The group averaging and the relational observables: examples from LQC

ILLEQGALS

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Motivation: canonical quantum gravity

Steps of the construction of the canonical quantum gravity:

- $\mathcal{H}_{\rm kin}$ – the kinematical Hilbert space
- $\hat{C}(N), \hat{C}(\vec{N})$ – the quantum constraint operators
- $\mathcal{H}_{\rm phys}$ – a physical Hilbert space of solutions
  - What are they defined?
  - Do they depend on the laps or shift functions?
  - What is their Hilbert space defined?
  - Proposal: The group averaging (Hawking?, Marolf, Louko, Giulini)
  - Equivalently, the spectral method (Thiemann, Bahr)
- Dirac observables on $\mathcal{H}_{\rm phys}$
  - the partial (relational) observables (Rovelli, Dittrich, Thiemann) from the kinematical ones
  - others?

We still do not control all the technical elements of this construction. Therefore we test the methods on simpler models: LQC FRW model (Bojowald, Ashtekar, Pawlowski, Singh, Szulc, Corrichi, Vandersloot, Chiou, Kaminski, Szulc)

Group Averaging $\Rightarrow$ Spectral Decomposition

- Group averaging:
  - $G \subset U(\mathcal{H}_{\rm kin})$ – a subgroup of the group of unitary operators.
  - $\mathcal{H}_{\rm kin} \ni \psi \mapsto \int dg\, \langle U_g \psi|$ – a distribution in some domain in $\mathcal{H}_{\rm kin}$
- Spectral Decomposition
  - $\hat{C}$ – the constraint operator in $\mathcal{H}_{\rm kin}$
  - Isomorphism

$$\mathcal{H}_{\rm kin} \ \ni \Psi \mapsto \ (\Psi_\lambda)_{\lambda \in \mathbb{R}}, \quad \Psi_\lambda \in \mathcal{H}_\lambda, \tag{1}$$

$$(\widehat{C}\Psi)_\lambda \ = \ \lambda \Psi_\lambda, \quad (\Psi|\Psi') \ = \int d\mu(\lambda)(\Psi_\lambda|\Psi'_\lambda)_\lambda. \tag{2}$$

  - continuity in λ if not an eigenvalue
  - $\mathcal{H}_{\rm phys} \ := \ \mathcal{H}_{\lambda=0}$
- by GA to SD (half heuristic, can be made rigorous for a point or absolutely continuous spectrum - Kaminski):
  $(\Psi_\lambda) \ \mapsto \int d\mu(t) e^{it\hat{C}}(\Psi_\lambda) \ = \ \int d\mu(t) e^{it\lambda}(\Psi_\lambda) \ = \ (\delta(\lambda)\Psi_\lambda)$

Example: the relativistic spin-less particle in 2D

- Kinematics:
  - $\mathcal{H}_{\text{kin}} = L^2(\mathbb{R} \otimes \mathbb{R})$
  - $\hat{x}^\mu \Psi(t,x) = x^\mu \Psi(t,x)$ and $\hat{p}^\mu \Psi(t,x) = -i\partial_\mu \Psi(t,x)$
  - $\Psi(t,x) = \frac{1}{2\pi} \int d\omega dk \; \Psi(\omega,k) e^{i\omega t - ikx}$,
- $\hat{C} = \hat{p}_t^2 - \hat{p}_x^2 - m^2$, and $\hat{p}_\mu T^\mu \geq 0$ for every future oriented T^μ
- The spectral decomposition ($\lambda := \omega^2 - k^2$)

$$\Psi(t,x) = \int d\lambda \frac{dk}{2\sqrt{\lambda + k^2}} (\Psi(\sqrt{\lambda + k^2}, k) e^{i\sqrt{\lambda+k^2}t + ikx} + \Psi(-\sqrt{\lambda + k^2}, k) e^{-i\sqrt{\lambda+k^2}t + ikx}) \quad (3)$$

- solutions for $\lambda = \lambda_0 \geq 0$: $\Psi(\omega,k) = \delta(\lambda - \lambda_0) \Psi(\sqrt{\lambda_0 + k^2}, k)$
- $(\Psi|\Psi')_{\lambda_0} = \int \frac{dk}{2\sqrt{\lambda_0 + k^2}} \overline{\Psi(\sqrt{\lambda_0 + k^2}, k)} \Psi'(\sqrt{\lambda_0 + k^2}, k)$
- due to the continuity in λ it makes sense to set $\lambda_0 = m^2$.

