ASYMPTOTIC ANALYSIS OF SPIN FOAM AMPLITUDE WITH TIMELIKE TRIANGLES:

TOWARDS EMERGING GRAVITY FROM SPIN FOAM MODELS

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ILQGS 19

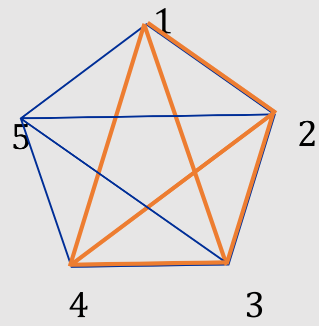
Based on arXiv:1810.09042
In collaboration with Muxin Han

Outline

- Introduction
- Amplitude and asymptotic analysis
- Geometric interpretation
- Amplitude at critical configurations
- Conclusion

Triangulation

The building block: 4 simplex

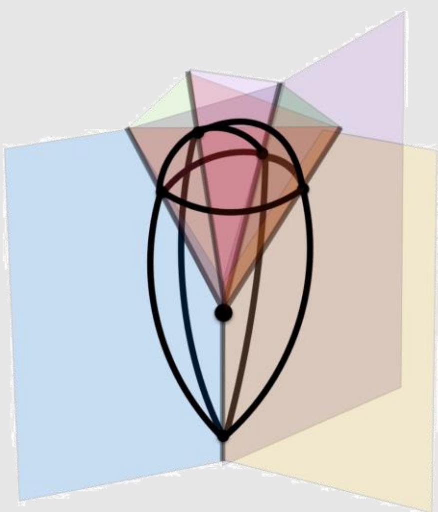


Boundary of a 4-simplex:
5 tetrahedron and 10 triangles

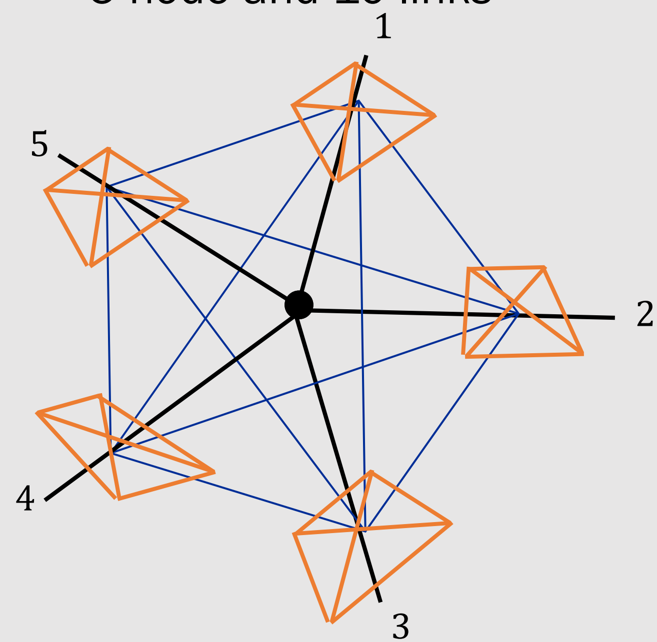
Dual graph

Boundary graph of a vertex:
5 node and 10 links

Dual Graph

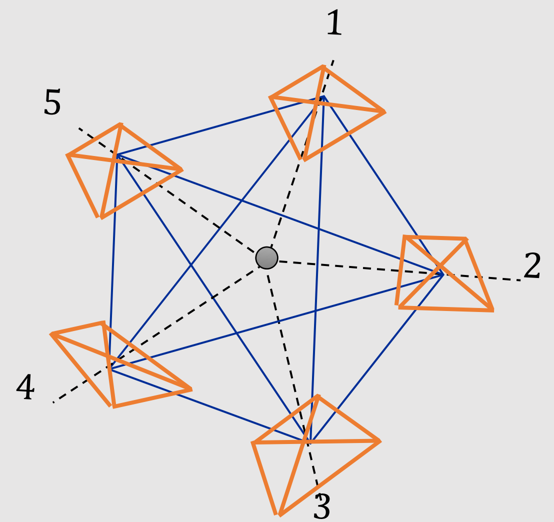


5 valent vertex and boundary

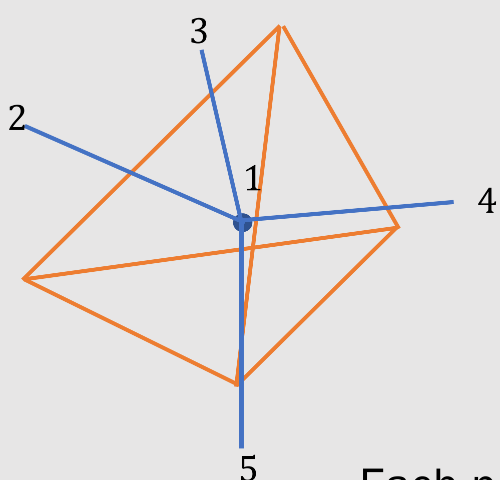


Picture from the book *covariant loop quantum gravity* (C. Rovelli F. Vidotto)

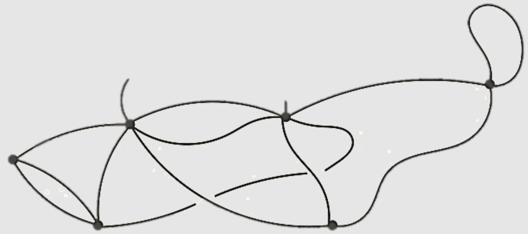
Boundary Graph:
Triangulation



Each link —— dual to a triangle :
Colored by spin!



