Dirac Observables and \((b, v)\)-Type Variables for Effective Polymer Black Holes

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based on


with

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Motivation

- BH are of interest in all main QG approaches
- May provide better understanding of full LQG
- Singularity resolution in AdS/CFT [Bodendorfer, Schäfer, Schliemann '16, Bodendorfer, FM, JM '18]

- Construct effective model inspired by LQG
- Lot of previous effort: [Ashtekar, Olmedo, Singh '18; Bianchi, Christodoulou, D'Ambrosio, Haggard, Rovelli '18; Olmedo, Saini, Singh '17; Perez '17; Corichi, Singh '16; Modesto '10; Pullin '07-,...]
- No consensus about BHs in LQG community

- BH interior has Kantowski-Sachs (cosmology) structure
- Use techniques of LQC (polymerisation)
- Subtleties: Dirac observables / Different schemes
Plan of the Talk

PART I: Dirac observables (Johannes)

- Classical theory
- Effective quantum theory and previous models

PART II: \((b, v)\)-type variables (Fabio)

- Adapted variables for effective polymer BHs
- Overcome previous limitations for physical viability

Conclusions and outlook
Mass and Horizon Dirac Observables in Polymer BH Models
Classical Setting

Spherically symmetric and static spacetime

[Vakili '18; Cavaglia, Alfaro, Filippov '94]

\[ ds^2 = -\bar{a}(r)dt^2 + N(r)dr^2 + \theta(r)^2d\Omega_2^2 \]

For \( \bar{a}, N < 0 \) t spacelike, \( r \) timelike \( \rightarrow \) Kantowski-Sachs-Cosmology

Regularisation: \( L_o \) and \( \mathcal{L}_o = \int_0^{L_o} \sqrt{\bar{a}} \bigg|_{r=r_{\text{ref}}} \ dt \)

\[ \sqrt{a} = \int_0^{L_o} \sqrt{-\bar{a}} \ dt = L_o \sqrt{-\bar{a}}, \quad n = Na \]

Connection variables for the interior

[Ashtekar, Olmedo, Singh '18; Olmedo, Saini, Singh '17; Corichi, Singh '16; Pullin '07-,...]

\[ ds^2 = -N_T^2(T)\,dT^2 + \frac{p_b^2(T)}{L_o^2|p_c(T)|} \, dx^2 + |p_c(T)| \, d\Omega_2^2 \]

\[ T = r, \quad x = t, \quad |p_c| = \theta^2, \quad p_b = -a\theta^2, \quad -N = N_T^2 \]
Classical Integration Constants

\[ b(T) = \pm \gamma \sqrt{Ae^{-T} - 1} , \quad c(T) = c_0 e^{-2T} \]

\[ p_b(T) \overset{H\approx 0}{=} - \frac{2c p_c}{b + \frac{\gamma^2}{b}} = \pm \frac{2c_0 p_c^0}{\gamma} \sqrt{\frac{e^T}{A} \left( 1 - \frac{e^T}{A} \right)} , \quad p_c(T) = p_c^0 e^{2T} . \]

Solving EoM ⇒ two integration constants \( c_o, p_c^0, A = e^{T_o} = 1 \) (w.l.o.g.)

Express \( a = \frac{p_b^2}{|p_c|} \) in terms of \( \mathcal{a} = \sqrt{|p_c|} \)

\[ \mathcal{a} = \sqrt{|p_c^0|} e^T , \quad a(\mathcal{a}) = \frac{4c_0^2 |p_c^0|}{\gamma^2 L_o^2} \left( \frac{\sqrt{|p_c^0|}}{\mathcal{a}} - 1 \right) \Rightarrow R_{\text{hor}} = \sqrt{|p_c^0|} \]

Line element

Redefining coordinates \( \tau = \sqrt{|p_c^0|} e^T, \quad y = \frac{2c_0 \sqrt{|p_c^0|}}{\gamma L_o} x \)

\[ ds^2 = -\frac{1}{R_{\text{hor}} \frac{\tau}{\tau} - 1} d\tau^2 + \left( \frac{R_{\text{hor}}}{\tau} - 1 \right) dy^2 + \tau^2 d\Omega_2^2 , \]
Dirac Observables

