INHOMOGENEOUS QUANTUM COSMOLOGY: THE GOWDY T^3 MODEL

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Outline

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Motivation

- CMB analysis calls for the development of inhomogeneous cosmology. Reliable perturbative quantum formalisms ought to be derived from inhomogeneous quantum cosmology.

- The extension of LQC to inhomogeneous spacetimes would validate present results and provide new predictions. Inhomogeneous quantum cosmological models would serve for comparison of techniques and interpretations.

- The linearly polarized Gowdy T^3 model is the simplest of all inhomogeneous cosmological systems.

- It can be treated as 2+1 gravity coupled to a free massless scalar field with axial symmetry, an interesting system from the viewpoint of QFT in curved backgrounds and parametrized field theory.

- The model is non-stationary and infinite dimensional. Issues of unitarity and uniqueness of the quantization are relevant.

The Gowdy T^3 model

- It descibes vacuum spacetimes with the spatial topology of a three-torus and two commuting, axial, and hypersurface orthogonal Killing vector fields.

- Classical solutions are inhomogenuous spacetimes with a cosmological singularity.

- After gauge fixing:

$$ds^2 = e^{\gamma} e^{-\phi/\sqrt{p}} \left(-dt^2 + d\theta^2\right) + e^{-\phi/\sqrt{p}} t^2 p^2 d\sigma^2 + e^{\phi/\sqrt{p}} d\delta^2 .$$

$t > 0$ is a positive time and $\theta, \sigma, \delta \in S^1$. $\partial_a = \{\partial_\sigma, \partial_\delta\}$ are Killing vector fields.

$p = -\oint P_\gamma/(2\pi) > 0$ and $\gamma = \gamma(Q, p, \phi, P_\phi)$.

Q and $P = ln\, p$ are canonically conjugate.

Deparametrization has been achieved by setting $det\left[(\partial_a)^i h_{ij} (\partial_b)^j\right] = t^2 p^2$.

- There is some arbitrariness in the choice of **time** and of the **field** ϕ.

The reduced model

- There is a **homogeneous constraint** left: $C_0 = \frac{1}{\sqrt{2\pi}} \oint P_\phi \phi' .$

This constraint affects only the field sector (ϕ, P_ϕ) .

- The reduced action is (with 4G=π)

$$S_r = \int_{t_i}^{t_f} dt \left(P \dot{Q} + \left[\oint P_\phi \dot{\phi} \right] - H_r \right), \qquad H_r = \frac{1}{2t} \oint \left[P_\phi^2 + t^2 (\phi')^2 \right].$$

Q and P are constants of motion.

The phase space can be decomposed as $\Gamma_r = \Gamma_0 + \tilde{\Gamma}$, corresponding to a "point particle" and the fieldlike degrees of freedom. We focus on $\tilde{\Gamma}$.

- The field is subject to the wave equation $\ddot{\phi} + \frac{\dot{\phi}}{t} - \phi'' = 0 .$

The reduced model (standard description)

- This is the Klein-Gordon equation of an axially symmetric free massless scalar field propagating in the fictitious 2+1 background $\left(\tilde{M}\simeq \mathbb{R}^{+}\times T^{2},\tilde{g}^{(B)}\right)$:

$$\tilde{g}^{(B)}_{ab}=-dt_a dt_b+d\theta_a d\theta_b+t^2 d\sigma_a d\sigma_b.$$

- Smooth solutions have the form

$$\varphi(t,\theta)=\sum_{n=-\infty}^{\infty}\left[A_n f_n(t,\theta)+A_n^* f_n^*(t,\theta)\right].\qquad f_n(t,\theta)=\bar{f}_n(t)e^{in\theta}\quad \forall n,$$

$$\bar{f}_0(t)=\frac{1-i\ln t}{\sqrt{4\pi}},\qquad \bar{f}_n(t)=\frac{H_0(|n|t)}{\sqrt{8}}\quad n\neq 0.$$

One obtains the covariant phase space $\tilde{\Gamma}_S$, with symplectic structure

$$\tilde{\Omega}_S(\varphi_1,\varphi_2)=\oint\left[\varphi_2 t\partial_t\varphi_1-\varphi_1 t\partial_t\varphi_2\right].$$