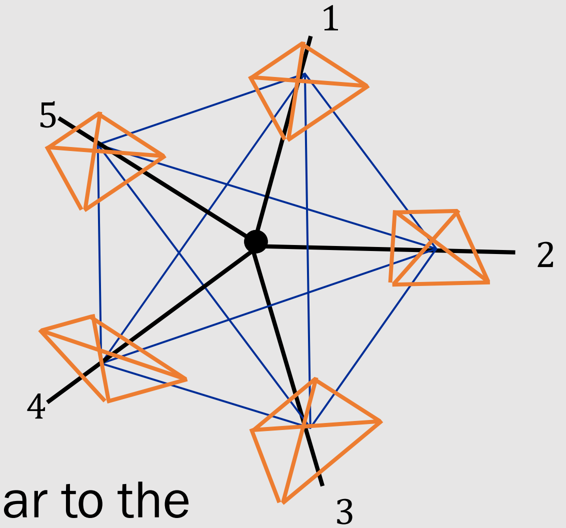
Gluing triangles



Spin network of boundary graph

Each node ● dual to a tetrahedron.
Gauge invariance: an intertwiner i_τ
(rank-4 invariant tensor)

Spin foam models



- A state sum model on $\mathcal{K}$ inspired from BF action

$$A(\mathcal{K}) = \sum_{J,i} \prod_f \mu_f(J_f) \prod_v A_v(J_f, i_e)$$

- Lorentzian theory:

  Gauge group $\mathrm{SL}(2,\mathbb{C})$

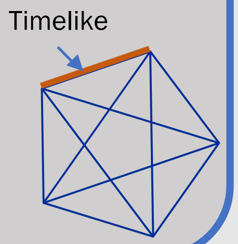
- Boundary gauge fixing: fix the normal perpendicular to the tetrahedron
  - *Time gauge* $u = (1,0,0,0)$ *(EPRL models)*
  - *Space gauge* $u = (0,0,0,1)$ *(Conrady-Hybrida Extension)*

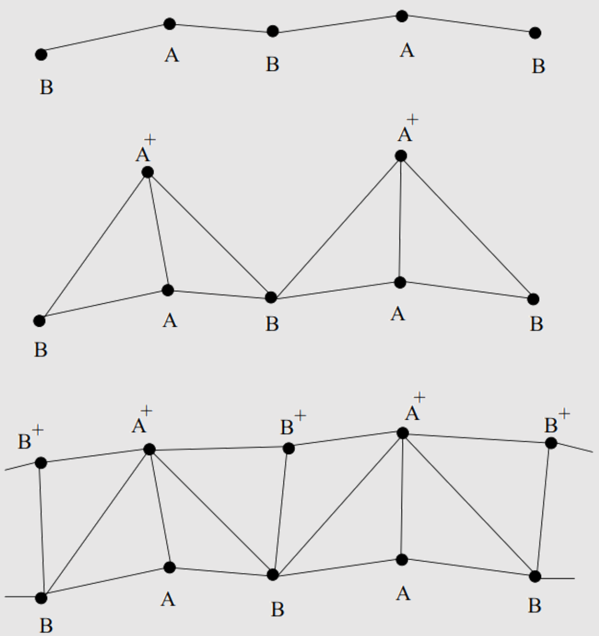
[Barrett, Rovelli, Dittrich, Engle, Livine, Freidel, Han, Dupris, Conrady, ...

Motivation: (3+1)-D Regge calculus Model

- A discrete formulation of General relativity.
- Discrete geometry: Sorkin triangulation
  - *Initial and final slice (Cauchy surface) triangulation (spacelike tetrahedra).*
  - *dragging vertices forward to construct the spacetime triangulation.*

Result : spacetime triangulation with
- Every 4-simplex contains both timelike and spacelike tetrahedra (e.g. 3 timelike, 2 spacelike)
- Every timelike tetrahedron contains both timelike and spacelike triangles





(1 + 1)-dimensional analogue of the Sorkin scheme

Known predictions: Gravitational wavers, Kasner solution, FLRW, etc…

[Gentle, Miller, Sorkin, Williams, Barrett, Collins, …

Asymptotics of spin foam models:
What we have so far?

- Spacelike tetrahedron (Euclidean model and Lorentzian model with Time gauge)
  - *Barrett-Crane model [Barrett and Steele, 03']*
  - *FK model [Conrady and L. Freidel, 08']*
  - *EPRL model [Barrett et al. 09']*
  - *EPRL model with many 4 -simplices [Han and Zhang 11']*
- Timelike tetrahedron with all faces spacelike [Kaminski et al, 17']
- Results:
  - *Asymptotics of the amplitude is dominated by critical configurations.*
  - *Critical configurations are simplicial geometry (possibly degenerate)*
  - *Asymptotic limit related to Regge action (discrete GR)*

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Non of them contains timelike triangles!!!

In all examples the Regge geometries contain timelike triangles.

Summary

- Asymptotic analysis of spin foam model with timelike triangles.
- The asymptotics of the amplitude is dominated by critical configurations.
- Critical configuration are again simplicial geometry.
- There will be no degenerate sector in the critical configuration.
- Asymptotic formula

$$A \sim N_+ e^{iS_{\mathcal{K}}} + N_- e^{-iS_{\mathcal{K}}}$$

$S_{\mathcal{K}}$ Regge action on on the simplicial complex.