Canonical degrees of freedom

4 phase space d.o.f + 1 first class constraint \( \Rightarrow 2 \) Dirac observables

\[
R_{\text{hor}} = \sqrt{|p_c|} \left( \frac{b^2}{\gamma^2} + 1 \right)^{\text{on-shell}} = R_{\text{hor}} = \sqrt{|p^o_c|}, \quad D = c p_c^{\text{on-shell}} = c_o p^o_c
\]

Only \( R_{\text{hor}} \) physical, \( D \) is irrelevant (not in metric + fiducial cell dep.)

Residual diffeomorphisms

Redefining coordinates \( y = \frac{2c_o \sqrt{|p^o_c|}}{\gamma L_o} x \) changes \( \tilde{a} \mapsto \left( \frac{2c_o \sqrt{|p^o_c|}}{\gamma L_o} \right)^{-2} \tilde{a} \)

\[
\sqrt{a} = L_o \sqrt{\tilde{a}} = \int_0^{L_o} dx \sqrt{\tilde{a}} = \int_{y(0)}^{y(L_o)} dy \sqrt{\tilde{a}} = y(L_o) \sqrt{\tilde{a}} = \sqrt{a}
\]

Transformation not present on phase space!

Canonical Dirac observables do **not know** about this remaining freedom. The physical metric is **affected**.
Conclusions Classical Theory

- Fiducial cell $\rightarrow$ residual diffeomorphisms
- Not present on the phase space $\rightarrow$ visible through $L_\rho$ dependence
- go back to metric $\rightarrow D$ can be absorbed

There are two Dirac observables, only one physically relevant

- $T, x$ rescaling absorbs one Dirac observable
- Identification: Physical observable is $R_{hor} = 2M_{BH}$

Warning!!!

- Observables of quantum theory in classical regime $\neq$ Observables of classical theory
- There, the second Dirac observable might be physically relevant!

$\rightarrow$ Examples
\((v_i, P_i)\)-Variables [Bodendorfer, FM, JM '19]

Define new variables

\[
(p_b)^2 = -8v_2, \quad |p_c| = \left(24v_1\right)^{\frac{2}{3}},
\]

\[
b = \text{sign}(p_b) \frac{\gamma}{4} \sqrt{-8v_2} P_2, \quad c = -\text{sign}(p_c) \frac{\gamma}{8} \left(24v_1\right)^{\frac{1}{3}} P_1.
\]

Polymerisation with constant \(\lambda_i\) and \(P_i \mapsto \sin(\lambda_i P_i)/\lambda_i\)

\[
\hat{\alpha} = \left(\frac{3v_1}{2}\right)^{\frac{1}{3}} = \frac{\mathcal{L}_o}{\lambda_2} \left(3DC^2\lambda_1^2\right)^{\frac{1}{3}} \left(\frac{\lambda_2^6}{16C^2\lambda_1^2\lambda_0^6} \left(\frac{\mathcal{L}_o r}{\lambda_2} + \sqrt{1 + \frac{\mathcal{L}_o^2 r^2}{\lambda_2^2}}\right)^6 + 1\right)^{\frac{1}{3}},
\]

\[
a = \frac{v_2}{2b^2}, \quad v_2 = 2\mathcal{L}_o^2 \left(\frac{\lambda_2}{\mathcal{L}_o}\right)^2 \left(1 + \frac{\mathcal{L}_o^2 r^2}{\lambda_2^2}\right) \left(1 - \frac{3CD}{2\lambda_2} \frac{1}{\sqrt{1 + \frac{\mathcal{L}_o^2 r^2}{\lambda_2^2}}}\right).
\]

Again two integration constants \(C, D\), now: can’t absorb \(D\) in \(r\)
Mass Dirac Observables

\[ F_Q(v_1, P_1, P_2) \text{ on-shell} = \left( \frac{3}{2}D \right)^4 \frac{C}{\mathcal{L}_o} \]

\[ \tilde{F}_Q(v_1, P_1, P_2) \text{ on-shell} = \frac{3CD\mathcal{L}_o}{\lambda_2^2} \left( 3DC^2\lambda_1^2 \right)^{\frac{1}{3}} \]

Both Dirac observables have physical meaning!

