New results: Operator Spin Foams, SL(2,C)

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papers:

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arXiv:1010.4787

Wojciech Kamiński arXiv:1010.5384

Plan

- Operator spin foams: a tool using the structure of the spin foams
  - definition
  - the equivalence relation
  - the glueing
  - the contraction at vertices
  - emergence of the face and the boundary edge amplitudes as consistency conditions
- Operator spin foam models of the maximal symmetry
- Application to the EPRL intertwiners
- The door to the SL(2,C) case opened with the finiteness (and understanding) of the evaluation.

Operator spin foam: definition

G - a given compact Lie group, (κ, ρ, P) - operator spin foam:

- κ - 2-cell complex (set of polygons, glueings along sites, no self-glueing)
- ρ - coloring of the faces with the irreducuble representations,

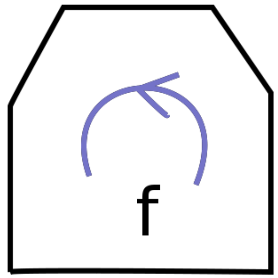
$$f \;\mapsto\; \rho_f \tag{1}$$

- P - coloring of each in ternal (contained in at least two faces) edge with operators

$$e \;\mapsto\; P_e \tag{2}$$

$$P_e \;:\; \mathcal{H}_e \;\rightarrow\; \mathcal{H}_e \tag{3}$$

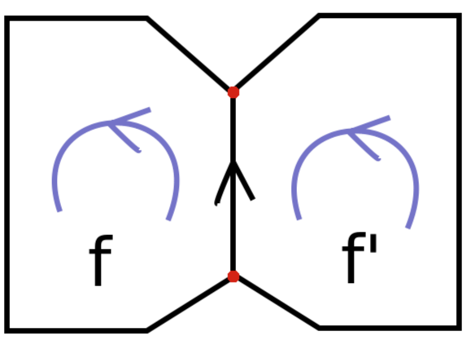
Operators spin foam: definition



$$\longmapsto \quad \rho_f \in \mathrm{Irr}(G)$$

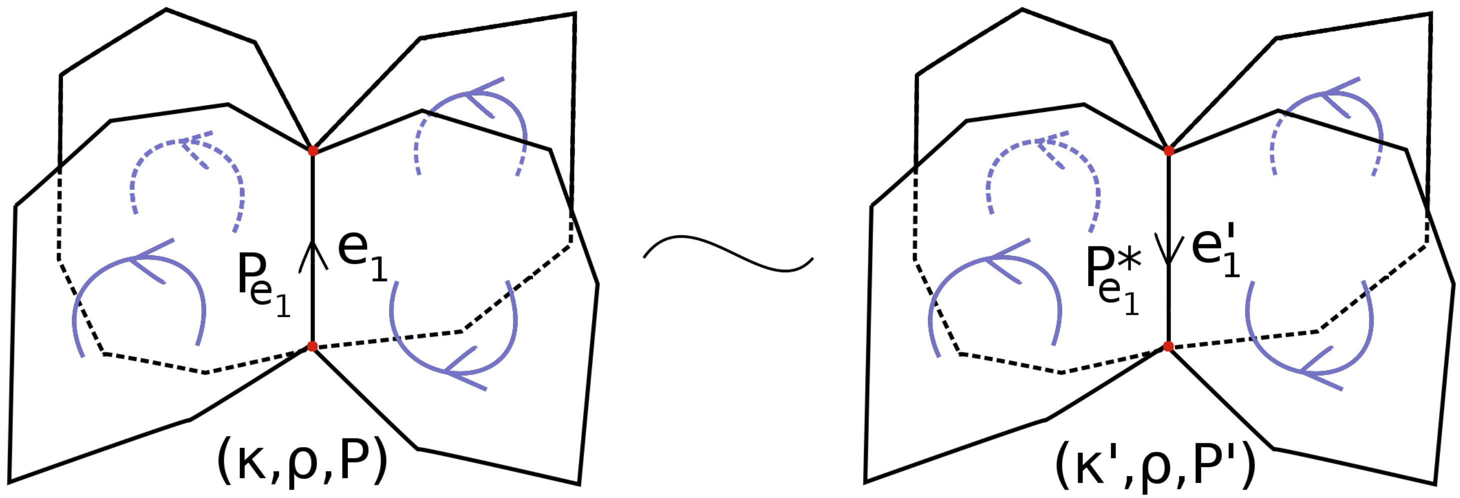
$$\longmapsto \quad P_e$$

$$P_e : \mathcal{H}_e \to \mathcal{H}_e, \qquad \mathcal{H}_e^* = \text{algebraic dual} \tag{4}$$



$$\mathcal{H}_e = \mathrm{Inv}(\mathcal{H}^*_{\rho_f} \otimes \ldots \otimes \mathcal{H}_{\rho_{f'}} \otimes \ldots)$$