Classical FRW

$$ds^2 \; = \; -dt^2 \; + \; a(t)^2 \, q^{(0)}_{ab} \, dx^a dx^b \tag{4}$$

coupled with the homogeneous scalar field:

$$\phi \qquad .$$

The constraint is either of:

$$C(1) \; = \; \frac{1}{2}\frac{\pi^2}{V} + C_{\rm gr} \quad \text{or} \quad \mathrm{C(V)} \; = \; \frac{1}{2}\pi^2 + \mathrm{V}C_{\rm gr} \tag{5}$$

Where, $\tilde{\pi}$ is the canonically conjugate momentum to ϕ, and:

$$\pi \; := \; \int_{\mathcal{U}_0} \tilde{\pi}, \qquad V \; := \; \int_{\mathcal{U}_0} a^3 \sqrt{\mathrm{det} q^{(0)}} \tag{6}$$

and $\mathcal{U}_0$ is a fixed finite region ("cell") in Σ.

- $C(1)$ and $C(V)$ are equivalent. The difference may be in the quantum theory
- $C_{\rm gr}$ has the cosmological constant term $\sim \; -\Lambda V^2$

Quantum FRW

The gravitational degrees of freedom

$$\mathcal{H}_{\rm gr} \;=\; \overline{{\rm Span}(|v\rangle \;:\; v \in \mathbb{R})}, \qquad \langle v|v'\rangle \;=\; \delta_{v,v'} \tag{7}$$

$$\hat{V}|v\rangle \;=\; V_0|v||v\rangle, \qquad \hat{h}_\nu|v\rangle \;=\; |v+\nu\rangle, \tag{8}$$

$V_0 = ...$ and $\hat{h}_\nu$ involves da/dt.
The scalar field degrees of freedom

$$\mathcal{H}_{\rm sc} \;=\; L^2(\mathbb{R}) \tag{9}$$

$$\hat{\phi}\psi(\phi) \;=\; \phi\psi(\phi), \qquad \hat{\pi}\psi(\phi) \;=\; \frac{\hbar}{i}\partial_\phi\psi(\phi). \tag{10}$$

$\mathcal{H}_{\rm kin} \;=\; \mathcal{H}_{\rm sc} \otimes \mathcal{H}_{\rm gr}$.
The quantum constraint operator is either of:

$$\widehat{C}(1) \;=\; \frac{1}{2}\hat{\pi}^2 \otimes \widehat{V^{-1}} + 1 \otimes \widehat{C_{\rm gr}} \quad \text{or} \quad \widehat{C}(V) \;=\; \frac{1}{2}\hat{\pi}^2 \otimes 1 + 1 \otimes \widehat{V^{-1}}^{\,-1/2}\widehat{C_{\rm gr}}\widehat{V^{-1}}^{\,-1/2} \tag{11}$$

Preliminary comparison between $\hat{C}(V)$ and $\hat{C}(1)$

- $\hat{C}(V)$
  - The operator is essentially self adjoint when $\Lambda < 0$ however it admits inequivalent self-adjoint extensions when $\Lambda > 0$ (Kaminski, L, P).
  - The Group Averaging framework gives the Ashtekar-Pawlowski-Singh model (Ashtekar, Pawlowski, Singh).
  - Non-equivalent s.a. extensions give non-equivalent quantum models.
- $\hat{C}(1)$
  - The operator is essentially self adjoint for every value of the cosmological constant, and for every type of the symmetry group, for every closed and for every open universe (L, Kaminski, Pawlowski).
  - The GA framework?
  - Comparison with the APS model?