- The basis of solutions $\{f_n,f_n^*\}$ is “orthonormal” in the product

$$(f_m,f_n)_{\varphi}=-i\tilde{\Omega}_S(f_m^*,f_n).$$

The reduced model (standard description)

- Given any Cauchy surface Σ_{t_0} (a constant time section $t=t_0$), there is a natural isomorphism $I_{t_0}: \tilde{\Gamma} \to \tilde{\Gamma}_S$ [such that $\phi(\theta)=\varphi(t_0,\theta)$, $P_\phi(\theta)=t_0\partial_t\varphi(t_0,\theta)$].

The evolution between Cauchy surfaces is given by $V(t_1,t_0):=I_{t_1}^{-1} I_{t_0}: \tilde{\Gamma} \to \tilde{\Gamma}$.

- Let us consider a given Σ_{t_0} and define

$$A_n^{(t_1)}[\phi, P_\phi] = \oint \frac{ie^{-in\theta}}{\sqrt{8}} \left[|n| t_1 H_1^*(|n| t_1)\phi + H_0^*(|n| t_1) P_\phi \right],$$

with $A_n^{*(t_1)} = \left[A_n^{(t_1)}\right]^*$. Then $A_n^{(t_0)}[\phi, P_\phi] = A_n = V(t_1,t_o) A_n^{(t_1)}[\phi, P_\phi]$. In this sense $A_n^{(t_0)}$ are constants of motion. Note the explicit dependence of $A_n^{(t_1)}$ on t_1.

- A complex structure is a *real* linear map $\tilde{J}$ on phase space such that $\tilde{J}^2=-1$. It is $\tilde{\Omega}$-compatible if it preserves $\tilde{\Omega}$ and $\tilde{\Omega}(\tilde{J}\,.,.)$ is positive definite.

We choose $\tilde{J}_0\left[A_n^{(t_0)}\right] = i\,A_n^{(t_0)}$, $\tilde{J}_0\left[A_n^{*(t_0)}\right] = -i\,A_n^{*(t_0)}$.

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Standard quantization

- $\frac{1-i\tilde{J}_0}{2}$ provides the projector on the positive frequency sector.

One can then construct the "one-particle" Hilbert space $\tilde{H}_0$, the symmetric Fock space $F(\tilde{H}_0)$, and the physical Hilbert space (by imposing the homogeneous constraint C_0).

- Let V be a symplectic transformation [e.g. $V(t_{1,}t_0)$] and $\tilde{J}_V = V\tilde{J}_0 V^{-1}$.

V is unitarily implementable on $F(\tilde{H}_0) \Leftrightarrow \tilde{J}_V - \tilde{J}_0$ is Hilbert-Schmidt $\Leftrightarrow$ $(\tilde{J}_V - \tilde{J}_0)(\tilde{J}_V + \tilde{J}_0)^{-1}$ is Hilbert-Schmidt $\Leftrightarrow \tilde{J}_V$ and $\tilde{J}_0$ provide unitarily equivalent quantizations $\Leftrightarrow V + \tilde{J}_0 V \tilde{J}_0$ is Hilbert-Schmidt.

The antilinear part of the Bogoliubov transformation that implements the evolution $V(t_{1,}t_0)$ has a single contribution for each mode $n \neq 0$ (in terms of $A_n^{(t_0)}$), that can be deduced employing the "orthonormalization" of the set $\{f_n, f_n^*\}$ in the product $(f_m, f_n)_\varphi = -i\tilde{\Omega}_S(f_m^*, f_n)$.

Standard quantization (unitarity problems)

- The Bogoliubov coefficients for the evolution $V(t_{1,}t_0)$ are

$$\tilde{\beta}_n(t_1,t_0)=i2\pi\left[\overline{f}^*_n(t_0)t_1\partial_t\overline{f}^*_n(t_1)-\overline{f}^*_n(t_1)t_0\partial_t\overline{f}^*_n(t_0)\right]$$

$$=\frac{i\pi|n|}{4}\left[H^*_0(|n|t_1)t_0H^*_1(|n|t_0)-H^*_0(|n|t_0)t_1H^*_1(|n|t_1)\right].$$

From Hankel's asymptotic expansions, it follows that for large n,

$$|\tilde{\beta}_n(t_1,t_0)|^2\approx\frac{(t_1-t_0)^2}{4t_1t_0}$$

and hence the Bogoliubov coefficients are not square summable.

