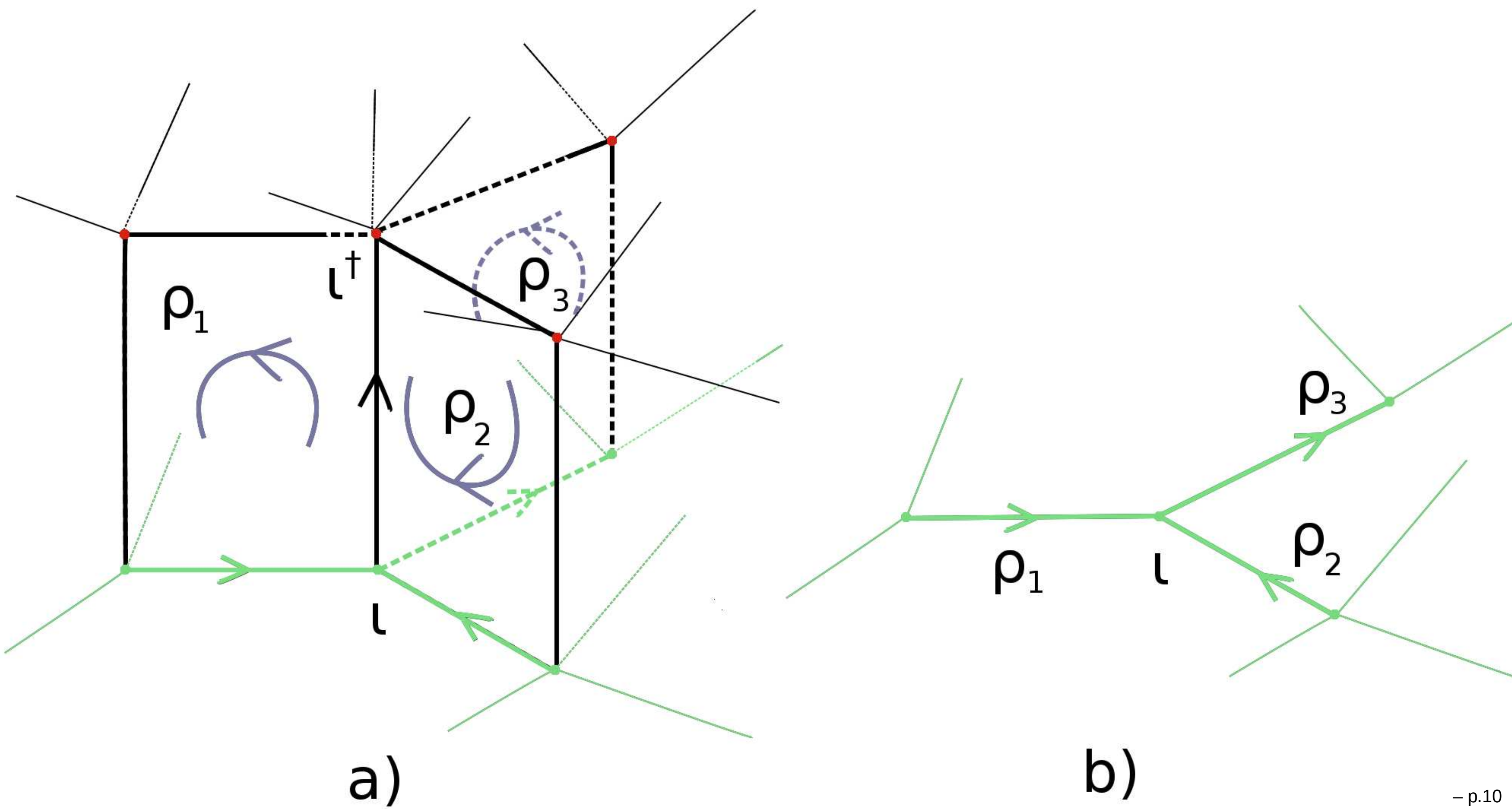
$$\iota \ : \ \mathcal{H}_f \otimes \ldots \ \longrightarrow \mathcal{H}_{f'} \otimes \ldots$$

Hermitian adjoint $\iota^\dagger \ : \ \mathcal{H}_{f'} \otimes \ldots \ \longrightarrow \mathcal{H}_f \otimes \ldots$