The Spectral Decomposition

Below we will restrict the Hilbert space $\mathcal{H}_{\rm gr}$ to a subspace - one of the APS super-selection sectors. This restriction will be irrelevant in the $\Lambda < 0$ case, and relevant only for the details of the $\Lambda > 0$ case:

$\mathcal{H}_{\rm kin} \ = \ \mathcal{H}_{\rm sc} \otimes \mathcal{H}_{\rm gr}$

$$\mathcal{H}_{\rm gr} \ = \ \overline{{\rm Span}(\ (|v\rangle \ + |-v\rangle) : \ v \in 4\mathbb{Z}\ )}, \qquad \mathcal{H}_{\rm sc} \ = \ L^2(\mathbb{R}) \tag{12}$$

$$\hat{\phi}\psi(\phi) \ = \ \phi\psi(\phi), \qquad \hat{\pi}\psi(\phi) \ = \ \frac{\hbar}{i}\partial_\phi\psi(\phi) \tag{13}$$

$$\hat{C} := \ \hat{C}(1) \ = \ \frac{1}{2}\hat{\pi}^2 \otimes \widehat{V^{-1}} \ + \ 1 \otimes \widehat{C_{\rm gr}} \tag{14}$$

What we look for is the spectral decomposition of $\hat{C}$:

$$\mathcal{H}_{\rm sc} \otimes \mathcal{H}_{\rm gr} \ \ni \Psi \mapsto \ \{\Psi_\lambda\}_{\lambda\in\mathbb{R}}, \quad \Psi_\lambda \in \mathcal{H}_\lambda, \tag{15}$$

$$(\widehat{C}\Psi)_\lambda \ = \ \lambda\Psi_\lambda, \quad (\Psi|\Psi') \ = \int d\mu(\lambda)(\Psi_\lambda|\Psi'_\lambda)_\lambda. \tag{16}$$

And:

$$\mathcal{H}_{\rm phys} \ := \ \mathcal{H}_{\lambda=0}.$$

The $\Lambda < 0$ case, decomposition

Derivation of the spectral decomposition

$\mathcal{H}_{\rm sc} \otimes \mathcal{H}_{\rm gr} = \int^{\oplus} d\lambda \mathcal{H}_\lambda, \quad \hat{C} = \frac{1}{2}\hat{\pi}^2 \otimes \widehat{V^{-1}} + 1 \otimes \widehat{C_{\rm gr}}$

For every $\pi \geq 0$, there is an orthonormal basis $\{\tilde{e}_{\lambda_n(\pi),\pi} \in \mathcal{H}_{\rm gr}\}_{n\in\mathbb{N}}$, such that

$$(\frac{1}{2}\pi^2\widehat{V^{-1}} + \widehat{C_{\rm gr}})\tilde{e}_{\lambda_n(\pi),\pi} = \lambda_n(\pi)\tilde{e}_{\pi_n(\lambda),\pi} \tag{17}$$

$$\Psi(\phi, v) = \int_{-\infty}^{\infty} d\pi \sum_{n=1}^{\infty} \tilde{\Psi}(\lambda_n(\pi), \pi) e^{i\pi\phi} \tilde{e}_{\lambda_n(\pi),|\pi|}(v) \tag{18}$$

$$= \int d\lambda \sum_{n>n_\lambda,\, \sigma=\pm 1} |\frac{d\lambda_n(\pi)}{d\pi}|^{-\frac{1}{2}} \tilde{\Psi}(\sigma\pi_n(\lambda), \lambda) e^{\sigma i \pi_n(\lambda)\phi} |\frac{d\lambda_n(\pi)}{d\pi}|^{-\frac{1}{2}} \tilde{e}_{\lambda,\pi_n(\lambda)}(v)$$