In fact, the unitarity problems persist on the physical Hilbert space (Torre).

- The problems can be traced back to the appearance of a factor t in the symplectic structure $\tilde{\Omega}_S(\varphi_1,\varphi_2)=\oint\left[\varphi_2t\partial_t\varphi_1-\varphi_1t\partial_t\varphi_2\right].$

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New field parametrization (preliminaries)

- It it is worth noticing that, for large n, our basis of solutions behaves as

$$f_n(t,\theta)=\frac{H_0(|n|t)}{\sqrt{8}}e^{in\theta}\approx\frac{e^{i\pi/4}}{\sqrt{4\pi|n|t}}e^{-i(|n|t-n\theta)}.$$

- The corresponding solutions for $\chi=\sqrt{t}\,\phi$ behave like the standard modes of a free massless scalar field propagating in a "Minkowski" background in 1+1 (or 2+1) dimensions.

Besides, the scaling of the field will absorb the problematic factor of t in the symplectic structure.

- Therefore we introduce the **TIME DEPENDENT** canonical transformation

$$\chi=\sqrt{t}\,\phi,\quad P_\chi=\frac{1}{\sqrt{t}}P_\phi.$$

A generating functional for this transformation is, e.g., $\tilde{F}=-\oint P_\phi\chi/\sqrt{t}.$

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New field parametrization (preliminaries)

- If $\dot{\phi}=\{\phi, H_r\}$, then $\dot{\chi}=\{\chi, H_r\}-\sqrt{t}\,\frac{d}{dt}\left(\frac{1}{\sqrt{t}}\right)\chi$. After the time dependent transformation, the new Hamiltonian is

$$\tilde{H}_r=H_r+\partial_t\tilde{F}=\frac{1}{2}\oint\left[P_\chi^2+(\chi')^2+\frac{P_\chi\chi}{t}\right].$$

- We can expand χ and P_χ in Fourier series:

$$\chi=\sum_{n=-\infty}^{\infty}\chi_{(n)}\frac{e^{in\theta}}{\sqrt{2\pi}},\qquad P_\chi=\sum_{n=-\infty}^{\infty}P_\chi^{(n)}\frac{e^{in\theta}}{\sqrt{2\pi}}.$$

The Fourier coefficients $\left(\chi_{(n)}, P_\chi^{(-n)}\right)$ are canonical pairs.

- We introduce "annihilation" (and "creation") variables for $n\neq 0$ [corresponding to a massless field]:

$$a_n=\frac{|n|\chi_{(n)}+iP_\chi^{(n)}}{\sqrt{2|n|}}.$$

- However, the corresponding vacuum (characterized by $\hat{a}_n|0\rangle=0$) does not belong to the domain of the Hamiltonian $\tilde{H}_r$.

New field parametrization

- The problem arises because of the term $\oint P_\chi \chi/(2t)$ in the Hamiltonian. We eliminate it by modifying the canonical transformation:

$$\xi = \sqrt{t}\,\phi, \quad P_\xi = \frac{1}{\sqrt{t}}\left(P_\phi + \frac{\phi}{2}\right).$$

- The reduced action and Hamiltonian become

$$S_r = \int_{t_i}^{t_f} dt \left(P\dot{Q} + \oint \left[P_\xi \dot{\xi}\right] - H_r^{(\xi)}\right), \quad H_r^{(\xi)} = \frac{1}{2}\oint\left[P_\xi^2 + \left(\xi'\right)^2 + \frac{\xi^2}{4t^2}\right].$$

This Hamiltonian is in fact that of a scalar field propagating in a flat (two-dimensional) background, though subject to a time dependent potential that **tends to zero at large times**. The field equations are $\ddot{\xi} - \xi'' + \frac{\xi}{4t^2} = 0$.