Asymptotic Schwarzschild regions

For \( r \to \pm \infty \) spacetime is asymptotically described by

\[ ds_-^2 \simeq -\left( 1 - \frac{F_Q}{\tilde{\beta}} \right) d\tau^2 + \frac{1}{1 - \frac{F_Q}{\tilde{\beta}}} d\tilde{\beta}^2 + \tilde{\beta}^2 d\Omega_2^2 \]

Similar for \( ds_+^2 \) with \( \tilde{F}_Q \).

Identify \( 2M_{BH} = F_Q \) and \( 2M_{WH} = \tilde{F}_Q \) \( \Leftrightarrow \) Initial cond. \( K(\tilde{\beta}_i) \) and \( R(\tilde{\beta}_i) \)
(generalised) $\mu_o$-schemes

[Ashtekar, Olmedo, Singh '18; Corichi, Singh '16;...] and similar [Modesto '09; '10]

$$\dot{b} = -\frac{1}{2} \left( \frac{\sin(\delta_b b)}{\delta_b} + \frac{\gamma^2 \delta_b}{\sin(\delta_b b)} \right), \quad \dot{c} = -2 \frac{\sin(\delta_c c)}{\delta_c},$$

$$\dot{p}_b = \frac{p_b}{2} \cos(\delta_b b) \left( 1 - \frac{\gamma^2 \delta_b^2}{\sin(\delta_b b)^2} \right), \quad \dot{p}_c = 2p_c \cos(\delta_c c).$$

Horizon Dirac observables

$$\mathcal{R}_{BH} = \left[ \frac{p_c \sin(\delta_c c)}{2} \left( \frac{\tan(\frac{\delta_c c}{2})}{B_o^2} \left( \frac{b_o + \cos(\delta_b b)}{b_o - \cos(\delta_b b)} \right) \frac{2}{b_o} + \frac{B_o^2}{\tan(\frac{\delta_c c}{2})} \left( \frac{b_o - \cos(\delta_b b)}{b_o + \cos(\delta_b b)} \right) \frac{2}{b_o} \right) \right]^{\frac{1}{2}},$$

$$\mathcal{R}_{WH} = \left[ \frac{p_c \sin(\delta_c c)}{2} \left( B_o^2 \tan(\frac{\delta_c c}{2}) \left( \frac{b_o + \cos(\delta_b b)}{b_o - \cos(\delta_b b)} \right) \frac{2}{b_o} + \frac{1}{B_o^2 \tan(\frac{\delta_c c}{2})} \left( \frac{b_o - \cos(\delta_b b)}{b_o + \cos(\delta_b b)} \right) \frac{2}{b_o} \right) \right]^{\frac{1}{2}}.$$

For [Modesto '09; '10] also mass observables can be constructed

Two d.o.f. $\Rightarrow$ Revisit arguments of [Ashtekar, Olmedo, Singh '18; Corichi, Singh '16;...]
Both scales **fiducial cell independent** [Bodendorfer, FM, JM '19] ⇒ Only one Dirac observable

### Fiducial cell dependence
- Two masses are unrelated
- More freedom
- Restriction of initial conditions
  \[ M_{WH} = f(M_{BH}) \] by quantum conditions
- Possibility to circumvent?