$$= \int d\lambda \sum_{n>n_\lambda,\, \sigma=\pm 1} \Psi(\sigma\pi_n(\lambda), \lambda) e^{\sigma i \pi_n(\lambda)\phi} e_{\lambda,\pi_n(\lambda)}(v) \tag{19}$$

$$e_{\lambda,\pi_n(\lambda)}(v) := |\frac{d\lambda_n(\pi)}{d\pi}|^{-\frac{1}{2}} \tilde{e}_{\lambda,\pi_n(\lambda)}(v), \tag{20}$$

$$(\Psi|\Psi')_{\rm sc\otimes gr} = \int d\lambda \sum_{n\geq n_\lambda, \sigma=\pm} \overline{\Psi(\lambda, \sigma\pi_n(\lambda))} \Psi'(\lambda, \sigma\pi_n(\lambda)) \tag{21}$$

The $\Lambda < 0$ case, new scalar product

$$\langle \widehat{C_{\rm gr}} \tilde{e}_{\lambda_n(\pi),|\pi|} | \tilde{e}_{\lambda_{n'}(\pi'),|\pi'|} \rangle \; - \langle \tilde{e}_{\lambda_n(\pi),|\pi|} | \widehat{C_{\rm gr}} \tilde{e}_{\lambda_{n'}(\pi'),|\pi'|} \rangle \; = \; 0 \tag{23}$$

$$(\lambda_n(\pi) - \lambda_{n'}(\pi')) \langle \tilde{e}_{\lambda_n(\pi),|\pi|} | \tilde{e}_{\lambda_{n'}(\pi'),|\pi'|} \rangle \; = \tag{24}$$

$$= \; \frac{1}{2}(\pi^2 - \pi'^2) \langle \tilde{e}_{\lambda_n(\pi),|\pi|} | \widehat{V^{-1}} \tilde{e}_{\lambda_{n'}(\pi'),|\pi'|} \rangle \tag{25}$$

we find

$$(\; (\lambda_n(\pi) = \lambda_{n'}(\pi')), \, \pi \neq \pi' \;) \; \Rightarrow \; 0 \; = \; \langle \tilde{e}_{\lambda_n(\pi),|\pi|} | \widehat{V^{-1}} \tilde{e}_{\lambda_{n'}(\pi'),|\pi'|} \rangle$$

$$\frac{d\lambda_n(\pi)}{d\pi} \; = \; \pi \langle \tilde{e}_{\lambda_n(\pi),|\pi|} | \widehat{V^{-1}} \tilde{e}_{\lambda_n(\pi),|\pi|} \rangle \tag{26}$$

and it implies

$$\langle e_{\lambda,\pi_n(\lambda)} \, | \, \widehat{V^{-1}} \pi_n(\lambda) \, e_{\lambda,\pi_{n'}(\lambda)} \rangle \; = \; \delta_{n,n'} \tag{27}$$

Hence, they are orthonormal in $\mathcal{H}_{\rm gr}$ with respect to a new product

$$\langle \cdot \, | \, \cdot \rangle_\lambda \; = \; \langle \cdot \, | \, \widehat{V^{-1}} H_\lambda \cdot \rangle, \qquad H_\lambda \; = \; \sqrt{\widehat{V^{-1}}^{\,-1} (\lambda - \widehat{C_{\rm gr}})}$$

The $\Lambda < 0$ case, transformation to APS

Now, each Hilbert space $\mathcal{H}_\lambda$ is spanned by the distributions

$$\Psi(\lambda', \pm\pi_n(\lambda')) \ = \ \delta(\lambda' - \lambda)\Psi(\pm\pi_n(\lambda))$$

with the scalar product

$$(\Psi|\Psi')_\lambda \ = \sum_{n>n_\lambda,\, \sigma=\pm} \overline{\Psi}(\sigma\pi_n(\lambda))\Psi'(\sigma\pi_n(\lambda)).$$