- There is one constraint, $C_0 = \frac{1}{\sqrt{2\pi}}\oint P_\xi \xi'$, which generates S^1-translations:

$$T_\alpha : \theta \to \theta + \alpha \quad \forall \alpha \in S^1.$$

New field parametrization: description

- We expand (ξ, P_ξ) in Fourier series, and introduce "annihilation" and "creation" variables for $n \neq 0$ [*corresponding to the "massless case"*] :

$$b_n = \frac{|n|\xi_{(n)} + iP_\xi^{(n)}}{\sqrt{2|n|}}, \quad b^*_{-n} = \frac{|n|\xi_{(n)} - iP_\xi^{(n)}}{\sqrt{2|n|}}.$$

We use these coordinates for the non-zero mode sector of phase space:

$$\{\boldsymbol{B}_m\}; \qquad \boldsymbol{B}_m = (b_m, b^*_{-m}, b_{-m}, b^*_m)^T \quad \forall\, m \in \mathbb{N}^+ .$$

- In terms of them, the S^1-translations have a diagonal action:

$$T_\alpha(b_{\pm m}) = e^{\pm i m \alpha} b_m, \quad T_\alpha(b^*_{\pm m}) = e^{-(\pm i) m \alpha} b^*_{\pm m}.$$

- Besides, the symplectic structure gets decomposed in 4x4 blocks:

$$\Omega(\{B_m^{(1)}\}, \{B_{\overline{m}}^{(2)}\}) = \sum_m (B_m^{(1)})^T \Omega_m B_m^{(2)}, \quad \Omega_m = \begin{pmatrix} \mathbf{0} & \omega_m \\ \omega_m & \mathbf{0} \end{pmatrix}, \quad \omega_m = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

New field parametrization: dynamics

- For ANY SYSTEM whose classical evolution commutes with S^1-translations and with the θ-reversal transformation $\left(b_m \leftrightarrow b_{-m},\, b_m^* \leftrightarrow b_{-m}^*\right)$, the evolution between constant-t sections is block diagonal and has the form:

$$\boldsymbol{B}_m(t_1) = U_m(t_1, t_0)\, \boldsymbol{B}_m(t_0),$$

$$U_m(t_1,t_0) = \begin{pmatrix} u_m(t_1,t_0) & \mathbf{0} \\ \mathbf{0} & u_m(t_1,t_0) \end{pmatrix}, \quad u_m(t_1,t_0) = \begin{pmatrix} \alpha_m(t_1,t_0) & \beta_m(t_1,t_0) \\ \beta_m^*(t_1,t_0) & \alpha_m^*(t_1,t_0) \end{pmatrix}.$$

$U_m(t_1, t_0)$ is a Bogoliubov transformation, with $|\alpha_m(t_1,t_0)|^2 - |\beta_m(t_1,t_0)|^2 = 1.$

- For the evolution generated by $H_r^{(\xi)}$ in our case, and defining $x_m^{(i)} = m\, t_i$:

$$\alpha_m(t_1,t_0) = c(x_m^{(1)})\, c^*(x_m^{(0)}) - d(x_m^{(1)})\, d^*(x_m^{(0)}), \quad \beta_m(t_1,t_0) = d(x_m^{(1)})\, c(x_m^{(0)}) - c(x_m^{(1)})\, d(x_m^{(0)}).$$

$$c(x_m^{(i)}) = \sqrt{\frac{\pi x_m^{(i)}}{8}} \left[\left(1 + \frac{i}{2x_m^{(i)}}\right) H_0(x_m^{(i)}) - i H_1(x_m^{(i)}) \right], \quad d(x_m^{(i)}) = \sqrt{\frac{\pi x_m^{(i)}}{2}}\, H_0^*(x_m^{(i)}) - c^*(x_m^{(i)}).$$

New field parametrization: quantization

- We introduce the complex structure $J_0(\boldsymbol{B}_m)=(J_0)_m \boldsymbol{B}_m$, with

$$(J_0)_m = diag(i,-i,i,-i).$$

We call H_0 the one-particle Hilbert space determined by J_0, $F(H_0)$ the symmetric Fock space, and $|0\rangle$ the vacuum (which satisfies $\hat{b}_{\pm m}|0\rangle=0$).