Conrady-Hnybida Extension:
EPRL/FK with timelike triangles

From “time” gauge to “space” gauge

- Timelike tetrahedron with normals $u = (0{,}0{,}0{,}1)$
- Stabilize group SU(1,1)

Y map $\mathcal{H}^j \to \mathcal{H}^{(\rho,n)}$: physical Hilbert space

$$\rho = \begin{cases} \gamma n \\ -n/\gamma \end{cases} \qquad j = \begin{cases} \frac{n}{2}, & \text{spacelike triangle} \\ -\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{n^2/\gamma^2 - 1} & \text{timelike triangle} \end{cases}$$

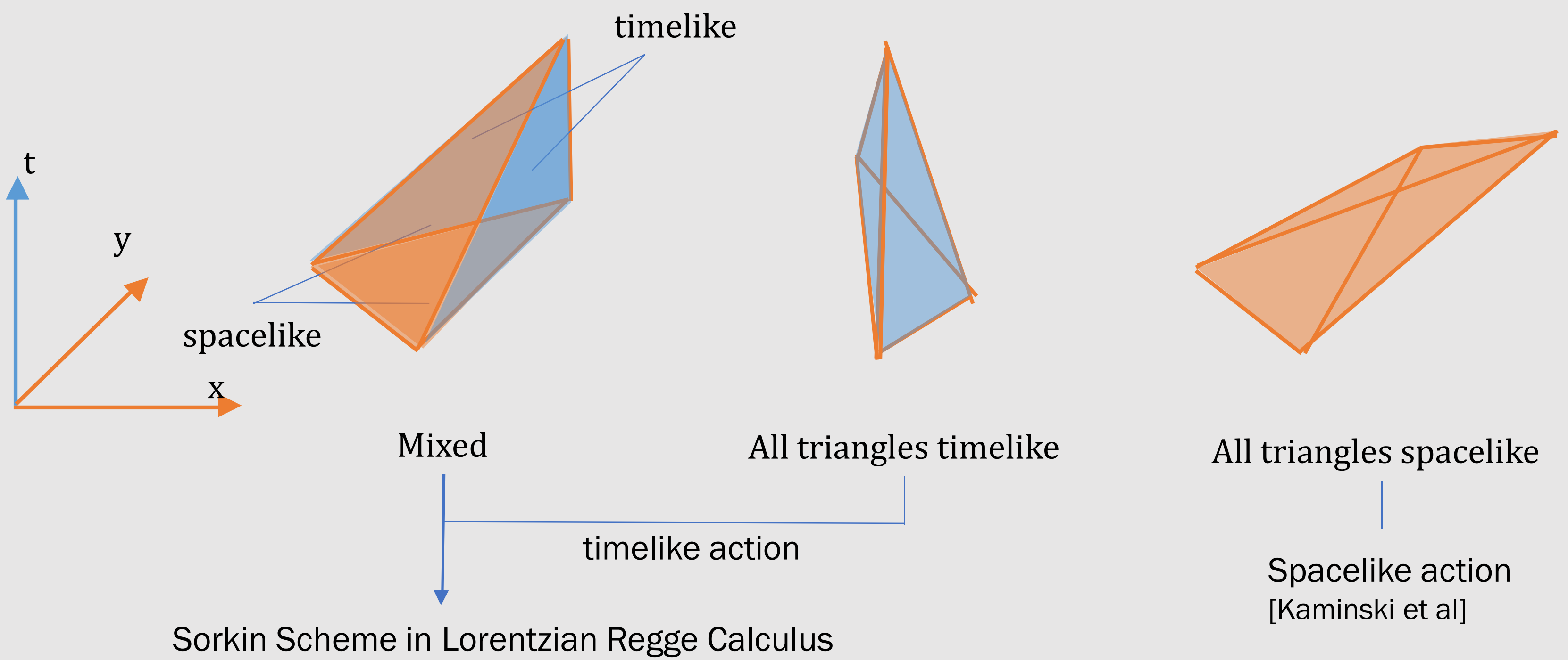
Area spectrum:

$$A_f = \begin{cases} \frac{n_f}{2} & \text{timelike triangle} \\ \gamma\sqrt{j_f(j_f+1)} & \text{spacelike triangle} \end{cases}$$

$\rho \in \mathbb{R}, n \in \mathbb{Z}/2$ labels of SL(2, ℂ) irreps

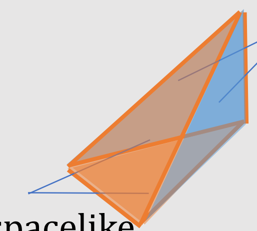
$j = \begin{cases} \mathbb{Z}/2 \\ -\frac{1}{2} + i\, s \in \mathbb{R} \end{cases}$ labels of SU(1,1) irreps

Conrady-Hnybida Extension:
Boundary tetrahedron



A timelike tetrahedron can contain both timelike and spacelike triangles!

SFM Amplitude



$\mathfrak{su}(1,1)$ generators: $\vec{F} = (J^3, K^1, K^2)$ ↑ t ↑ x^1 ↑ x^2

K_1 eigenstates: $K_1|j,\lambda,\pm\rangle = \lambda\,|j,\lambda,\pm\rangle,\ \lambda \in \mathbb{R}$

J_3 eigenstates: $J_3|j,m\rangle = m|j,m\rangle,\ m \in \mathbb{Z}/2$

Coherent states: on each triangle is defined as (with eigenstate of K_1 and J_3)

$$\Psi_{ef} \in \mathcal{H}^{(\rho,n)} = \begin{cases} Y\,D^j(v)|j,-s,+\rangle, & timelike \\ Y\,D^{j\pm}(v)|j,\pm j\rangle, & spacelike \end{cases}, v \in SU(1,1)$$

The amplitude now given by

$$A(K) = \sum_{j_f} \prod_f \mu(j_f) \prod_{(\nu,e)} \int_{\mathbb{SL}(2,\mathbb{C})} dg_{\nu e} \prod_{(e,f)} \int_{S^2} dN_{ef} \prod_{\nu\in f} \left\langle \Psi_{\rho_f n_f}(N_{ef}) \right| D^{(\rho_f,n_f)}(g_{ev}g_{ve'}) \left| \Psi_{\rho_f n_f}(N_{e'f}) \right\rangle$$