On the other hand, from each of the distributions we can construct

$$\Psi^\pm(\phi, v) \ = \sum_{n>n_\lambda} \Psi(\pm\pi_n(\lambda))e_{\lambda,\pi_n(\lambda)}(v)e^{\pm i\pi_n(\lambda)\phi},$$

and we have (we use any instant of ϕ)

$$(\Psi|\Psi')_\lambda \ = \ \langle\, \Psi^+_\phi \,|\, \Psi'^+_\phi \,\rangle_\lambda \ + \ \langle \Psi^-_\phi \,|\, \Psi'^-_\phi \,\rangle_\lambda.$$

In particular, for $\lambda = 0$ we derive a unitary map

$$\mathcal{H}_{\lambda=0} \to \mathcal{H}_{\rm APS}.$$

$\Lambda < 0$, the Dirac observables, a new trick!

The observable $\hat{\pi}$ is well defined in (and preserves) each $\mathcal{H}_\lambda$

$$(\hat{\pi}\Psi)(\lambda, \pm\pi_n(\lambda)) \;=\; \pm\pi_n(\lambda)\Psi(\lambda, \pm\pi_n(\lambda)).$$

Relational observables: Given a function $(V, p_V) \mapsto \mathcal{O}(V, p_V)$, and choosing ϕ to be the reference time function, the associated Dirac observable is

$$\mathcal{O}_{\phi_0} \;:=\; \mathcal{O}(t)_{t=t_0,\,\mathrm{s.t.}\,|\phi(t_0)=\phi_0} \qquad (28)$$

$$\mathcal{O}(t) \;=\; \mathcal{O} + t\{\mathcal{O}, C\} + \frac{1}{2}t^2\{\{\mathcal{O}, C\}, C\} + \;... \qquad (29)$$

To quantize it we write $\mathcal{O} = (\theta(\pi) + \theta(-\pi))\mathcal{O}$, and use the kinematic $\hat{\mathcal{O}}$:

$$\theta(\pi)\mathcal{O}_{\phi_0} \;=\; \int dt\delta(t-t_0)(\theta(\pi)\mathcal{O})(t) \;=\; \int dt(\;|\{\phi, C\}|\delta(\phi-\phi_0)\theta(\pi)\mathcal{O}\;)(t) \;=$$

$$=\; \int dt(\;V^{-1}|\pi|\delta(\phi-\phi_0)\theta(\pi)\mathcal{O}\;)(t) \qquad (30)$$

$$\widehat{\theta(\hat{\pi})\mathcal{O}_{\phi_0}} \;:=\; \int dt e^{it\hat{C}}|\theta(\hat{\pi})\hat{\pi}|^{\frac{1}{2}}\;\delta(\phi-\phi_0)|\theta(\hat{\pi})\hat{\pi}|^{\frac{1}{2}}\widehat{V^{-1}}^{1/2}\hat{\mathcal{O}}\widehat{V^{-1}}^{1/2}e^{-it\hat{C}},$$

$\Lambda < 0$, *the equivalence with APS*

Finally

$$\widehat{\mathcal{O}_{\phi_0}}(\lambda, \sigma\pi_n(\lambda), \lambda', \sigma'\pi_n(\lambda')) = \delta(\lambda - \lambda')\delta_{\sigma,\sigma'}(e^{\sigma i(\pi_n(\lambda) - \pi_{n'}(\lambda))\phi_0}). \\ \cdot |\pi_n(\lambda)|^{\frac{1}{2}} |\pi_{n'}(\lambda)|^{\frac{1}{2}} \langle e_{\lambda,\pi_n(\lambda)} \,|\, \widehat{V^{-1}}^{1/2} \widehat{\mathcal{O}} \widehat{V^{-1}}^{1/2} e_{\lambda,\pi_{n'}(\lambda)} \rangle \tag{32}$$

It preserves each Hilbert space $\mathcal{H}_\lambda$ and induces (acting by duality) therein an operator $\widehat{\mathcal{O}_{\phi_0}}_\lambda$,

$$\widehat{\mathcal{O}_{\phi_0}} \{\Psi_\lambda\}_{\lambda \in \mathbb{R}} = \{\widehat{\mathcal{O}_{\phi_0}}_\lambda \Psi_\lambda\}_{\lambda \in \mathbb{R}}$$

In particular, in $\mathcal{H}_{\text{phys}} = \mathcal{H}_{\lambda=0}$ this result coincides with the APS construction, that is with $\hat{C}(V)$.