### Fiducial cell independence
- One observable encodes both masses
- A relation
  \[ M_{WH} = f(M_{BH}) \] is selected by dynamics
- \( M_{BH} \leftrightarrow M_{WH} \) symmetry
  \[ M_{WH} \propto M_{BH}^{\pm 1} \]
- We found a model for \(-1\)
PART II

(b,v)-Type Variables for Effective Polymer Black Holes
New Canonical Variables: Curvature Variables

Previous model

\[
\frac{P_1(\ell)}{\mathcal{L}_0} = \left( \frac{2}{3D} \right)^{\frac{1}{3}} \frac{2M_{BH}}{\ell^3} \propto \sqrt{\mathcal{K}} \quad \text{iff} \quad D \text{ mass-independent}
\]

\[
\downarrow
\]

mass indep. curvature upper bound for specific relations \( M_{WH}(M_{BH}) \)

New variables directly related to curvature

\[
v_k = \left( \frac{3}{2} v_1 \right)^{\frac{2}{3}} \frac{1}{P_2}, \quad v_j = v_2 - \frac{3v_1 P_1}{2P_2}, \quad k = \left( \frac{3}{2} v_1 \right)^{\frac{1}{3}} P_1 P_2, \quad j = P_2
\]

s.t.

\[
k(\ell) \approx R_{\mu\nu\alpha\beta} e^{\mu\nu} e^{\alpha\beta} = \frac{2M_{\text{Misner-Sharp}}(\ell)}{\ell^3} \quad \text{on-shell} \quad \frac{2M_{BH}}{\ell^3} \propto \sqrt{\mathcal{K}}
\]

physical viability independently of the relation between the masses

Similar to \((b, v)\)-variables in LQC, where \( R \propto b^2 \).
Classical Theory in \((v_k, k, v_j, j)\)-Variables

Hamiltonian

\[
H_{cl} = \sqrt{nH_{cl}}, \quad H_{cl} = 3v_k k j + v_j j^2 - 2 \approx 0
\]

One fiducial cell indep. Dirac observable (for two integration constants)

\[
F = k \left( v_k j \right)^{\frac{3}{2}} = (D)^{\frac{3}{2}} \cdot C
\]

Express \( a = \frac{v_j j + v_k k}{2v_k j^2} \) in terms of \( \bar{a} = \sqrt{v_k j} \)

\[
\bar{a} = \sqrt{Dr} \quad , \quad a(\bar{a}) = \frac{1}{D} \left( 1 - \frac{F}{\bar{a}} \right) \quad \Rightarrow \quad 2M_{BH} = F
\]

Line element

Coordinate redefinition \( \tau = t / \sqrt{D} \), \( \bar{a} = \sqrt{Dr} \)

\[
ds^2 = - \left( 1 - \frac{2M_{BH}}{\bar{a}} \right) d\tau^2 + \frac{1}{1 - \frac{2M_{BH}}{\bar{a}}} d\bar{a}^2 + \bar{a}^2 d\Omega_2^2
\]
Polymerisation of the Model

- Interior: \( r \) time-like \( \rightarrow \) homog. Cauchy slices with topology \( \mathbb{R} \times S^2 \)

- On-shell interpretation:

\[
k(\mathcal{O}) = \frac{2M_{BH}}{\mathcal{O}^3} \propto \sqrt{\mathcal{K}} \quad , \quad j(\mathcal{O}) \mathcal{L}_o = \frac{1}{\mathcal{O}} \left( \frac{3D}{2} \right) ^{\frac{1}{3}}
\]

- Polymerisation (constant \( \lambda \rightarrow \mu_o \)-scheme):

\[
k \leftrightarrow \frac{\sin(\lambda_k k)}{\lambda_k} \quad , \quad j \leftrightarrow \frac{\sin(\lambda_j j)}{\lambda_j}
\]

- \( [\lambda_k] = L^2 \) inverse curvature scale (large curv. qu.-effects)
- \( [\lambda_j / \mathcal{L}_o] = L \) length scale (small areal radius qu.-effects)

- Effective Hamiltonian

\[
H_{eff} = \sqrt{n}H_{eff} \quad , \quad H_{eff} = 3v_k \frac{\sin(\lambda_k k)}{\lambda_k} - v_j \frac{\sin^2(\lambda_j j)}{\lambda_j^2} - 2 \approx 0
\]
Solutions of the Effective Dynamics