Amplitude appears in integration form (in a large j approximation)

Timelike face action

$$A_v(\mathcal{K}) = \int_{\mathbb{SL}(2,\mathbb{C})} \prod_e dg_{ve} \prod_{f\in t} \int_{CP_1} \frac{\Omega_{z_{vf}}}{h_{vef}h_{ve'f}} \left(e^{S_{vf+}} + e^{S_{vf+}} + e^{S_{vf+}} + e^{S_{vf+}}\right) \prod_{f\in s} \int_{CP_1} \Omega_{z_{vf}} e^{S_{vf-sp}}$$

Spacelike face action
[Kaminski 17']

- Actions for timelike triangles are pure imaginary!
- ½ order singularity appears in denominator h

Basic variables and gauge transformations

Actions

- Timelike action for timelike triangles

$$S_{vf\pm} = S_{ve'f\pm} - S_{vef\pm}, \quad S_{vfx\pm} = S_{ve'f\pm} - S_{vef\mp}$$

$$S_{vef\pm} = \frac{n_f}{2} \ln \frac{\langle Z_{vef}, l^{\pm}_{ef}\rangle}{\langle l^{\pm}_{ef}, Z_{vef}\rangle} \mp \mathrm{i}s \ln \langle Z_{vef}, l^{\pm}_{ef}\rangle\langle l^{\pm}_{ef}, Z_{vef}\rangle + \mathrm{i}(\rho_f \pm s) \ln \langle Z_{vef}, Z_{vef}\rangle$$

Pure imaginary!!

$\langle l, m\rangle = l^{\dagger}\sigma_3 m$ SU(1,1) inner product

- Spacelike action for spacelike triangles, complex [Kaminski et al.]

$$S^{\pm}_{vf-sp} = \mathrm{i}\gamma j_f \ln \frac{\langle Z_{vef}, Z_{vef}\rangle}{\langle Z_{ve'f}, Z_{ve'f}\rangle} - j_f \ln \frac{\langle \xi^{\pm}_{e'f}, Z_{ve'f}\rangle^2 \langle Z_{vef}, \xi^{\pm}_{ef}\rangle^2}{\langle Z_{vef}, Z_{vef}\rangle\langle Z_{ve'f}, Z_{ve'f}\rangle}$$

Variables

$$z_{vf} \in \mathbb{C}P^1, \quad g_{ve} \in SL(2,\mathbb{C}), \quad Z_{vef} = g^{\dagger}_{ve}\bar{z}_{vf} \in \mathbb{C}^2, \quad v_{ef} \in SU(1,1),$$

$$l^{\pm}_{ef}, \xi^{\pm}_{ef} \in \mathbb{C}^2, \quad l^{\pm}_{ef} = v_{ef}\binom{1}{\pm 1}, \quad \xi^{+}_{ef} = v_{ef}\binom{1}{0}, \quad \xi^{-}_{ef} = v_{ef}\binom{0}{-1}$$

s.t. $\langle l^+, l^-\rangle = \langle \xi^+, \xi^+\rangle = \langle \xi^-, \xi^-\rangle = 1, \langle \xi^+, \xi^-\rangle = \langle l^+, l^+\rangle = \langle l^-, l^-\rangle = 0.$

Null basis in $\mathbb{C}^2$

Basic variables and gauge transformations

Actions

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$$S_{vf\pm} = S_{ve'f\pm} - S_{vef\pm}, \quad S_{vfx\pm} = S_{ve'f\pm} - S_{vef\mp}$$

$$S_{vef\pm} = \frac{n_f}{2} \ln \frac{\langle Z_{vef}, l^{\pm}_{ef} \rangle}{\langle l^{\pm}_{ef}, Z_{vef} \rangle} \mp \mathrm{i}s \ln \langle Z_{vef}, l^{\pm}_{ef} \rangle \langle l^{\pm}_{ef}, Z_{vef} \rangle + \mathrm{i}(\rho_f \pm s) \ln \langle Z_{vef}, Z_{vef} \rangle$$

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Gauge transformations

$$g_{ve} \to g_v g_{ve}, \qquad z_{vf} \to \lambda_{vf} (g_v^T)^{-1} z_{vf}$$

$$g_{ve} \to s_{ve} g_{ve}, \qquad s_{ve} = \pm 1$$

$$g_{ve} \to g_{ve} v_e, \qquad l^{\pm}_{ef} \to v_e l^{\pm}_{ef}$$

Asymptotics of the amplitude
stationary phase approximation

- Semiclassical limit of spinfoam models:
  - *Area spectrum:* $A_f = l_p^2 n_f / 2$
  - *Keep area fixed & Take* $l_p^2 \to 0$
  - *Results in scaling* $n_f \sim s_f \to \infty$ *uniformly.* *Large - j asymptotics*

SU(1,1) continuous series

$$j = -\frac{1}{2} + i\, s,\ n \sim \gamma s, \rho \sim -s$$

- Recall integration form of the amplitude (timelike face part only)

$$A_v(\mathcal{K}) = \int_{CP_1} \frac{\Omega_{z_{vf}}}{h_{vef-} h_{ve'f+}} \left(e^{S_{vf+}} + e^{S_{vf-}} + e^{S_{vfx+}} + e^{S_{vfx-}} \right)$$

S linear in j_f and pure imaginary

½ order singularity appears in denominator h

Stationary phase approximation with branch point!