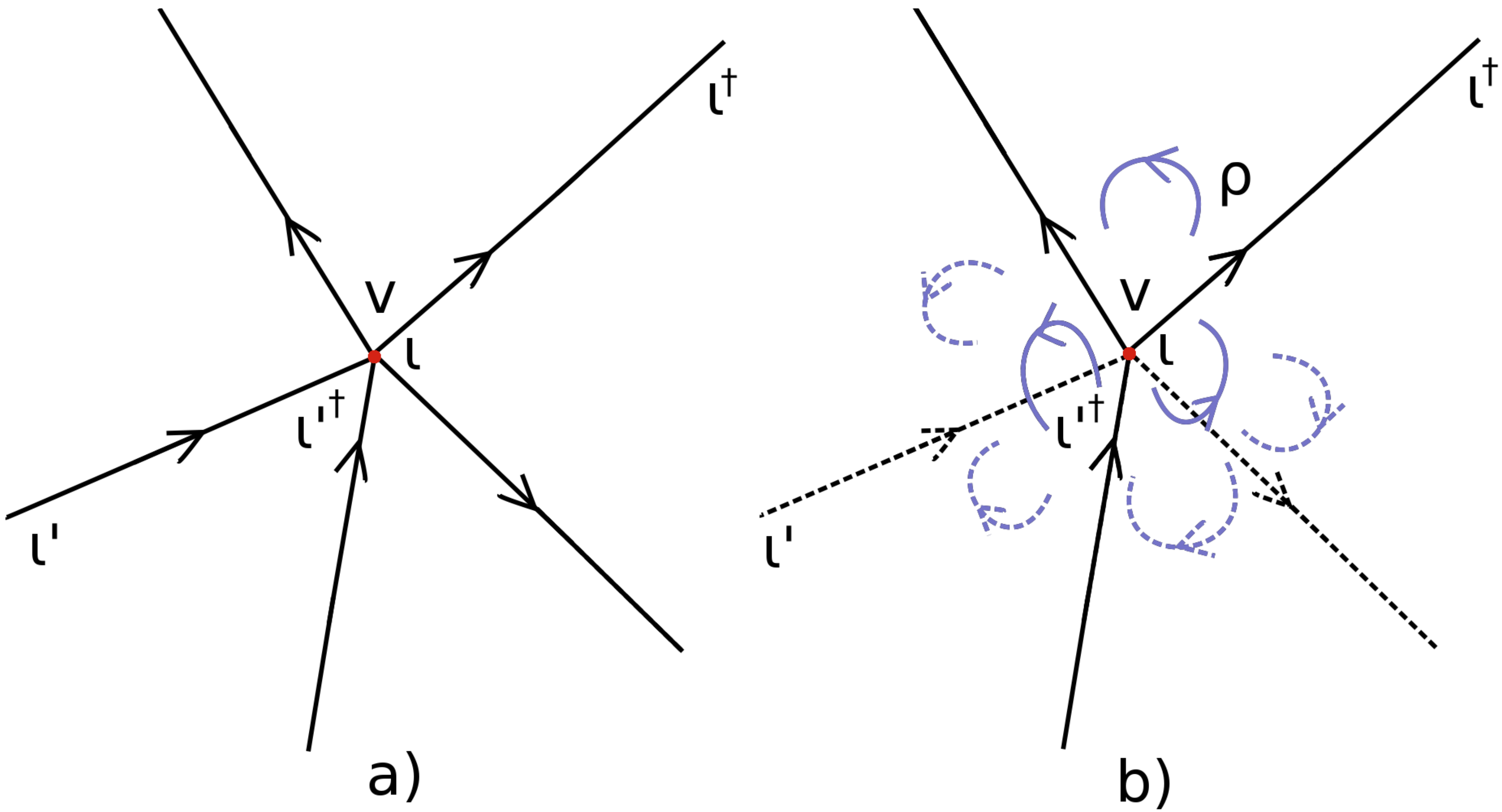
Spin-network induced on the boundary

Spin-foam with boundary

Induced spin-network

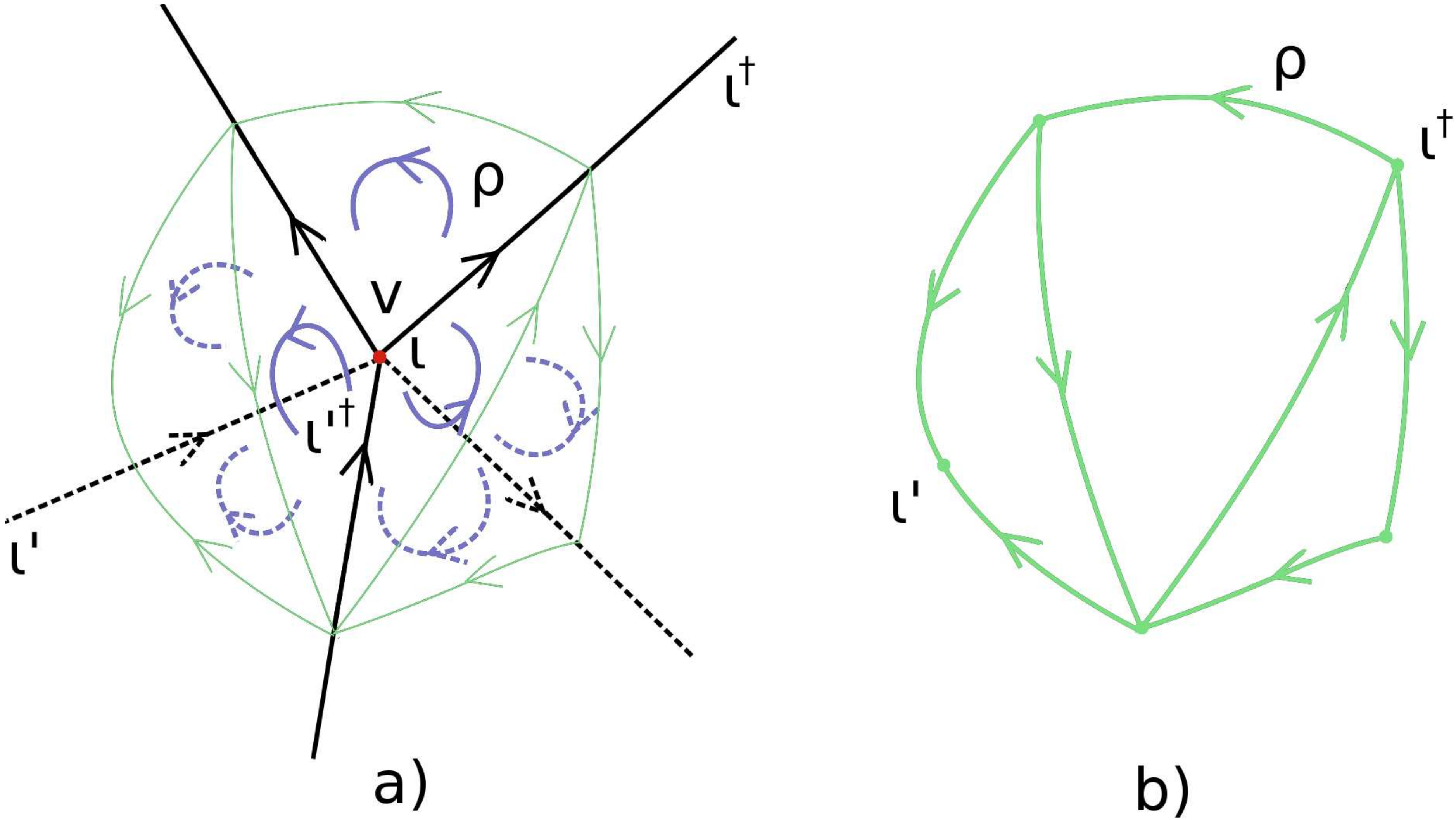


Faces meetting at a vertex