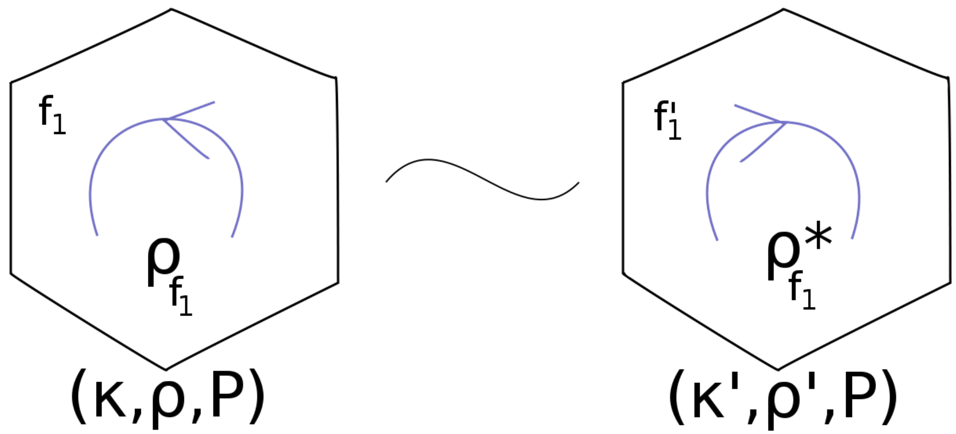
Equivalence relation: edge reorientation



Given an operator spin foam (κ, ρ, P), the following operator spin foam (κ', ρ, P') is considered equivalent:

- κ' obtained from κ by flipping an edge: $e_1' = e_1^{-1}$.
- $$P'_{e_1'} = P^*_{e_1},$$
- the remaining colorings unchanged: $P_e' = P_e$ for $e \neq e_1'$, $\rho' = \rho$

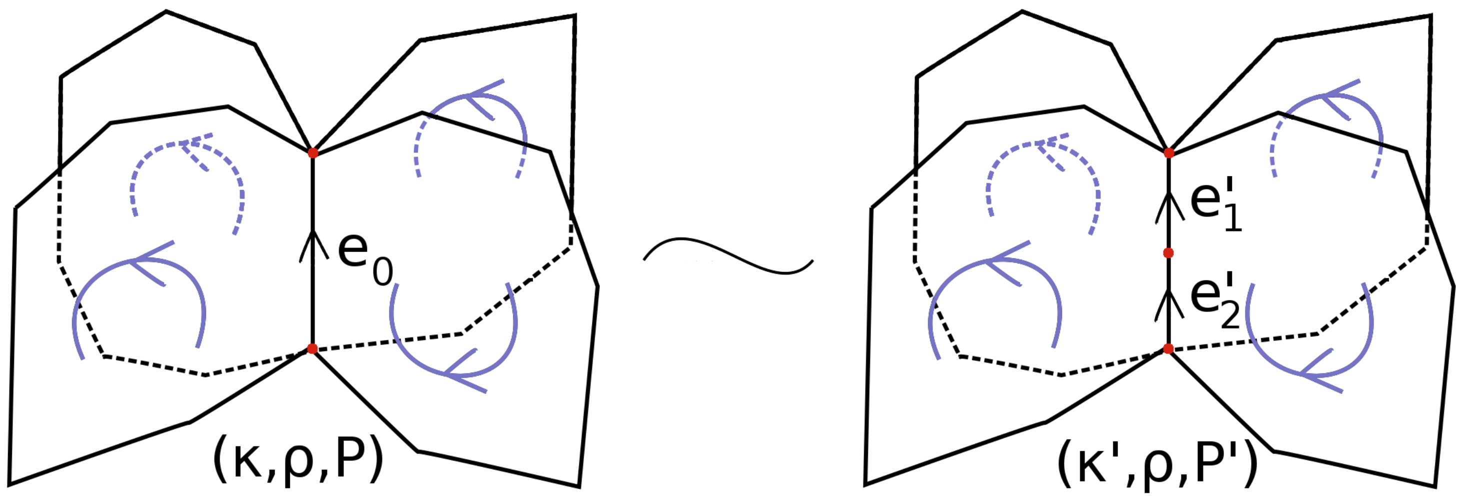
Equivalence relation: face reorientation