Stationary phase analysis
with branch points

Asymptotics of integration I: $\Lambda \to \infty$

$$I = \int \mathrm{d}x \frac{1}{\sqrt{x - x_0}} g(x) \mathrm{e}^{\Lambda S(x)}$$

Critical point x_c: solutions of $\delta_x S(x) = 0$

x_0: ½ order singularity

Stationary phase approximation with branch point:

- Critical point locates at the branch point ← Our case ✔

$$I \sim g(x_c) \frac{\pi \mathrm{e}^{\mathrm{i}\pi(\mu-2)/8}}{\Gamma(3/4)} \left(\frac{2}{\Lambda |S_{xx}(x_c)|} \right)^{1/4} \mathrm{e}^{\Lambda S(x_c)}$$

- Critical point and branch point are separated ✗

Multivariable case: iterate evaluation.
Measure factor is more invovled.

Critical point equations

for a general timelike tetrahedron

Decomposition of Z_{vef} using $l^{\pm}_{ef}$: change of variables

$Z_{vef} \in \mathbb{C}^2$

$$Z_{vef} = \zeta_{vef}(l^{\pm}_{ef} + \alpha_{vef} l^{\mp}_{ef})$$

$$Z_{vef} = \zeta_{vef}(\xi^{\pm}_{ef} + \alpha_{vef} \xi^{\pm}_{ef})$$

$\zeta_{vef} \in \mathbb{C}$

$\alpha_{vef} \in \mathbb{C}$

$Z_{vef} = g^{\dagger}_{ve} \bar{z}_{vf}$ impose:

$$g_{ve}\ J\ (l^{\pm}_{ef} + \alpha_{vef} l^{\mp}_{ef}) = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'}\ J\ (l^{\pm}_{e'f} + \alpha_{ve'f} l^{\mp}_{ef})$$

Real condition of spacelike action : $\alpha_{vef} = 0$

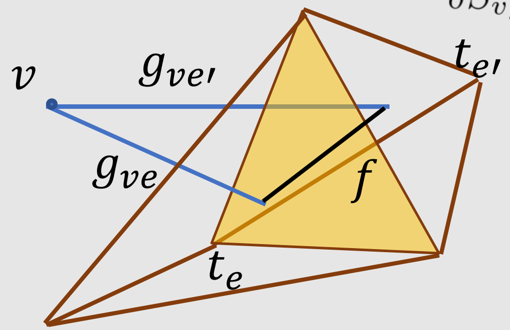
$$g_{ve} J \xi^{\pm}_{ef} = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'} J \xi^{\pm}_{e'f}$$

Variation respect to z_{vf} : gluing condition on edges $e \to e'$

Timelike:

$$\delta S_{vf+} = (\gamma - \mathrm{i}) s_f \left(\frac{g_{ve} \eta l^{+}_{ef}}{\bar{\zeta}_{vef}} - \frac{g_{ve'} \eta l^{+}_{e'f}}{\bar{\zeta}_{ve'f}} \right) = 0 \qquad \text{with} \quad Z = \zeta(l^{-} + \alpha l^{+})$$

$$\delta S_{vf-} = -\mathrm{i} s_f \left(\frac{g_{ve} \eta n_{vef}}{\mathrm{Re}(\alpha_{vef}) \bar{\zeta}_{vef}} - \frac{g_{ve'} \eta n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f}) \bar{\zeta}_{ve'f}} \right) = 0 \qquad \text{with} \quad Z = \zeta(l^{+} + \alpha l^{-})$$



$$n_{vef} := l^{+}_{ef} + \mathrm{i}\left(\gamma\,\mathrm{Re}(\alpha_{vef}) + \mathrm{Im}(\alpha_{vef})\right) l^{-}_{ef} \qquad \langle n, l^{+} \rangle = 0$$

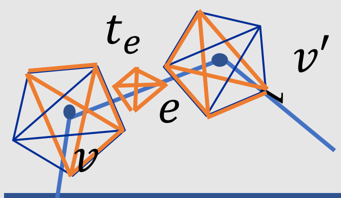
Spacelike:

$$g_{ve} \eta \xi^{\pm}_{ef} = \frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}} g_{ve'} \eta \xi^{\pm}_{e'f}$$

Critical point equations

for a general timelike tetrahedron

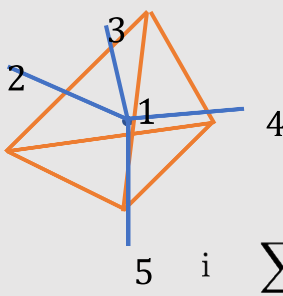
Variation respect to l_{ef} : gluing condition on vertices $v \to v'$



Timelike action: $\gamma\,\mathrm{Re}(\alpha_{vef}) \mp \mathrm{Im}(\alpha_{vef}) = \gamma\,\mathrm{Re}(\alpha_{v'ef}) \mp \mathrm{Im}(\alpha_{v'ef})$

Spacelike action: $\alpha_{vef} = 0$ ← Real Condition

Variation respect to g_{ve} : closure on triangules f



null vectors

$$(1+\gamma^2) \sum_{f \text{ /w } S_{+(x)}} s_f\,\mathrm{Re}(\alpha_{vef})\langle l_{ef}^+, F^i l_{ef}^+\rangle + \sum_{f \text{ /w } S_{-(x)}} s_f \frac{\langle n_{ef}, F^i n_{ef}\rangle}{\mathrm{Re}(\alpha_{vef})} = 0$$

$$\mathrm{i} \sum_{f \text{ /w } S_{+(x)}} s_f \langle l_{ef}^- + \mathrm{i}\,\mathrm{Im}(\alpha_{vef}) l_{ef}^+, F^i l_{ef}^+\rangle + \mathrm{i} \sum_{f \text{ /w } S_{-(x)}} s_f \langle l_{ef}^-, F^i n_{ef}\rangle + \sum_{f \text{ /w } S_{sp}} j_f \langle \xi_{ef}^{\pm}, F^i \xi_{ef}^{\pm}\rangle = 0$$

spacelike vectors

timelike vectors

Check the paper for cases when all triangles are timelike

Geometric interpretation
of critical configuration

Critical point solutions $\mathbb{C}^2, \mathrm{SL}(2,\mathbb{C})$ → Geometrical interpretation $\mathbb{R}^{(1,3)}, \mathrm{SO}(1,3)$ → Simplicial geometry