Summary: this derivation was very general. The only information we used, was the existence of orthonormal basis in H_{gr} of the eigenvectors of the operator $\hat{C}_{\mathrm{gr}}$. The result coincides with the one obtained from $\hat{C}(V)$.

However: If we do not introduce $1 = \theta(\pi) + \theta(-\pi)$, then the positive and negative frequencies talk to each other.

The $\Lambda > 0$ case (non-unique in APS)

The first step: the spectral decomposition $\mathcal{H}_{\rm sc} \otimes \mathcal{H}_{{\rm gr},\epsilon=0} = \int^{\oplus} d\lambda \mathcal{H}_\lambda$:
For every $\pi \geq 0$, there is a normalized to δ basis $\{\tilde{e}_{\lambda,\pi}\}_{\lambda\in\mathbb{R}}$ in $\mathcal{H}_{\rm gr}$,

$$(\frac{1}{2}\pi^2\widehat{V^{-1}} + \widehat{C_{\rm gr}})\tilde{e}_{\lambda,\pi} = \lambda\tilde{e}_{\lambda,\pi}, \qquad \langle\tilde{e}_{\lambda,\pi}\,|\,\tilde{e}_{\lambda',\pi}\rangle = \delta(\lambda-\lambda')$$

$$\Psi(\phi,v) = \int d\lambda d\pi \tilde{\Psi}(\lambda,\pi)e^{i\pi\phi}\tilde{e}_{\lambda,\pi}(v), \quad (\Psi|\Psi')_{\rm sc\otimes gr} = \int d\lambda d\pi \overline{\tilde{\Psi}(\lambda,\pi)}\tilde{\Psi}'(\lambda,\pi)$$

Given λ_0, the Hilbert space $\mathcal{H}_{\lambda_0}$ is the space of distributions:

$$\tilde{\Psi}(\lambda_0,\pi) = \delta(\lambda-\lambda_0)\tilde{\Psi}(\pi)$$

with the scalar product

$$(\Psi|\Psi')_{\lambda_0} = \int d\pi \overline{\Psi(\pi)}\Psi'(\pi).$$

But: Can we express a solution as

$$\Psi(\phi,v) := \text{''} \int d\pi \tilde{\Psi}(\pi)e^{i\pi\phi}\tilde{e}_{\lambda=0,\,\pi}(v)\text{''} \quad ?$$

$\Lambda > 0$, *the normalizations*

We need to study the family $\{\tilde{e}_{\lambda,\pi}\}_{\pi\geq 0}$ corresponding to a fixed λ.
This time, we find that rather then 0 on the right hand side...,

$$\lim_{N\to\infty}\sum_{n=1}^{4N}\overline{\tilde{e}_{\lambda,\pi}(4n)}\widehat{C_{\rm gr}}\tilde{e}_{\lambda,\pi'}(4n)-\overline{\widehat{C_{\rm gr}}\tilde{e}_{\lambda,\pi}(4n)}\tilde{e}_{\lambda,\pi'}(4n) \;=\; b\sin(a(\pi,\lambda){-}a(\pi',\lambda))$$

where a is some function ($\lim_{\pi\to\infty} a(\pi,\lambda)=\infty$). This implies

$$\frac{1}{2}(\pi^2-\pi'^2)\sum_{n=1}^{\infty}\overline{\tilde{e}_{\lambda,\pi}(4n)}\widehat{V^{-}1}\tilde{e}_{\lambda,\pi'}(4n) \;=\; b\sin(a(\pi,\lambda)-a(\pi',\lambda)) \qquad (33)$$