Given an operator spin foam (κ, ρ, P), the following operator spin foam (κ', ρ', P) is considered equivalent:

- κ' obtained from κ by flipping the orientation of a face f_1, and denoting the result f_1'
- $$\rho'_{f_1'} = \rho^*_{f_1},$$
- the remaining colorings unchanged: $\rho'_f = \rho_f$ for $f \neq f_1'$, $P' = P$

Equivalence relation: edge splitting



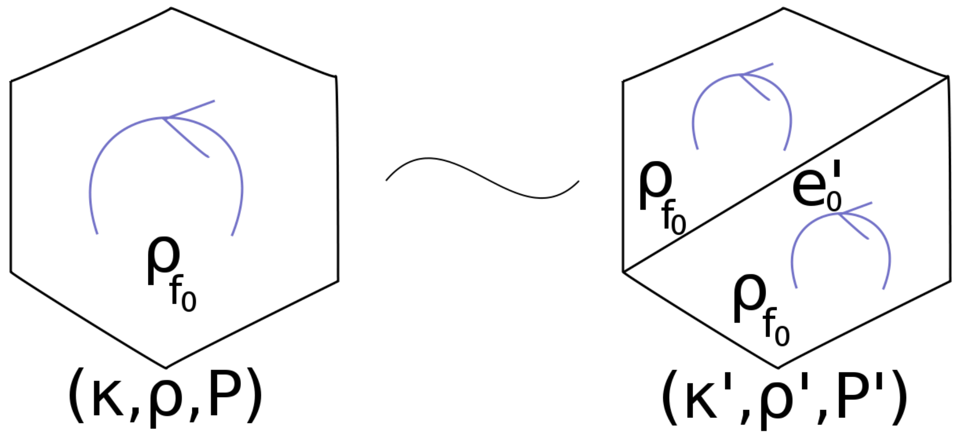
Given an operator spin foam (κ, ρ, P), any operator spin foam (κ', ρ', P) defined below is considered equivalent:

- κ' obtained from κ by splitting an edge: $e_0 = e'_1 \circ e'_2$
- $P'_{e'_1}$ and $P'_{e'_2}$ constrained by:

$$P'_{e'_1} P'_{e'_2} \;=\; P_{e_0}\,, \tag{5}$$

- the remaining colorings unchanged: $P'_e = P_e$ for $e \neq e'_1, e'_2$, $\rho' = \rho$

Equivalence relation: face splitting



Given an operator spin foam (κ, ρ, P), the following operator spin foam (κ', ρ', P') is considered equivalent:

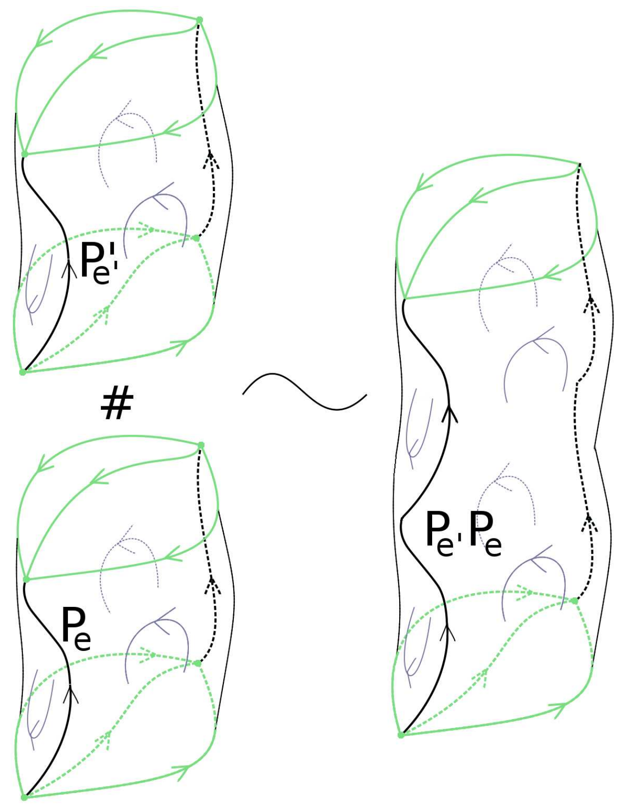
- κ' obtained from κ by splitting a face f_0 into f_1' and f_2' oriented as f_0 and glued along a new edge e_0',
- $$\rho'_{f_1'} = \rho'_{f_2'} = \rho_{f_0}, \quad P'_{e_0'} \;=\; \mathrm{id} \tag{6}$$
- the remaining colorings unchanged: $\rho'_f = \rho_f$ for $f \neq f_1', f_2'$, $P'_e = P_e$ for $e \neq e_0'$.