Metric coefficients (with $x = \mathcal{L}_o r / \lambda_j$):

$$b^2(x) = \frac{1}{2} \left( \frac{\lambda_k}{M_{BH} M_{WH}} \right)^{\frac{2}{3}} \frac{1}{\sqrt{1 + x^2}} \frac{M_{BH}^2 \left( x + \sqrt{1 + x^2} \right)^6 + M_{WH}^2}{(x + \sqrt{1 + x^2})^3}$$

$$\frac{a(x)}{\lambda_j^2} = 2 \left( \frac{M_{BH} M_{WH}}{\lambda_k} \right)^{\frac{2}{3}} \left( 1 - \left( \frac{M_{BH} M_{WH}}{\lambda_k} \right)^{\frac{1}{3}} \frac{1}{\sqrt{1 + x^2}} \right) \frac{(1 + x^2)^{\frac{3}{2}} \left( x + \sqrt{1 + x^2} \right)^3}{M_{BH}^2 (x + \sqrt{1 + x^2})^6 + M_{WH}^2}$$

**Main features**

- Solution extends to exterior, $r \in (-\infty, \infty)$, two asymptotic regions
- Two horizons at $r = r_{s}^{(\pm)}$ s.t. $a(r) = 0$:
  $$b_{\pm} \simeq 2 M_{BH}/WH + \text{quant. corrections} \ (\to 0 \ \text{as} \ \lambda \to 0)$$
- $b$ has non-zero minimum (*transition surface*)
- Two integration constants $M_{BH}, M_{WH}$, both physically relevant
- $\lambda_j$ does not appear in the final metric
### Integration Constants

#### Dirac observables

\[
F_Q = \frac{\sin(\lambda_k k)}{\lambda_k} \cos \left( \frac{\lambda_k k}{2} \right) \left( \frac{2v_k}{\lambda_j \cot \left( \frac{\lambda_j j}{2} \right)} \right)^{\frac{3}{2}},
\]

\[
\bar{F}_Q = \frac{\sin(\lambda_k k)}{\lambda_k} \sin \left( \frac{\lambda_k k}{2} \right) \left( \frac{2v_k}{\lambda_j \cot \left( \frac{\lambda_j j}{2} \right)} \right)^{\frac{3}{2}}
\]

#### Asymptotic Schwarzschild regions

For \( r \to \pm \infty \) spacetime is asymptotically described by

\[
ds^2_+ \simeq - \left( 1 - \frac{F_Q}{\ell} \right) d\tau^2 + \left( 1 - \frac{F_Q}{\ell} \right)^{-1} db^2 + \ell^2 d\Omega_2^2
\]

Similarly for \( ds^2_- \) with \( \bar{F}_Q \).

\[\implies \text{Identify} \quad 2M_{BH} = F_Q \quad \text{and} \quad 2M_{WH} = \bar{F}_Q\]

Note: \( M_{BH} \) and \( M_{WH} \) independent!
Quantum-Corrected Effective Spacetime Structure

- Eff. metric smooth in the whole $r$-domain $r \in (-\infty, +\infty)$
- Quantum effects relevant in large curvature regime
- Singularity resolved by transition surface $\mathcal{T}$
  - (BH-to-WH transition)
- Infinitely many trapped (BH) and anti-trapped (WH) regions
- Asymp. Schwarzschild spacetimes with alternating masses $M_{BH}, M_{WH}$
Curvature Upper Bound

**Requirement**

Unique (mass-indep.) Planckian upper bound for curvature invariants

- Analyse Kretschmann scalar at transition surface
- All mass relations fine
- Special is the range

\[
\frac{1}{8} < \frac{M_{BH}}{M_{WH}} < 8
\]

\[\lambda_k = \lambda_j = 1\]

- \(M_{WH} = 8M_{BH}\),
- \(M_{WH} = \frac{1}{8}M_{BH}\).
Onset of Quantum Effects

For $M_{WH} = mM_{BH}$, $\lambda_k = \lambda_j = 1$

- For $\frac{1}{8} < \frac{M_{BH}}{M_{WH}} < 8$
  
  $j$-sector dominates

- For $m = 1$ only $\lambda_k$ scale relevant

$$\mathcal{K}_{cl}^{BH} = \frac{48 M_{BH}^2}{\mathcal{C}_+^6} \ll \frac{3}{4 \lambda_k^2} \left( \frac{M_{BH}}{M_{WH}} \right)^2$$
Quantum Theory: A Sketch