- J_0 is invariant under the S¹-translations T_α.

One obtains an invariant unitary implementation of this gauge group (with $\hat{T}_\alpha|0\rangle=|0\rangle$, $\hat{T}_\alpha \hat{b}_{\pm m} \hat{T}_\alpha^{-1}=e^{\pm i m}\hat{b}_{\pm m}$, $\hat{T}_\alpha \hat{b}^*_{\pm m} \hat{T}_\alpha^{-1}=e^{-(\pm i) m}\hat{b}^*_{\pm m}$).

- Furthermore, one can check that $\sum_m |\beta_m(t_1,t_0)|^2<\infty \quad \forall\, t_1, t_0 \neq 0.$

Hence, the **dynamical evolution is unitarily implemented** on $F(H_0)$ (and on the physical Hilbert space).

Uniqueness of the quantization

- Is this the only possible Fock quantization?

  We consider compatible complex structures that are **invariant** under the group of S^1-translations.

➢ Any such complex structure can be obtained as $J = K_J J_0 K_J^{-1}$ where K_J is a symplectic transformation with the block diagonal form in the $\{\boldsymbol{B}_m\}$ basis

$$(K_J)_m = \begin{pmatrix} (k_J)_m & \mathbf{0} \\ \mathbf{0} & (k_J)_m \end{pmatrix}, \quad (k_J)_m = \begin{pmatrix} \kappa_m & \lambda_m \\ \lambda_m^* & \kappa_m^* \end{pmatrix}, \quad \kappa_m > 0, \ |\kappa_m|^2 - |\lambda_m|^2 = 1\,.$$

➢ The evolution $U(t_1, t_0)$ is unitarily implementable w.r.t. J iff so is $U^J(t_1,t_0) := K_J^{-1} U(t_1,t_0) K_J$ w.r.t. J_0. We get the blocks

$$U_m^J(t_1,t_0) = \begin{pmatrix} u_m^J(t_1,t_0) & \mathbf{0} \\ \mathbf{0} & u_m^J(t_1,t_0) \end{pmatrix}, \quad u_m^J(t_1,t_0) = \begin{pmatrix} \alpha_m^J(t_1,t_0) & \beta_m^J(t_1,t_0) \\ \beta_m^{J*}(t_1,t_0) & \alpha_m^{J*}(t_1,t_0) \end{pmatrix},$$

$$\alpha_m^J(t_1,t_0) = |\kappa_m|^2 \alpha_m(t_1,t_0) - |\lambda_m|^2 \alpha_m^*(t_1,t_0) + \kappa_m^* \lambda_m^* \beta_m(t_1,t_0) - \kappa_m \lambda_m \beta_m^*(t_1,t_0),$$

$$\beta_m^J(t_1,t_0) = 2i\,\Im[\alpha_m(t_1,t_0)]\,\kappa_m^* \lambda_m + (\kappa_m^*)^2 \beta_m(t_1,t_0) - \lambda_m^2 \beta_m^*(t_1,t_0).$$

Uniqueness of the quantization

- From the expression of $\beta^J_m(t_1,t_0)$ it is easy to see that, $\forall\, a\geq 0,\ \ N\in\mathbb{N}^+$,

$$\sum_{m=1}^{N}\frac{2a|\lambda_m|^2}{1+a}\left\{1-(\Re[\alpha_m(t_1,t_0)])^2-a|\beta_m(t_1,t_0)|^2\right\}\leq\sum_{m=1}^{N}\left(2|\beta_m(t_1,t_0)|^2+|\beta^J_m(t_1,t_0)|^2\right).$$

The infimum of the coefficients of $|\lambda_m|^2$ might be non-positive.
For a free massless field, $\alpha_m(t_1,t_0)=e^{im(t_1-t_0)}$, $\beta_m(t_1,t_0)=0$.
The idea is to average over $\tau=t_1-t_0$ on a subset of $[0,\pi]$.

✔ We assume that the evolution is unitary not only w.r.t. J_0 but also w.r.t. J.