Equivalence relation: the other cases

Given an operator spin foam (κ, ρ, P), any of the following operator spin foams $(\kappa', \rho', P'),\ (\kappa'', \rho'', P'')$ and $(\kappa''', \rho''', P''')$ is considered equivalent:

1. Adding a trivial face: κ' obtained from κ by adding a face f_0 colored by the trivial representation, $\rho'_{f_0} =$ trivial, and $\rho'_f = \rho_f$ for $f \neq f_0$, $P' = P$.
2. Rescalling of the operators: $\kappa'' = \kappa$, $\rho'' = \rho$, P' obtained by rescaling, that is $P'_e = a_e P_e$ such that $\prod_e a_e = 1$
3. Replacing a representation by an equivalent one: $\kappa''' = \kappa$, for f_0, one of the faces, use a unitary map $\mathcal{I} : \mathcal{H}_{\rho_{f_0}} \to \mathcal{H}'''$, to define an equivalent representation $\rho'''_{f_0} = \mathcal{I} \circ \rho_{f_0} \circ \mathcal{I}^{-1}$, and the corresponding operators $P'''_{e_1}, P'''_{e_2}, P'''_{e_3}, ...$ for every edge $e_1, e_2, e_3, ...$ contained in f_0.

Glueing the operator spin foams

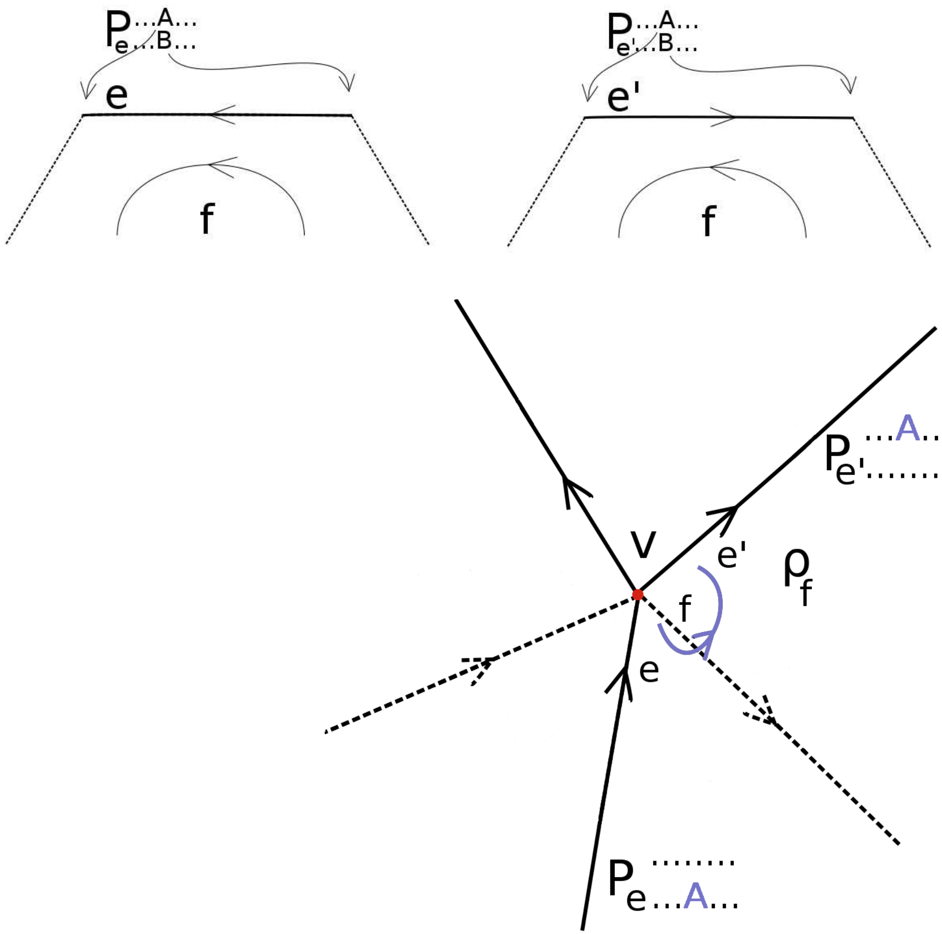


Given (κ, ρ, P) and (κ', ρ', P'), glueing the colored 2-cell complexes into $(\kappa\#\kappa', \rho\#\rho')$ along links of the boundary is accompanied by the composition of the operators coloring each pair e', e of edges connected into a single edge of $\kappa'\#\kappa$,

$$P'_{e'} P_e \;=\; (P'\#P)_{e'\circ e} \tag{7}$$