a) Edges b) Every face meeting v contains exactly two of the edges

The vertex structure



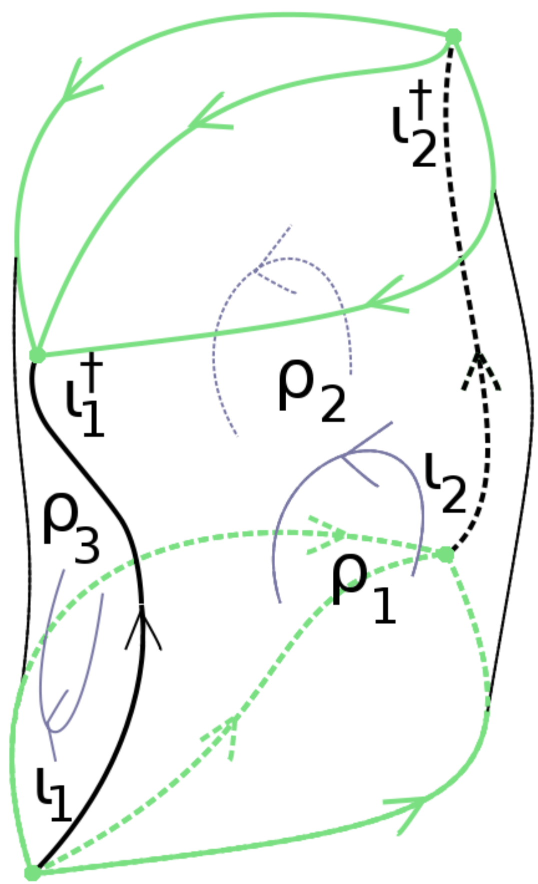
Neigbourhood U of v

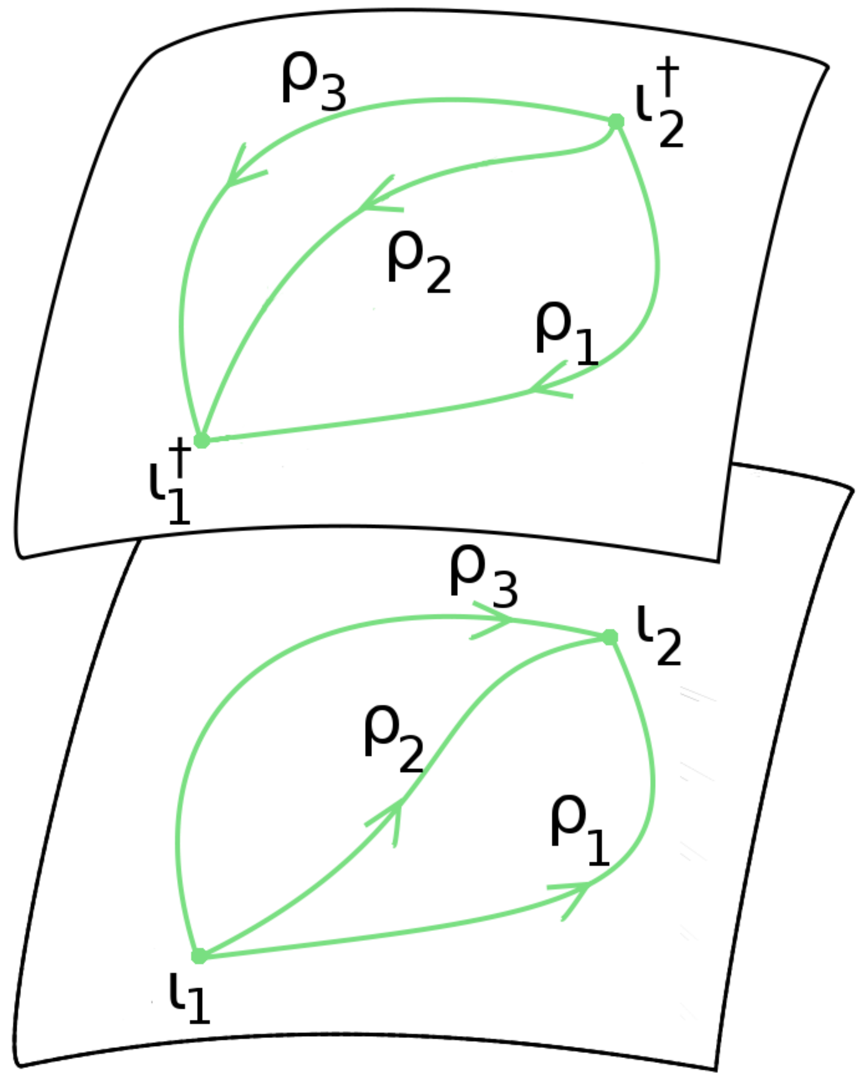
Spin-network induced on ∂U,

$a(v) := \overline{\mathrm{trace}}$ of the spin-network $= \overline{\Psi(0)}$

A no vertex spin-foam

The spin-foam structure: A general spin-foam is glued from vertex neghbourhouds and no vertex spin-foams. A no vertex spin-foam:





The scheme of (Euclidean) SF models

- $G =$SU$(2)\times$SU(2)
- the kinematical Hilbert space: spanned by the spin-network states, the spin-networks embedded in a given 3-manifold Σ
- histories of spin-network states: embedded spin-foams in $\Sigma \times \mathbb{R}$
- the Spin Foam aplitude:

$$\prod_{\text{internal vertexes } v} a(v) \cdot \prod_{\text{faces } f} \dim(\rho_f) \cdot \prod_{\text{boundary vertexes } v} \|\iota_v\|$$

- Simplicity constraints imposed on the intertwiners ι:
  - the Barrett-Crane intertwiner – we know that this choice was too restrictive, but since it was the obliging interwtiner untill recently, it is good to know it is easily generalizated to an arbitrary spin-foam
  - the EPRL intertwiner – likelly to be the right, even easier to be generalized
- Summing the SF amplitudes: with respect to elements of orthonormal basis in the subspaces of simple (EPRL) intertwiners

The Barrett-Crane intertwiner

- a representation of SU$(2)\times$SU(2) is a pair of SU(2) representations
$$(\rho_{j^+}, \rho_{j^-}) \text{ in } \mathcal{H}_{j^+} \otimes \mathcal{H}_{j^-}$$
- an intertwiner is $\iota \;\in\; \mathrm{Inv}\left((\rho_{j_1^+}, \rho_{j^-}) \otimes ... \otimes (\rho_{j_n^+}, \rho_{j_n^-})\right) \;=$
$$\mathrm{Inv}(\rho_{j_1^+} \otimes ... \otimes \rho_{j_n^+}) \otimes \mathrm{Inv}(\rho_{j_1^-} \otimes ... \otimes \rho_{j_n^-})$$
- Briefly speaking, the BC intertwiner is... the identity map
$$\mathrm{Inv}(\rho_{j_1^-} \otimes ... \otimes \rho_{j_n^-}) \;\rightarrow\; \mathrm{Inv}(\rho_{j_1^+} \otimes ... \otimes \rho_{j_n^+})$$
provided $j_1^+ = j_1^- =: j_1, \quad ..., \quad j_n^+ = j_n^- =: j_n$
- Exactly, let $\iota_1, ..., \iota_k \in \mathrm{Inv}(\rho_{j_1} \otimes ... \otimes \rho_{j_n})$ be any orthonormal basis.
$$\iota_{\mathrm{BC}}{}^{A_1^+ ... A_n^+ A_1^- ... A_n^-} \;=\; \sum_{i=1}^k \iota_i^{A_1^+ ... A_n^+} \otimes \iota^\dagger_{iB_1^- ... B_n^-} \epsilon_{j_1}^{B_1^- A_1^-} ... \epsilon_{j_n}^{B_n^- A_n^-}$$
where $\epsilon_j \;\in\; \mathrm{Inv}(\mathcal{H}_j \otimes \mathcal{H}_j)$.
- What is ι_{BC} good for?
$$(\mathcal{O} \otimes 1)\,\iota_{\mathrm{BC}} \;=\; \pm(1 \otimes \mathcal{O})\,\iota_{\mathrm{BC}}$$
whenever $(\mathcal{O} \otimes 1)\,\epsilon_{j_1} \otimes ... \otimes \epsilon_{j_n} \;=\; \pm(1 \otimes \mathcal{O})\,\epsilon_{j_1} \otimes ... \otimes \epsilon_{j_n}$
- Given a graph colored by SU(2) representations ρ_{j_I}, the coresponding Hilbert space of the BC spin-networks is either 0 or 1

The EPRL intertwiner

- a representation of SU$(2)\times$SU(2) is a pair of SU(2) representations (ρ_{j^+}, ρ_{j^-}) in $\mathcal{H}_{j^+} \otimes \mathcal{H}_{j^-}$
- an intertwiner is $\mathrm{EPRL} \in \mathrm{Inv}\left((\rho_{j_1^+}, \rho_{j^-}) \otimes ... \otimes (\rho_{j_n^+}, \rho_{j_n^-})\right) =$

$$\mathrm{Inv}(\rho_{j_1^+} \otimes ... \otimes \rho_{j_n^+}) \otimes \mathrm{Inv}(\rho_{j_1^-} \otimes ... \otimes \rho_{j_n^-})$$

$$\mathrm{EPRL}^{A_1^+ ... A_n^+ A_1^- ... A_n^-}(\iota) = \sum_{i=1}^{k} \iota_i^{A_1^+ ... A_n^+} \otimes \iota^{\dagger}_{i B_1^+ ... B_n^+} c_1^{B_1^+ A_1^- D_1} ... c_n^{B_n^- A_n^- D_n} \iota_{D_1 ... D_n}$$