[Ashtekar, Corichi, Singh '07; Martín-Benito, Mena Marugán, Olmedo '09]

- For $\lambda_k = \lambda_j = 2$ and $\sqrt{n} = v_j$, we choose the ordering:

$$H_{\text{eff}} = 3\sqrt{v_k} \left( \frac{\sin(2k)}{4} \text{sign}(v_k) + \text{sign}(v_k) \frac{\sin(2k)}{4} \right) \sqrt{v_k}$$

$$\times \sqrt{v_j} \left( \frac{\sin(2j)}{4} \text{sign}(v_j) + \text{sign}(v_j) \frac{\sin(2j)}{4} \right) \sqrt{v_j}$$

$$+ \left( \sqrt{v_j} \left( \frac{\sin(2j)}{4} \text{sign}(v_j) + \text{sign}(v_j) \frac{\sin(2j)}{4} \right) \sqrt{v_j} \right)^2 - 2v_j \approx 0$$

- Hilbert space: $|\chi\rangle = \sum_{v_k, v_j \in \mathbb{Z}} \tilde{\chi}(v_k, v_j) |v_k, v_j\rangle$

$$\hat{w}_k |v_k, v_j\rangle = v_k |v_k, v_j\rangle \quad , \quad e^{-ip k} |v_k, v_j\rangle = |v_k + \rho, v_j\rangle \quad (\rho \in \mathbb{Z})$$

- Rescaling $\tilde{\chi}(v_k, v_j) = \sqrt{|v_k v_j|} \tilde{\psi}(v_k, v_j)$ and changing variables $y_i = \log(\sinh(x_i))$ with $x_1 = \log(\tan(k/2))$ and $x_2 = \log(\tan(j/2))$:

$$\hat{H} = \left( -3 \partial_{y_1} - \partial_{y_2} + 4i \cosh(y_2) \right) \partial_{y_2}$$

- Solution of $\hat{H} |\psi\rangle = 0$:

$$\psi_{\text{phys}}(y_1, y_2) = g(y_1) + \int_{y_2}^{y_2} dy'_2 e^{4i \sinh(y'_2)} f \left( y'_2 - \frac{1}{3} y_1 \right)$$
Conclusion and Outlook

Summary

- Key role of Dirac observables in effective polymer BH models
- new canonical variables for quantum-corrected Schwarzschild BH (constant polymerisation scales)
- singularity resolved by BH-to-WH transition
- physical viability for all mass relations (symmetric bounce preferred)
- remarkably simple Hamiltonian → Quantum theory analytically solvable!

Future work

- complete quantum theory (role of Dirac obs?)
- relation with full LQG
- other spacetimes:
  - gravitational collapse
  - higher dimensions and asymptotically AdS → holography?
Conclusion and Outlook

Summary

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Thank you for your attention!
Our model in connection variables

Our \((v_k, k, v_j, j)\)-variables are related to connection variables as:

\[
p_b^2 = -\frac{8(v_k k + v_j j)}{j}, \quad |p_c| = 4 \frac{2^2}{3} v_k j,
\]

\[
b = \text{sign}(p_b) \frac{\gamma}{4} \sqrt{-8(v_k k + v_j j) j}, \quad c = -\text{sign}(p_c) \frac{\gamma}{4} \frac{2^3}{3} \frac{k}{j}.
\]

Demanding

\[
\lambda_j j = \lambda_2 P_2 = \delta_b b, \quad \lambda_k k = \lambda_1 P_1 = \delta_c c
\]

our model with const. \(\lambda\)'s corresponds to the following (generalised) \(\bar{\mu}\)-scheme:

\[
\delta_b = \pm \frac{4\lambda_j}{\gamma |p_b|}, \quad \delta_c = \pm \frac{64}{\gamma^2} \frac{2^3}{3} \frac{b}{p_b} \lambda_k
\]