The contraction $\mathrm{Tr}(\kappa, \rho, P)$

$$(P_e w)^{A'\dots}_{A\dots} \;=\; P_e{}^{A'\dots B\dots}_{A\dots B'\dots}\, w^{B'\dots}_{B\dots}. \tag{8}$$



indices assigned to vertices.

$\mathrm{Tr}_{v,f}\, P_{e'} \otimes P_e$

$$\prod_{v,f} \mathrm{Tr}_{v,f} \bigotimes_e P_e \;=:\; \mathrm{Tr}(\kappa,\rho,P) \;\in \bigotimes_e \mathcal{H}_e \otimes \bigotimes_{e'} \mathcal{H}^*_{e'} \tag{9}$$

e/e' are incoming to/outgoing from $\partial\kappa$.

Consistency with the equivalence and glueing

$\mathrm{Tr}(\kappa, \rho, P)$ is inconsistent with:

- the face splitting equivalence relation
- with the glueing

However, for every (κ', ρ', P') obtained by splitting a face of any (κ, ρ, P),

$$(\prod_f A_f)\mathrm{Tr}(\kappa, \rho, P) \;=\; (\prod_{f'} A_{f'})\mathrm{Tr}(\kappa', \rho', P') \;\Rightarrow A_{f''} \;=\; \mathrm{dim}\rho_{f''}. \qquad (10)$$

In that way, the face amplitude emerges.
For the consistency with the glueing we introduce the boundary edge amplitude,

$$\mathcal{Z}(\kappa, \rho, P) \;=\; (\prod_{l\in\partial\kappa} A'_l)(\prod_f A_f)\mathrm{Tr}(\kappa, \rho, P). \qquad (11)$$

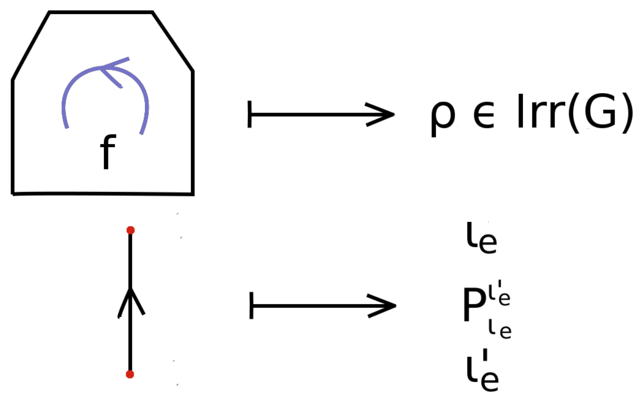
For any pair of the operator spin foams glued along a graph γ,

$$\mathcal{Z}(\kappa'\#\kappa, \rho'\#\rho, P'\#P) \;=\; \mathrm{Tr}_\gamma \mathcal{Z}(\kappa, \rho, P)\mathcal{Z}(\kappa, \rho, P) \;\Rightarrow A'_l = \frac{1}{\sqrt{d_{f_l}}}. \qquad (12)$$