- Define maps $\mathbb{C}^2 \to \mathbb{R}^{(1,3)}$, $\pi: \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)$
- With gauge transformation $g \to -g$, we can always gauge fix $G = \pi(g) \in \mathrm{SO}_+(1,3)$
- Geometric vector and bivectors

$$V = v^i F^i = -\eta l^- \otimes (l^+)^\dagger + \frac{1}{2}\langle l^+, l^- \rangle I_2$$

spin 1 →

$$\begin{pmatrix} 0 & -v^1 & -v^2 & 0 \\ v^1 & 0 & v^0 & 0 \\ v^2 & -v^0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = *(v \wedge u)^{IJ}$$

$u = (0,0,0,1)$

$$v^i = 2\langle l^+, F^i l^- \rangle$$

→

$$v^I := (v^0, -v^2, v^1, 0) = \mathrm{i}(\langle l^- | \hat{\sigma}^I | l^+ \rangle + u^I)$$

normal of triangles in a tetrahedron

Geometric solution

for a tetrahedron with both timelike and spacelike triangles

A non-degenerate tetrahedron geometry exists only when timelike triangles is with action S_{vf+}

We define a bivector

$$B_{ef} = 2A_f V_{ef} = 2A_f * (v_{ef} \wedge u)$$

with vectors and areas

$$v^I_{ef} = \begin{cases} -\mathrm{i}(\left\langle l^+_{ef}\right| \sigma^I \left| l^-_{ef}\right\rangle - u^I) & \text{for timelike triangle} \\ \left\langle \xi^\pm_{ef}\right| \sigma^I \left|\xi^\pm_{ef}\right\rangle - \langle \xi^\pm, \xi^\pm \rangle u^I & \text{for spacelike case} \end{cases}, \quad A_f = \begin{cases} \gamma s_f = n_f/2 & \text{for timelike triangle} \\ \gamma j_f = \gamma n_f/2 & \text{for spacelike triangle} \end{cases}$$

Define: $B_{ef}(v) = G_{ve} B_{ef} G^{-1}_{ve}$ $\quad N_e(v) = G_{ve} u$

Critical point equations

Parallel transport

$$B_{ef}(v) = B_{e'f}(v) = B_f(v)$$

$$N_e(v) \cdot B_{ef}(v) = 0 \qquad \sum_{f \in t_e} \epsilon_{ef}(v) B_{ef}(v) = 0$$

Simplicity condition

Closure

Geometric Reconstruction
single non-degenerate 4 simplex

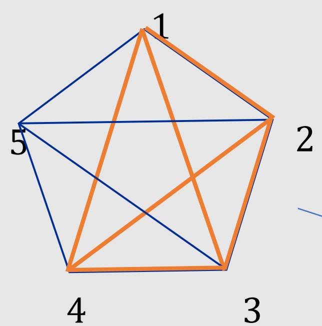
A geometrical 4-simplex ⟷ (Cut) ⟷ Geometric solutions (non-degenerate)

Outpointing normals: N_i^Δ ⟷ $N_i = \pm N_i^\Delta$ ⟷ Normals N_i every 4 out of 5 linearly independent

Bivectors B_{ij}^Δ ⟷ $B_{ij} = \pm B_{ij}^\Delta$ ⟷ Bivectors B_{ij}

Geometric rotations: $G^\Delta \in O(1,3)$ ⟷ $G_i = G_i^\Delta I^{s_i}(IR_u)^s$ ⟷ Geometrical solution $G \in SO(1,3)$

Determine tetrahedron edge lengths uniquely



Length matching condition

Orientation matching condition

[Barrett et al, Han et al, Kaminski et al,....

Geometric Reconstruction
single 4 simplex

Reconstruction theorem

The non-degenerate geometrical solution exists if and only if the lengths and orientations matching conditions are satisfied.

There will be two gauge in-equivalent geometric solutions $\{G_{ve}\}$, such that the bivectors $B_f(v) = * \left(G_{ve} v_{ef} \wedge G_{ve} u\right)$ correspond to bivectors of a reconstructed 4 simplex as

$$B_f(v) = r(v) B_f^{\Delta}(v), \;\; r(v) = I^{s(v)} = \pm 1$$

And normals

$$N_e(v) = I^{s_e(v)} N_e^{\Delta}(v) = \pm N_e^{\Delta}(v)$$

The two gauge equivalence classes of geometric solutions are related by

$$\tilde{G} = R_{e_\alpha} G R_{u_e} I^{s_e} \in SO_+(1,3)$$

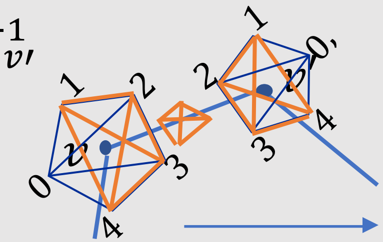
Geometric Reconstruction
Simplicial complex with many 4 simplicies

Gluing condition $e = (v, v')$

$$N_e(v) = G_{v\,v'} N_e(v')$$
$$B_f(v) = G_{v\,v'} B_f(v') G_{v\,v'}^{-1}$$

Consistent Orientation:

$v: [p_0, p_1, p_2, p_3, p_4]$
$v': -[p_0{}', p_1, p_2, p_3, p_4]$



Reconstruction at v, v'

$$N_e(v) = I^{s_e(v)} N_e^{\Delta}(v)$$
$$B_f(v) = r(v) B_f^{\Delta}(v)$$

$$sgn\big(V(v)\big) r(v) = const$$

Reconstruction theorem

The simplicial complex $\mathcal{K}$ can be subdivided into sub-complexes $\mathcal{K}_1, \ldots, \mathcal{K}_2$, such that (1) each $\mathcal{K}_i$ is a simplicial complex with boundary, (2) within each sub-complex $\mathcal{K}_i$, $sgn\big(V(v)\big)$ is a constant. Then there exist

$$G_f^{\Delta} \in SO(1,3) = I^{\sum_{v \in f} 1} I^{\sum_{v,e \in f} s_e(v)} G_f = \pm G_f.$$

such that G_f^{Δ} are the discrete spin connection compatible with the co-frame.