Therefore $\tilde{e}_{\lambda,\pi}\in\mathcal{H}'_{\rm gr}$ endowed with $\langle\cdot|\cdot\rangle'=\langle\cdot|\widehat{V^{-1}}\cdot\rangle$, and

$$\langle\tilde{e}_{\lambda,\pi}\,|\,\widehat{V^{-}1}\tilde{e}_{\lambda,\pi'}\rangle \;=\; 2b\frac{\sin(a(\pi,\lambda)-a(\pi',\lambda))}{\pi^2-\pi'^2} \qquad (34)$$

$$\langle\tilde{e}_{\lambda,\pi}|\widehat{V^{-1}}\tilde{e}_{\lambda,\pi}\rangle \;=\; \frac{b}{\pi}\frac{\partial a}{\partial\pi} \qquad (35)$$

$$a(\pi',\lambda)=a(\pi,\lambda)\;+\;n\,3.14\ldots\;\;\Rightarrow\;\;\langle\tilde{e}_{\lambda,\pi}\,|\,\widehat{V^{-}1}\tilde{e}_{\lambda,\pi'}\rangle \;=\; 0. \qquad (36)$$

$\Lambda > 0$, *the first relation with APS*

Hence, for every value $a \in [0, 3.14...)$ and $\lambda = 0$, (also true for non-zero λ) we have a sequence $(\pi_n(a,0))_{n>n_{a,0}}$ such that

$$a(\pi_n(a,0),0) \;=\; a - n\,3.14... \tag{37}$$

and a sequence of orthogonal eigenfunctions $\big(\tilde{e}_{\lambda,\pi_n(a,0)}\big)_{n\geq n_{a,0}}$,

$$\langle \tilde{e}_{\lambda=0,\pi_{n'}(a,\lambda=0)} | \widehat{V^{-1}} \tilde{e}_{\lambda=0,\pi_n(a,\lambda=0)} \rangle \;=\; 0, \quad \text{when } \mathrm{n}' \neq \mathrm{n}. \tag{38}$$

What is that sequence?

Let us go back to the APS formulation. Every $a \in [0, 3.14...)$ labels a self-adjoint extension of the operator $-2\widehat{V^{-1}}^{-1}\widehat{C_{\rm gr}}$ in $\mathcal{H}'_{\rm gr}$. The operator is not positive. The sequence $\big(\tilde{e}_{\lambda,\pi_n(a,\lambda)}\big)_{n\geq n_{a,\lambda}}$ defined by $\lambda = 0$ spans the "physical" part of $\mathcal{H}'_{\rm gr}$ corresponding to the non-negative part of the spectrum. This is exactly the subspace in which the APS physical states $\phi \mapsto \Psi_\phi \in \mathcal{H}'_{\rm gr}$ take the values.

$\Lambda > 0$, *relations and contrast with APS*

We can write the spectral decomposition in the following way

$$\begin{aligned}\Psi(\phi, v) &= \int d\lambda d\pi \tilde{\Psi}(\lambda, \pi) e^{i\pi\phi} \tilde{e}_{\lambda,\pi}(v) = \\ &= \int d\lambda da \sum_{n \geq n_{a,\lambda}, \sigma=\pm} (\frac{\partial a}{\partial \pi})^{-\frac{1}{2}} \tilde{\Psi}(\lambda, \sigma\pi(a,\lambda))(\frac{\partial a}{\partial \pi})^{-\frac{1}{2}} \tilde{e}_{\lambda,\pi}(v) e^{i\sigma\pi_n(a,\lambda)\phi} = \\ &= \int d\lambda da \sum_{n \geq n_{a,\lambda}, \sigma=\pm} \Psi(\lambda, \sigma\pi_n(a,\lambda)) e_{\lambda,\pi_n(a,\lambda)}(v) e^{i\sigma\pi_n(a,\lambda)\phi}\end{aligned} \tag{39}$$

$$(\Psi|\Psi')_{\mathrm{sc}\otimes\mathrm{gr}} = \int d\lambda da \sum_{n \geq n_{a,\lambda}, \sigma=\pm} \overline{\Psi(\lambda, \pi_n(a,\lambda))} \Psi'(\lambda, \pi_n(a,\lambda)). \tag{40}$$