Passage to the spin foams

$$P_e = \sum_{\iota_e \in \mathcal{B}_e} \sum_{\iota'_e \in \mathcal{B}^\dagger_{e'}} P^{\iota'_e}_{\iota_e}\, \iota_e \otimes \iota'_e \tag{13}$$

in any basis, $\mathcal{B}_e \subset \mathcal{H}_e$ and the conjugate $\mathcal{B}^\dagger_e \subset \mathcal{H}^*_e$



$$A_v = \prod_{f\,:\,f \ni v} \mathrm{Tr}_{vf} \left(\bigotimes_{e \text{ incoming}} \iota_e \otimes \bigotimes_{e' \text{ outgoing}} \iota'_{e'} \right), \tag{14}$$

$$\mathcal{Z}(\kappa, \rho, P) = \sum_{\iota} \prod_{e} P^{\iota'_e}_{\iota_e} \prod_{f} d_f \prod_{v} A_v \prod_{\tilde{l}} \frac{1}{\sqrt{d_{f_{\tilde{l}}}}} \bigotimes_{\tilde{e}} \iota_{\tilde{e}} \otimes \bigotimes_{\tilde{e}'} \iota'_{\tilde{e}'} \tag{15}$$

Middle summary

- We have introduced operator spin foams (OSF) (κ, ρ, P)
- The equivalence relations identify pairs of OSF, alow to reorient and refine OSFs
- The natural contraction at the vertices applied to $\otimes_e P_e$ leads to the 'operator' $\mathcal{Z}(\kappa, \rho, P) \ \in \mathcal{H}_{\partial\kappa}$
- The operator involves the face amplitude $A_f = d_{\rho_f}$ for the consistency with the equivalence with respect to the face splitting
- The operator involves the boundary amplitude $A_l = 1/\sqrt{d_l}$ for the consistency with the glueing of OSF
- The operator consist of the spin foams, the contractions give the familiar vertex amplitudes (evaluation of vertex spin-networks), the edge operators give conditions of glueing different vertices
- This is just an algebraic combinatorial framework, no "physical" postulates have been made thus far, the framework can be used to express formulae of any spin foam model

Natural operator spin foam models

We define a G operator spin foam model to be an assignment

$$(\kappa, \rho) \;\mapsto (\kappa, \rho, P)\,, \tag{16}$$

of an operator coloring P to each 2-cell complex κ colored by ρ with the irreducible representations of a compact group G; the assignment is assumed to be consistent with the equivalence relation in the set of the operator spin foams and with the glueing of the operator spin foams.

The set of maximal symmetry assumptions defines a natural operator spin foam model:

- operator coloring $e \mapsto P_e$ depends only on the unordered sequence of labels ρ_f coloring the faces f such that $e \subset f$
- for every (internal) edge e

$$P_e^\dagger \;=\; P_e. \tag{17}$$

Natural OSFM: a general solution

The consistency with

- the face splitting (see transparency 8) determines

$$\mathcal{H}_e = \mathcal{H}_{\rho_f} \otimes \mathcal{H}^*_{\rho_f} \;\Rightarrow\; P_e \;=\; \mathrm{id} \tag{18}$$

  that is $P^{AA'}_{BB'} \;=\; \frac{1}{d_{\rho_f}} \delta^A_B \delta^{A'}_{B'}$

- the edge splitting equivalence (see transparency 7)

$$\Rightarrow \qquad P_e P_e \;=\; P_e, \;\; \text{for every } e\,. \tag{19}$$

Hence, each operator P_e is the orthogonal projection onto a subspace

$$\mathcal{H}^{\mathrm{s}}_e \subset \mathcal{H}_e. \tag{20}$$

For every e, the subspace depends only on the unordered sequence of the representations ρ_f given by the faces f containing e.

Examples of the natural OSFM

- The spin foam model of the G 3d BF theory:

$$\mathcal{H}^{\mathrm{s}}_e \;=\; \mathcal{H}_e, \quad \text{and} \quad P_e \;=\; \mathrm{id} \tag{21}$$

- The Barrett-Crane model: $G = SU(2)^+ \times SU(2)^-$, $P_e = 0$ unless $\rho^+_f = \rho^-_f$, then

$$P_e \;=\; \iota_{\mathrm{BC},e} \otimes \iota^\dagger_{\mathrm{BC},e}. \tag{22}$$

- The EPRL or FK solutions to the simplicity constraints: $G = SU(2)^+ \times SU(2)^-$. For every edge e, there is defined the subspace

$$\mathcal{H}^{\mathrm{EPRL/FK}}_e \;\subset\; \mathcal{H}_e \tag{23}$$

  of intertwiners which satisfy the suitably defined simplicity constraints. Notation: $\mathcal{H}_{\rho_f} = \mathcal{H}_{j^-_f} \otimes \mathcal{H}_{j^+_f}$.