For two non gauge inequivalent geometric solutions with different r:

$$\widetilde{G_f} = R_u G_f R_u$$

Degenerate Solutions?

Degenerate Condition: All normals are parallel to each other: $G_{ve} \in SO(1,2)$

if the 4-simplex contains both timelike and spacelike tetrahedra

→ **Can not be degenerate!!**

Vector geometries

$$G_{ve}\, v_{ef} = G_{ve'} v_{e'f}, \qquad \sum_f \varepsilon_{ef}(v)\ v_{ef} = 0$$

Flipped signature solution

1-1 correspondence to solutions in split signature space M' with (-,+,+,-)

- pair of two non-gauge equivalent vector geometries $G_{ve}^{\pm}$,
- Geometric $SO(M')$ non-degenerate solution G'_{ve}.

The two vector geometries are obtained from $SO(M')$ solutions with map $\Phi^{\pm}$:

$$\Phi^{\pm}(G'_{ve}) = G_{ve}^{\pm}$$

Reconstruction → shape matching → Reconstruction 4-simplex in M'

Geometric $SO(M')$ solution G'_{ve} degenerate: vector geometries $G_{ve}^{\pm}$ gauge equivalent

Action at critical configuration

timelike triangles

Recall stationary phase formula $I \sim g(x_c)\frac{\pi e^{i\pi(\mu-2)/8}}{\Gamma(3/4)}\left(\frac{2}{\Lambda|S_{xx}(x_c)|}\right)^{1/4} e^{\Lambda S(x_c)}$

Action after decomposition: A phase

$$S_f = \sum_{v\in\partial f} S_{vf} = -2is_f\left(\sum_{v\in\partial f}\theta_{e'vef} + \gamma\sum_{v\in\partial f}\phi_{e'vef}\right)$$

Asymptotics of amplitude: determine $\theta_{e'vef}$ and $\phi_{e'vef}$ at critical points

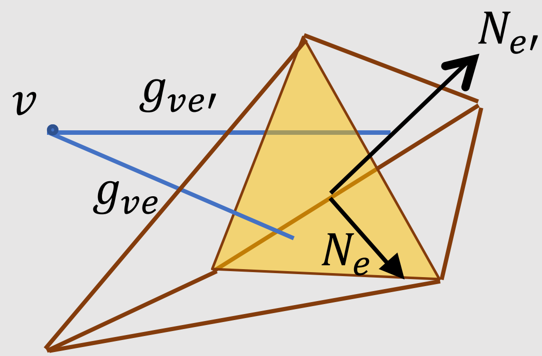
$$\theta_{e'vef} := \ln\frac{|\zeta_{ve'f}|}{|\zeta_{vef}|}, \qquad \phi_{e'vef} := \arg(\zeta_{ve'f}) - \arg(\zeta_{vef})$$

U(1) ambiguity from boundary coherent states $\Psi = D\big(v(N)e^{i\,u\,K_1}\big)|j,-s\rangle$

$$e^S \sim \langle\Psi|D(g^{-1}g')|\Psi'\rangle$$

Phase difference at two critical points $\Delta S = S(G) - S(\tilde{G})$

Phase difference
on one 4 simplex with boundary triangles



$$G_f(v) = G_{e'v} G_{ve}$$

Boundary tetrahedron normals

$$\cos\theta_f = N_e^\Delta \cdot N_{e'}^\Delta, \qquad \theta_f \in (0,\pi)$$

Reconstruction theorem

Two solutions for given boundary data.

$$\tilde{G} = R_{e_\alpha} G R_{u_e} I^{s_e} \in SO_+(1,3)$$

$$G_{ve} \tilde{G}_f^{-1} G_f G_{ev} = R_{N_e} R_{N_{e'}}$$

Reflection

Rotation in a spacelike plane

From critical point equations

$$g_{ve} (\tilde{g}_{e'v} \tilde{g}_{ve})^{-1} g_{e'v} = \mathrm{e}^{-2\Delta\theta_{e'vef} X_f + 2\mathrm{i}\Delta\phi_{e'vef} X_f}$$

Reconstruction

$$\mathrm{e}^{2r\Delta\theta_{e'vef} \frac{*(N_{e'}^\Delta \wedge N_e^\Delta)}{|N_{e'}^\Delta \wedge N_e^\Delta|} + 2r\Delta\phi_{e'vef} \frac{N_{e'}^\Delta \wedge N_e^\Delta}{|N_{e'}^\Delta \wedge N_e^\Delta|}} = R_{N_e^\Delta} R_{N_{e'}^\Delta} = \mathrm{e}^{2\theta_f \frac{N_e^\Delta \wedge N_{e'}^\Delta}{|N_e^\Delta \wedge N_{e'}^\Delta|}}$$

$$\Delta\theta_{e'vef} = 0, \qquad -r\Delta\phi_{e'vef} = \theta_f \mod \pi$$

Phase difference
on simplicial complex with many 4 simplices

Face holonomy: $G_f(v) = \prod_v G_{e'v} G_{ve}$

Reconstruction theorem: for two gauge inequivalent solutions (with different uniform orientation r)

$$\widetilde{G_f} = R_u G_f R_u \qquad G_{ve}\tilde{G}_f^{-1} G_f G_{ev} = R_{N_e} R_{N_{e'}}$$