In the consequence, each physical state in the SD sense $\Psi(\lambda, \pi) := \delta(\lambda)\tilde{\Psi}(\pi)$ defines a family of the APS physical states labeled by $a \in [0, 3.14..)$,

$$\Psi^{\pm,a}(\phi, v) = \sum_{n > n_{a,\lambda=0}} \Psi(\pm\pi_n(a, \lambda = 0)) e_{\lambda=0,\pi_n(a,\lambda=0)}(v) e^{\pm i\pi_n(a,\lambda=0)\phi},$$

$\Lambda > 0$, *the Dirac observables*

And the SD physical scalar product compared to the APS one, reads

$$(\Psi|\Psi')_{\lambda=0} \; = \int da (\Psi^a|\Psi'^a)_{\mathrm{APS}}$$

Still, if this decomposition were preserved by the observables, we could call it "super-selection" (improper, because non-normalizable). It turns out it is not the case. Given a function $(V, p_V) \mapsto \mathcal{O}(V, p_V)$ and the corresponding operator $\widehat{\mathcal{O}}$ in $\mathcal{H}_{\mathrm{gr}}$, we construct the ϕ_0 dependent observable $\int dt(\, |V^{-1}\pi|\delta(\phi - \phi_0)\mathcal{O} \,)(t)$ and the operator $\widehat{\mathcal{O}}_{\phi_0}$ defined in $\mathcal{H}_{\lambda=0}$. It turns out that,

$$(\Psi|\widehat{\mathcal{O}}_{\phi_0}\Psi')_{\lambda=0} \; = \int dada' (\Psi^a|\widehat{\mathcal{O}}_{\phi_0}\Psi'^{a'})_{\mathrm{APS}}.$$

Notice that if $\mathcal{O} = 1$, then

$$1_{\tau_0}(V, p_V, \phi, \pi) \; = 0, 1$$

depending on whether $\phi(t)$ achieves ϕ_0 or not.

Summary

- In LQC FRW, the definition of a physical solution does not depend on the choice of laps $N = 1, \sqrt{q}$ if $\Lambda < 0$
- In LQC FRW, the definition of a physical solution does depend on the choice of laps $N = 1, \sqrt{q}$ if $\Lambda > 0$.
- We showed a tricky formula to define quantum relational observables, useful when we can not analytically calculate a given classical relational observable.
- Our results almost automatically generalize to the full theory of the gravitational field coupled with the Brown-Kuchar dusts (modulo the analysis of the operator $\hat{C}_{\mathrm{gr}}$).

THANK YOU

The End

Appendix: The Ashtekar-Pawlowski-Singh model

The physical states are

$$\mathbb{R} \ni \phi \;\mapsto\; \Psi^{\pm}(\phi, \cdot) \in \mathcal{H}_{\rm gr} \tag{41}$$

which satisfy the equation

$$\frac{\hbar}{i}\frac{d}{d\phi}\Psi^{\pm}(\phi, v) \;=\; \pm\sqrt{2\widehat{V^{-1}}^{\,-1}\widehat{C_{\rm gr}}}\,\Psi^{\pm}(\phi, v) \tag{42}$$

where the scalar product is

$$\langle \cdot \,|\, \cdot \rangle' \;:=\; \langle \cdot \,|\, \widehat{V^{-1}} \cdot \rangle$$

The final scalar product between two physical states is

$$(\Psi \,|\, \Psi') \;:=\; (\Psi^{+}(\phi, \cdot) \,|\, \widehat{V^{-1}}\, \Psi'^{+}(\phi, \cdot)) \;+\; (\Psi^{-}(\phi, \cdot) \,|\, \widehat{V^{-1}}\, \Psi'^{-}(\phi, \cdot)).$$

evaluated at any $\phi = \phi_0$ (the result is invariant w.r. to ϕ).