  $\mathcal{H}^{\mathrm{EPRL}}_e \subset \mathrm{Inv}(\otimes_f \mathcal{H}_{\rho_f})$ non-empty, provided $j^\pm_f \;=\; \frac{|1\pm\gamma|}{2} k_f$, $k_f \in \frac{1}{2}\mathbb{N}$.

  One can use $\mathcal{H}^{\mathrm{EPRL}}_e$ to define a natural operator spin foam model.

Natural OSFM for the EPRL intertwiners

$\mathcal{H}_e^{\mathrm{EPRL}}$ is defined to be the immage of the EPRL map

$$\iota_e^{\mathrm{EPRL}} \ : \ \mathrm{Inv}_{\mathrm{SU}(2)} \bigotimes_{f\supset e} \mathcal{H}_{k_f} \ \longrightarrow \ \mathrm{Inv}_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \bigotimes_{f\supset e} \mathcal{H}_{j_f^-} \otimes \mathcal{H}_{j_f^+} \tag{24}$$

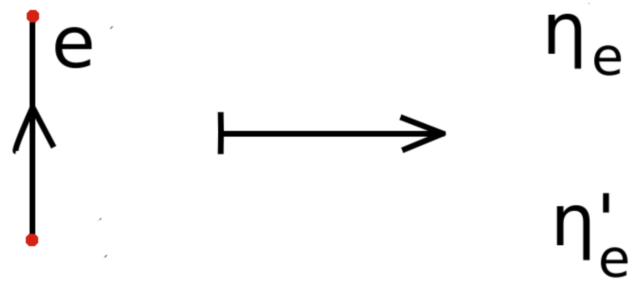
The map ι_e^{EPRL} is one-to-one, but not an isometry *Kamiński,Kisielowski,L 2009*. Maps an orthonormal basis $\tilde{\mathcal{B}}_e \ \subset \ \mathrm{Inv}_{\mathrm{SU}(2)} \bigotimes_{f\supset e} \mathcal{H}_{k_f}$ into a non-orthonormal basis of $\mathcal{H}_e^{\mathrm{EPRL}}$. Therefore,

$$P_e = \sum_{\eta_e \in \tilde{\mathcal{B}}_e, \eta'_e \in \tilde{\mathcal{B}}_e^\dagger} P^{\eta_e}_{\eta'_e} \ \iota_e^{\mathrm{EPRL}}(\eta_e) \otimes \iota_e^{\mathrm{EPRL}}(\eta'_e)^\dagger, \tag{25}$$

$$\sum_{\eta'_e} P^{\eta_e}_{\eta'_e} \left(\iota_e^{\mathrm{EPRL}}(\eta'_e) | \iota_e^{\mathrm{EPRL}}(\eta''_e) \right)_e \ = \ \delta^{\eta_e}_{\eta''_e} \, . \tag{26}$$

Natural OSFM for the EPRL in terms of SFs

Coloring η: two independent $SU(2)$ intertwiners η_e, η'_e assigned to each e.



$$\mathcal{Z}(\kappa,\rho,P) = \sum_\eta \prod_e P^{\eta_e}_{\eta'_e} \prod_f (2j^+_f+1)(2j^-_f+1) \prod_v A_v$$
$$\prod_{\tilde{l}} \frac{1}{\sqrt{(2j^+_{f_{\tilde{l}}}+1)(2j^-_{f_{\tilde{l}}}+1)}} \bigotimes_{\tilde{e}} \iota^{\mathrm{EPRL}}_e(\eta_e) \otimes \bigotimes_{\tilde{e}'} \left(\iota^{\mathrm{EPRL}}_e(\eta'_{e'})\right)^\dagger \tag{27}$$

f - the set of faces, v - the set of the internal vertices, l - the set of the boundary edges (links), and $\tilde{e}$/$\tilde{e}'$ - the set of edges which intersect $\partial\kappa$ at the end/beginning point. A_v is the vertex amplitude and f_l is the (unique) face containing l.

Asymptotics 1

define $\sigma = \text{EPRL}$.
define $\iota^\sigma(\eta) = \iota^{\text{EPRL}}(\eta)$
Compare the originaly proposed state sum that uses

$$\tilde{P}_\sigma = \sum_\eta \iota^\sigma(\eta)\iota^{\sigma\,\dagger}(\eta) \tag{28}$$

with the natural state sum that uses

$$P_\sigma = \sum_{\eta,\eta'} \iota^\sigma(\eta)h(\eta,\eta')\iota^{\sigma\,\dagger}(\eta') \tag{29}$$

with $h(\eta,\eta')$ such that $P_\sigma\iota^\sigma(\eta) = \iota^\sigma(\eta)$.