Rotation in a spacelike plane

Now the normals become

$$N_e = G_{ve} u, \qquad N_{e'} = G_{ve}(G_f^{-1} u)$$

parallel transported vector along the face

From critical point equations

$$g_{ve}\tilde{G}_f^{-1} G_f g_{ev} = \mathrm{e}^{-2\sum_{v\in\partial f}\Delta\theta_{e'vef} X_f + 2\mathrm{i}\sum_{v\in\partial f}\Delta\phi_{e'vef} X_f}$$

Reconstruction

$$\mathrm{e}^{2r\sum_{v\in\partial f}\Delta\theta_{e'vef}\frac{*(N^\Delta_{e'}\wedge N^\Delta_e)}{|N^\Delta_{e'}\wedge N^\Delta_e|} + 2r\sum_{v\in\partial f}\Delta\phi_{e'vef}\frac{N^\Delta_{e'}\wedge N^\Delta_e}{|N^\Delta_{e'}\wedge N^\Delta_e|}} = R_{N^\Delta_e} R_{N^\Delta_{e'}} = \mathrm{e}^{2\theta_f \frac{N^\Delta_e \wedge N^\Delta_{e'}}{|N^\Delta_e\wedge N^\Delta_{e'}|}}$$

$$\sum_{v\in\partial f}\Delta\theta_{e'vef} = 0, \qquad -r\sum_{v\in\partial f}\Delta\phi_{e'vef} = \theta_f \mod \pi$$

Phase difference

$A_f = \gamma s_f = n_f/2 \in Z/2$

Now phase difference

$$\Delta S_f = 2\mathrm{i}rA_f\theta_f \quad \mathrm{mod} \quad \mathrm{i}\pi$$

$i\pi$ ambiguity relates to the lift ambiguity!

Some of them may be absorbed to gauge transformations $g_{ve} \to -g_{ve}$

Fixed at each vertex $\sum_{v\in\partial f} \Delta\phi_{e'vef} = -\sum_{v\in\partial f} \theta_f(v) \quad \mathrm{mod}\ 2\pi$

θ_f here a rotation angle $\cos\theta_f = N_e^\Delta \cdot N_{e'}^\Delta, \qquad \theta_f \in (0,\pi)$

No. of simplicies in the bulk

Related to dihedral angles $\Theta_f(v) = \pi - \theta_f(v)$

deficit angles $\epsilon_f(v) = 2\pi - \sum_{v\in f}\Theta_f(v) = (1 - m_f)\pi - \theta_f(v)$

The total phase difference

$$\exp(\Delta S_f) = \exp\left\{2\mathrm{i}r\sum_{f\ \mathrm{bulk}} A_f\left[(2-m_f)\pi - \varepsilon_f\right] + 2\mathrm{i}r\sum_{f\ \mathrm{boundary}} A_f\left[(1-m_f)\pi - \theta_f\right]\right\}$$

When m_f even, Regge action up to a sign

In all known examples of Regge calculus, m_f is even

Degenerate solutions

From critical point equations: two non-gauge equivalent solution $g_{ve}^{\pm}$

$$g_{ev}^{\pm}g_{ev}^{\mp}g_{ve'}^{\mp}g_{e'v}^{\pm} = \mathrm{e}^{\mp 2\Delta\theta_{e'vef}X_f^{\pm} \pm 2\mathrm{i}\Delta\phi_{e'vef}X_f^{\pm}}$$

Degenerate: $g_{ve}^{\pm} \in SU(1,1)$ → $2\Delta\phi_{e'vef} = 0 \mod 2\pi$ ← Lift ambiguity

1-1 Correspondence to flipped signature non-degenerate solutions:

$$\Phi^{\pm}(G'_{ev}\tilde{G}'_{ev}\tilde{G}'_{ve'}G'_{e'v}) = G_{ev}^{\pm}G_{ev}^{\mp}G_{ve'}^{\mp}G_{e'v}^{\pm} = \Phi^{\pm}(R_{N_e}R_{N_{e'}})$$

$$\Phi^{\pm}(\mathrm{e}^{2\Delta\theta_{e'vef}*'X_f}) = \mathrm{e}^{\mp 2\Delta\theta_{e'vef}X_f^{\pm}}$$

bivectors in flipped signature space

$$\mathrm{e}^{-r2\Delta\theta_{e'vef}\frac{N_{e'}^{\Delta}\wedge N_e^{\Delta}}{|*(N_{e'}^{\Delta}\wedge N_e^{\Delta})|}} = R_{N_e}R_{N_{e'}} = \mathrm{e}^{2\theta_f\frac{N_e^{\Delta}\wedge N_{e'}^{\Delta}}{|N_e^{\Delta}\wedge N_{e'}^{\Delta}|}},$$

$$\Delta\theta_{e'vef} = -r\theta_f \quad \longrightarrow \quad \exp(\Delta S_f) = \exp\left(2\mathrm{i}r\frac{1}{\gamma}A_f\theta_f\right)$$

Regge action

Conclusion & Outlook

- The asymptotic analysis of spin foam model is now complete (with non degenerate boundaries)
- The asymptotics of the amplitude is dominated by critical configurations which are simplicial geometry
- The model excludes degenerate geometry sector
- The asymptotic limit of the amplitude is related to Regge action

Outlooks

- Continuum limit and Results in Lorentzian Regge calculus: emerging gravity from spin foams (Kasner and FLRW cosmology model with Han.)
- Black holes in semi-classical limit of spin foam models
- Graviton propagator
- Causal structure?
- ....

Thanks for your attention