✔ We admit that $\alpha(t_0+\tau,t_0)$ and $\beta(t_0+\tau,t_0)$ are measurable functions of τ on $[0,\pi]$. This is the case for the Gowdy model.

- Then (using Egorov's theorem), one can show that, $\forall\,\delta>0$, $\exists\, E_\delta\subset[0,\pi]$ with Lebesgue measure $\mu(E_\delta)>\pi-\delta$ and $\exists\, I_\delta>0$ such that the integral over E_δ of the r.h.s. of the above inequality is bounded by I_δ $\forall\, N\in\mathbb{N}^+$.

Uniqueness of the quantization

- We assume that, for (a given value of) t_0, $\exists\,\delta\in(0,\pi)$ such that, $\forall\, E\subset[0,\pi]$ with $\mu(E)>\pi-\delta$, $\exists\,\Delta(E)>0$ that satisfies

$$\int_E d\tau\left\{1-(\Re[\alpha_m(t_0+\tau,t_0)])^2\right\}>\Delta(E)\quad\forall\, m\in\mathbb{N}^+.$$

This condition is fulfilled in the Gowdy model (and in the free massless case).

- Integrating our inequality over the set E_δ (provided by Egorov's theorem), and taking $0<a<2\Delta(E_\delta)/I_\delta$, we then get

$$\sum_{m=1}^{N}|\lambda_m|^2\leq\frac{(1+a)\,I_\delta}{a[2\Delta(E_\delta)-a\,I_\delta]}<\infty\quad\forall\, N\in\mathbb{N}^+.$$

Hence, the symplectic transformation K_J is implemented unitarily on $F(H_0)$.

In conclusion, all compatible complex structures that are invariant under S^1-translations and allow a unitary implementation of the dynamics determine unitarily equivalent Fock representations.

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Remarks

- The proof of uniqueness is valid for more general field models. This is the case, e.g., when the dynamics coincides with the free massless one in the entire future of a Cauchy surface.

- Uniqueness holds if one replaces the condition of unitary implementation of the dynamics by the requirement that the Fock vacuum belong to the domain of the Hamiltonian.

- Apparently, compactness of the spatial sections plays an important role in the proof. In contrast, the spatial dimension does not seem to be so relevant.

- Let $J_{M(t_0)}$ be the natural complex structure corresponding to a free field with mass equal to the instantaneous value $M(t_0)=1/(4t_0^2)$. With compact topology, J_0 and $J_{M(t_0)}$ lead to unitarily equivalent Fock representations.

However, the complex structures that $J_{M(t_0)}$ induces by time evolution differ from $J_{M(t)}$.

Remarks

- Torre has recently shown that, in the Schrödinger representation of the standard field description of the Gowdy T^3 model, Gaussian measures at different times are mutually singular. This problem disappears with the new field parametrization.

- The new field parametrization can be extended to other models. In the Gowdy T^3 model with general polarization, the reduced system consists of two coupled scalar fields; one of them propagates again in a flat (two-dimensional) background with a time dependent mass $M(t_0)=1/(4t_0^2)$.

In the case of the Einstein-Rosen waves (linearly polarized cylindrical waves) a similar scaling $\xi=\sqrt{r}\,\phi$ seems to cure the unitarity problems detected in parametrized field theory by Cho and Varadarajan. This scaling was already considered by Niedermaier, who defined the fundamental field on AdS_2.

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Conclusion

- We have introduced a new field parametrization for the Gowdy metric and completed it into a time dependent canonical transformation. In this way, we have redistributed the time dependence into an explicit and an implicit part, the latter being unitarily implementable in the quantum theory.

- There exists certain freedom in the choice of deparametrization (time gauge) and field parametrization for the model. Once these choices are made, the Fock representation is essentially unique if one demands unitary implementation of the dynamics and invariance under S^1-translations.

- The Fock vacuum is not left invariant by the time evolution. As a consequence, there is some "particle" production, that attenuates for asymptotically large times. However, these "particles" have no genuinely physical interpretation.

- It would be interesting to construct geometric operators, define a proper time and study the singularity quantum mechanically, comparing the results with those of other possible approaches to the quantization.