Both operators have the same range: $P_\sigma\tilde{P}_\sigma = \tilde{P}_\sigma$

Asymptotics 2

Coherent state basis

$$\eta(j_i, n_i) = \int \mathrm{d}g \bigotimes_i g|n_i\rangle_{j_i}$$

(or for simplicity $\eta(n_i)$) then has for $\gamma < 1$:

$$\tilde{P}_\sigma = \int \mathrm{d}n_i \ \iota^\sigma\Big(\eta(n_i)\Big) \ \iota^{\sigma\,\dagger}\Big(\eta(n_i)\Big) \tag{30}$$

and

$$P_\sigma = \int \mathrm{d}n_i \mathrm{d}n_i' \ \iota^\sigma\Big(\eta(n_i)\Big) \ \tilde{h}\Big(\eta(n_i), \eta(n_i')\Big) \ \iota^{\sigma\,\dagger}\Big(\eta(n_i')\Big) \tag{31}$$

Now from

$$P_\sigma \ \iota^\sigma\Big(\eta(n_i)\Big) = \iota^\sigma\Big(\eta(n_i)\Big)$$

we have

$$\int \mathrm{d}n_i' \ \tilde{h}\Big(\eta(n_i), \eta(n_i')\Big) \ \ \Big\langle \iota^\sigma(\eta(n_i'))\Big| \iota^\sigma(\eta(n_i''))\Big\rangle = \delta(n_i, n_i''). \tag{32}$$

Asymptotics 3

Coherent state basis

$$\eta(n_i) = \int \mathrm{d}g \bigotimes_i g|n_i\rangle_{j_i}$$

then has for $\gamma < 1$:

$$\tilde{h}^{-1}\Big(\eta(n_i), \eta(n_i')\Big) = \int \mathrm{d}g^{\pm} \prod_i \langle n_i|g^+|n_i'\rangle^{(1+\gamma)j_i} \langle n_i|g^-|n_i'\rangle^{(1-\gamma)j_i} \qquad (33)$$

Some observations:

- Goes to zero exponentially unless $\sum_i j_i n_i = 0$ and $\eta(n_i) \sim \eta'(n_i)$.
- On the diagonal we have

$$\tilde{h}^{-1}\Big(\eta(j_i, n_i), \eta(j_i, n_i)\Big) = |\eta((1+\gamma)j_i, n_i)|^2 |\eta((1-\gamma)j_i, n_i)|^2.$$

Asymptotics 4

- For normalised coherent intertwiners $\tilde{\eta}(j_i, n_i)$, $\tilde{h}^{-1}\Big(\tilde{\eta}(j_i, n_i), \tilde{\eta}(j_i, n_i)\Big)$ is $\sim |\eta(n_i)|^2$.

$\Rightarrow$ Asymptotically the two proposals are very similar, differ by edge meassure factor. Original proposal is polynomially small for large j.

The SL(2,C) case

Brief outline of what we have given a graph γ:

- The map SU(2) spin network $\mapsto$ a function on SL(2,C)$^\gamma$, solution to the EPRL constraints,

$$s \mapsto \psi_s$$

- The immage defines a space of functions $\mathcal{F}$
- A natural scalar product in $\mathcal{F}$
- A rigging map $\eta : \mathcal{F} \to \mathcal{F}^*$. Each $\eta(\psi(s))$ is an SL(2,C) invariant distribution on $\mathcal{F}$.
- Evaluation of each $\eta(\psi(s))$ at $(I, ..., I) \in$SL(2,C)$^\gamma$ which is proven to be taking finite values: the long standing conjecture by Baez and Barrett, passed to the EPRL case by Engle and Pereira proved *Kamiński 2010*.

Thank you

Summary

- The framework of operator spin foams encodes the structure of the spin networks and spin foams.
- Emergence of the face amplitude as consistency condition
- A class of the natural operator spin foam models (NOSFM) defined by maximal symmetry.
- The EPRL simplicity constraints can be used to define an example of the NOSFM. The result is inequivalent to the original EPRL proposal.
- We have calculated the asymptotics of that model for large j and compaired with the original EPRL model.
- The door to the SL(2,C) extensions open due to the finiteness of the evaluation (suitably defined).