- where:
  - $c_i \in \mathrm{Inv}(\mathcal{H}_{j_i^+} \otimes \mathcal{H}_{j_i^-} \otimes \mathcal{H}_{k_i})$,
  - $j_i^+ = \frac{1}{2}|1+\gamma|k_i, \quad j_i^- = \frac{1}{2}|1-\gamma|k_i$
  - $\iota \in \mathrm{Inv}(\rho_{k_1^*} \otimes ... \otimes \rho^*_{k_n})$.

The injectivity of $\mathcal{I} \mapsto \iota_{\mathrm{EPRL}}$

The result: For every n-valent vertex, the EPRL map

$$\mathrm{SU}(2) \quad \text{intertwiners} \quad \rightarrow \quad \mathrm{EPRL} \quad \text{intertwiners}$$

is injective. The proof splits into two cases:

- $\gamma \geq 1$ case KKL 2009
  - only $j^+ \; = \; j^- \; + \; k$ is used
  - the key observation: the map

    $$\mathcal{H}_{j^-} \otimes \mathcal{H}_k \; \rightarrow \; \mathcal{H}_{j^-+k}$$

    does not kill simple tensor products
- $0 \; < \; \gamma \; < \; 1$ case Kaminski
  - not only $j^- \; + \; j^+ \; = \; k$
  - the full $j^\pm \; = \; \frac{1}{2}(1 \pm \gamma)k$ was used

The sketch of the proof for $\gamma < 1$

- given $\iota \;\in\; \mathrm{Inv}(\rho_{k_1} \otimes \ldots \otimes \rho_{k_n})$
- let $P_{k_{12},k_1k_2}\iota \neq 0$ be the projection onto the lowest possible k_{12}, where $P_{k_{12},k_1k_2} : \mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2} \;\to\; \mathcal{H}_{k_{12}}$
- we find $\iota' \;\in\; P_{j^-_{12},j^-_1 j^-_2} \otimes P_{j^+_{12},j^+_1 j^+_2} \mathrm{Inv}(\rho_{j^-_1} \otimes \ldots \otimes \rho_{j^-_n}) \otimes \mathrm{Inv}(\rho_{j^+_1} \otimes \ldots \otimes \rho_{j^+_n})$ such that

$$j^-_{12} + j^+_{12} \;=\; k_{12},$$

and

$$(\mathrm{EPRL}(\iota)\,|\,\iota') \;\neq\; 0.$$

The limits

- $\gamma \to \pm\infty$. Exists at each level: the Holst action converges to the Palatini action, $j^+ = j^-$, $k = 0$, the EPRL intertwiner converges to the BC intertwiner, all the EPRL derivation converges to a finite limit.
- The limit that can not be extended to the entire derivation although the Holst action does converge perfectly well to the self dual action is

$$\gamma = \pm 1. \tag{1}$$

However the EPRL intertwiner has a limit in that case:

$$j^{\mp} = 0, k = j^{\pm},$$

and moreover

$$\mathrm{EPRL}(\iota) = \iota,$$

where ι is an arbitrary SU(2) intertwiner, and the amplitude turns into the SU(2) BF amplitude. So the limit theory is the SU(2) BF theory. Strange: the self dual action still defines the same Einstein's (Euclidean) gravity.

Limits (continued)

- The limit in which the Holst action is no longer equivalent to the Palatini action and (upon the rescaling by γ) defines the SU(2)$\times$SU(2) BF theory, is

$$\gamma \, = \, 0 \, . \tag{2}$$

Then

$$j^{+} \, = \, j^{-} , \quad k \, = \, j^{+} \, + \, j^{-} \tag{3}$$

but, quite surprisingly, the EPRL theory does not resemble the SU(2)$\times$SU(2) theory at all.

THANK YOU